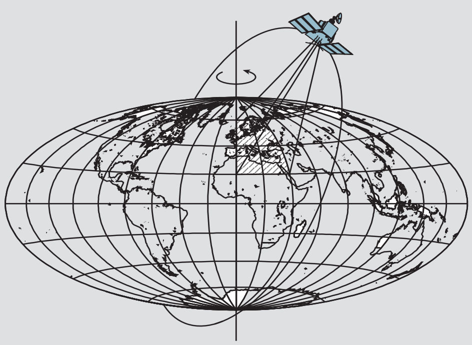
Spline Representations of Functions on a Sphere for Geopotential Modeling

by

Christopher Jekeli



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Final Technical Report

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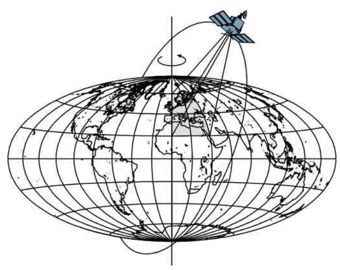

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Preface

This report was prepared with support from the National Geospatial-Intelligence Agency under contract NMA302-02-C-0002 and serves as the final technical report for this project.

Abstract

Three types of spherical splines are presented as developed in the recent literature on constructive approximation, with a particular view towards global (and local) geopotential modeling. These are the tensor-product splines formed from polynomial and trigonometric *B*-splines, the spherical splines constructed from radial basis functions, and the spherical splines based on homogeneous Bernstein-Bézier (BB) polynomials. The spline representation, in general, may be considered as a suitable alternative to the usual spherical harmonic model, where the essential benefit is the local support of the spline basis functions, as opposed to the global support of the spherical harmonics. Within this group of splines each has distinguishing characteristics that affect their utility for modeling the Earth's gravitational field. Tensor-product splines are most straightforwardly constructed, but require data on a grid of latitude and longitude coordinate lines. The radial-basis splines resemble the collocation solution in physical geodesy and are most easily extended to three-dimensional space according to potential theory. The BB polynomial splines apply more generally to any sphere-like surface (e.g., the geoid or the Earth's surface) and have a strong theoretical legacy in the field of spline approximations. This report provides a review of these three types of splines, their application to the geodetic boundary-value problem, and formal expressions for determining the model coefficients using data with observational errors.

1. Introduction

Surface and near surface (that is, airborne) gravity data are still the essence of very high-resolution global gravity models (e.g., EGM96 as described by Lemoine et al., 1998) that have become standard in physical geodesy and associated disciplines. In contrast, satellite-derived gravity observations define the global, longer-wavelength components. They form the basis for spherical harmonic expansions of the field up to degree 120 with GRACE (Tapley et al., 2004) and are expected to yield a model up to degree 250 and higher with the upcoming satellite gradiometer mission, GOCE (Rummel et al., 2002). However, a global model with resolution of 10 km, corresponding to spherical harmonic degree 2160, is now being constructed (Pavlis et al., 2004), based on surface and near-surface gravity data and satellite altimetry data in ocean areas. The surface gravity data do not necessarily possess uniform resolution, and in many parts of the world's continents they do not yet support a global 2160-degree spherical harmonic expansion. Conversely, some areas are endowed with even greater resolution requiring some form of decimation or averaging in order to be consistent with a 2160-degree harmonic expansion.

When considering alternatives to the spherical harmonic model for the Earth's gravity field it is necessary to understand its disadvantages, as well as the desirable attributes that any global model should possess. Perhaps the greatest *disadvantage* of spherical harmonics, from the point of view of modeling, is that they are basis functions with global support. That is, each spherical harmonic function is significantly different from zero almost everywhere on the entire sphere. This means that each spherical harmonic coefficient in the model, corresponding to a basis function, depends essentially on all the data over the entire sphere. Change the data at a single point, and in theory all the coefficients change. A second disadvantage (which actually is a consequence of the global support for the basis functions) is that the evaluation of the field at any point requires that all coefficients be accumulated even though the field at one point hardly depends on its values half way around the world. Related to these disadvantages is the intractability of rigorously propagating the errors in the data through the spherical harmonic coefficients. Approximations (Han, 2004) or alternative techniques (Pavlis, and Saleh, 2004) must be employed to calculate the error statistics for derived quantities from those of the data. Finally, as already noted, spherical harmonic models depend on regularly gridded data, and specifically on latitude-longitude grids. Aside from the artificial requirements associated with the meridian convergence (where many more data per unit area are needed near the poles than near the equator), it is clear that globally the surface data are far from uniformly distributed.

Although a high-resolution global gravity model is based on surface data over the entire Earth, it must be more than a representation of a surface function. The gravitational potential is a scalar function in three-dimensional space and many of the gravitational quantities that we measure or wish to interpret geophysically or geodetically are the derivatives of the potential, not just the horizontal derivatives, but, in fact, mainly the vertical derivatives. Since the gravitational potential is harmonic in free space, we wish the model to be harmonic as well, although this is not as crucial from a practical viewpoint if the ability to compute accurate derivatives is fulfilled. Since our data are always discrete and contain observation error, any modeling of the potential from the data will yield only an approximation that may be greater than that associated with the non-harmonicity of the model. It is quite certain, that in the foreseeable future, spherical harmonic models will always be constructed from satellite gravity mapping missions, such as GRACE and GOCE and their successors. It is equally clear that these models will have limited resolution (i.e., relatively low maximum degree). Therefore, it is desirable that any high-resolution model be easily combinable with a low-degree spherical harmonic model. In theory this is always possible by referring data to a given spherical harmonic model and representing the residuals by any other model.

To overcome the disadvantages of the spherical harmonic model, an alternative model would allow local data to be changed or new data to be incorporated without changing the model globally. Data would not need to be uniformly distributed and the model would locally reflect the significant resolution of the function. That is, where the data are sparse, the model would correspondingly have low resolution; and, where the function has significant variation (and the data to support it), the model would have high resolution. It quickly becomes clear that such characteristics can only be fulfilled if the basis functions have local support. That is, the basis functions are zero everywhere except in a local region of the sphere. Such representations are well developed for functions in Cartesian space (on the line, the plane, etc.) and fall under the broad category of *splines*. Table 1 provides a brief overview of the desirable characteristics of a global geopotential model and how spherical harmonics compare with spline representations.

Table 1: Two types of global geopotential models in terms of desirable properties.

Property	**Spherical harmonic expansion**	**Spline representation**
Faithful representation of data (resolution and accuracy)	always based on gridded data (result subject to aliasing and spectral leackage)	basis functions depend on data distribution
Reliable error statistics	rigorous error propagation is difficult, since it passes through the global spectrum	error propagation depends only on local error sources
Predictive capability (e.g., geoid undulation from gravity anomalies)	straightforward using spectral relationships (spherical approximation)	computationally more difficult, based on space domain convolutions
Update capability (model modification with new data)	requires re-computation of the entire model	affects only that part of the model representing the data update region
Manipulative capability (computational & mathematical)	relatively easy, but all coefficients of the model must be acumulated	only relevant part of model requires computation
Potential theory foundation	3-D harmonic structure of the model is implicit	surface function model serves as input to the boundary-value problem

While it is the opinion of the author that spherical harmonic series will continue to dominate global gravitational field modeling, even to extraordinarily high degree (e.g., to degree 10,000 within the next 15-20 years), the following discussion presents an alternative representation based on recent work in *spherical* splines accomplished by mathematicians at Vanderbilt University (Alfeld et al., 1996c). This is compared to spherical splines also developed recently by a group at Kaiserslautern University (Freeden et al., 1998), and to tensor-product splines, also investigated, among others, by the Vanderbilt group (Schumaker and Traas, 1991).

The literature on splines, finite elements, and constructive approximation in general is vast, though only a few decades in the making to a mature discipline, and it is oriented primarily to CAGD (computer-aided geometric design). The present report does not offer a comprehensive review of these techniques, nor a review of geodetic applications that have surfaced sporadically in concert with these developments. Instead, an attempt is made only to examine a few recent advancements by the Vanderbilt group and to compare these to the developments by the Kaiserslautern group, which has promoted the radial basis functions in the geodetic community for some time. The motivation for this analysis was the outcome of the Geopotential Modeling Workshop, organized by DARPA, and held on 15-16 March 2001 in Arlington, Virginia.

2. Preliminaries

Let points in 3-D space, $\mathbb{R}^3$, be denoted by *v*, and their restriction to the unit sphere, W, by x (or also h), whose Cartesian components may be thought of in terms of familiar spherical polar coordinates, co-latitude, q, and longitude, l:

$$x = \begin{pmatrix} x_x \\ x_y \\ x_z \end{pmatrix} = \begin{pmatrix} \sin q \cos l \\ \sin q \sin l \\ \cos q \end{pmatrix} . \tag{2.1}$$

Suppose the unit sphere is populated by a finite set of points, $\{h_i\}_{i=1,\ldots,V}$, with no particular distribution imposed at the moment. At each of these points, or vertices, the value of a function, *f*, is given:

$$y_i = f(h_i), \quad i = 1,\ldots,V \ . \tag{2.2}$$

It is desired to represent the function, *f*, on the sphere using some interpolating function, one that represents the function between the data points with some degree of smoothness and reproduces the data values at the data points. We may also allow errors in the function values if they represent observations, then

$$\tilde{y}_i = f(h_i) + e_i, \quad i = 1,\ldots,V \ . \tag{2.3}$$

The desired interpolating function, in this case, approximates the data values at the data points in a least-squares sense (see Section 7). In view of the introductory discussion, the interpolating functions to be considered are splines, *s*, constructed from locally supported functions, w_n, on the sphere:

$$s(x) = \sum_{n=1}^{M} c_n w_n(x) \ . \tag{2.4}$$

As a matter of terminology, we call s a spline representation of the function, *f*; and, also the component functions, w_n, that usually are the basis functions of the spline space, are called splines.

It is noted that spherical splines of the Vanderbilt group were developed more generally for *sphere-like* surfaces, e.g., the geoid or even the Earth's surface (if it can be viewed as a sufficiently smooth radial projection from the sphere). We consider only spheres in this review.

A brief overview of spherical harmonics is necessary in order to contrast this conventional representation with the spherical spline models. A polynomial, $p^{(d)}$, of degree *d* is called *homogeneous* if for constant, a:

$$p^{(d)}(a\,v) = a^{d}\, p^{(d)}(v) \ . \tag{2.5}$$

A general, homogenous, degree-*d* polynomial in $v \in \mathbb{R}^3$ may be written as

$$p^{(d)}(v) = \sum_{p+q+r=d} a_{pqr}\, x^p y^q z^r \ \ , \tag{2.6}$$

where $v = (x, y, z)^{\mathrm{T}}$. The space of homogeneous (trivariate) polynomials of degree *d* has dimension $(d+2)(d+1)/2$ (which can be verified by counting the number of possible distinct summands in equation (2.6)). Now consider *harmonic*, homogeneous polynomials, which belong to a subset of the homogeneous polynomials and satisfy Laplace’s equation:

$$\nabla^2 p^d(v) = 0 \ , \tag{2.7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Restricted to the sphere, these harmonic homogeneous polynomials are defined to be the s*pherical harmonics* (Müller, 1966). Thus, a spherical harmonic of degree ℓ is a harmonic homogeneous polynomial (of degree ℓ), expressible as

$$\bar{Y}_{\ell m}(x) = \sum_{p+q+r=\ell} a_{pqr}^{(m)} (x_x)^p (x_y)^q (x_z)^r, \quad m = -\ell, \ldots, \ell \ , \tag{2.8}$$

where the order, *m*, identifies the $2\ell+1$ independent polynomials of degree ℓ. A more conventional expression of a spherical harmonic function (polynomial) of degree ℓ and order *m* is given by

$$\bar{Y}_{\ell m}(x) = \bar{P}_{\ell|m|}(\cos q)\, e^{iml}\,; \quad m = -\ell, \ldots, \ell; \quad \ell = 0, 1, \ldots \ ; \tag{2.9}$$

where the associated Legendre functions (of the first kind) are fully normalized so that the square of a spherical harmonic integrated over the unit sphere is equal to $4p$:

$$\bar{P}_{lm}(\cos q) = \begin{cases} \sqrt{2(2n+1)\dfrac{(n-m)!}{(n+m)!}}\, \sin^m q\, \dfrac{d^m}{d(\cos q)^m} P_n(\cos q), & 0 < m \le \ell; \\ \sqrt{(2n+1)}\, P_n(\cos q), & m = 0; \end{cases} \tag{2.10}$$

where P_n is the usual Legendre polynomial of degree, *n*.

As noted, the dimension of the space of harmonic homogeneous polynomials (i.e., spherical harmonics) of degree ℓ is $2\ell+1$, which is well known to geodesists and proved in (Müller, 1966) and (Freeden et al., 1998). Therefore, the set of spherical harmonics

$\left\{\bar{Y}_{\ell m}\right\}_{m=-\ell,\ldots,\ell;\ \ell=0,\ldots,L}$ is a basis for harmonic polynomials of maximum degree L on the sphere. Clearly, the spherical harmonics, $\bar{Y}_{\ell m}$, have global support as illustrated in Figure 1, meaning that their magnitude is different from zero almost everywhere on the unit sphere.

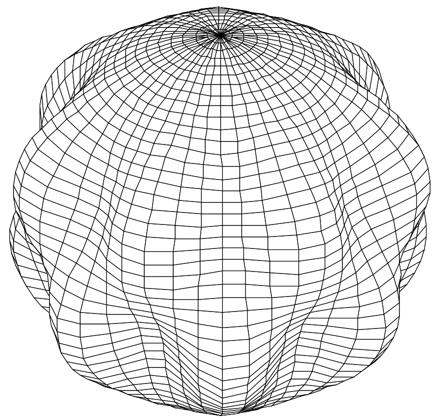

Figure 1: Spherical harmonic function, $\bar{Y}_{8,7}$.

The function with values given by equation (2.2) may be represented by a finite series of spherical harmonics:

$$s_{SH}(x)=\sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell}c_{\ell m}\bar{Y}_{\ell m}(x)\ , \tag{2.11}$$

where the coefficients can be determined by setting

$$s_{SH}(h_i)=y_i\ , \tag{2.12}$$

and hence:

$$\begin{pmatrix} c_{0,0} \\ \vdots \\ c_{L,L} \end{pmatrix}=\begin{pmatrix} \bar{Y}_{0,0}(h_1) & \cdots & \bar{Y}_{L,L}(h_1) \\ \vdots & \ddots & \vdots \\ \bar{Y}_{0,0}(h_V) & \cdots & \bar{Y}_{L,L}(h_V) \end{pmatrix}^{-1}\begin{pmatrix} y_1 \\ \vdots \\ y_V \end{pmatrix}. \tag{2.13}$$

Here, it is assumed that V, the number of data points, equals the number of spherical harmonics, $(L+1)^2$, and that the points, h_i, represent an admissible set, in the sense that the matrix in equation (2.13) is non-singular. Furthermore, errors in the data values have been ignored. The spherical harmonic model (2.11) resembles our desired spline representation (2.4), except that the harmonics have global support.

3. Tensor-Product Spherical Splines

Borrowing directly from splines on the line, Schumaker and Traas (1991) developed spherical splines by adapting conventional polynomial *B*-splines. *B*-splines are piecewise polynomials associated with a finite set of knots, or sequential, possibly identical points. The degree of the polynomial pieces is equal to one less than the number of knots over which the *B*-spline is defined. The number of knots plus one also defines the order of the spline and the support of the spline is the interval associated with these knots (for a thorough mathematical treatment of *B*-splines, see Schumaker, 1981, Chapter 4.3). If $N_k^n(q)$, $k=1,\ldots,K$, denotes a set of splines of order n on a meridian, and $T_j^m(l)$, $j=1,\ldots,J+m-1$ denotes a set of m^{th}-order splines on a circle of latitude, the *tensor-product*

$$s_{TP}(x) \equiv s_{TP}(l,q) = \sum_{j=1}^{J+m-1} \sum_{k=1}^{K} c_{j,k} T_j^m(l) N_k^n(q) \tag{3.1}$$

is a bivariate spline on the sphere. The spline is to be fitted to given function values, as in equation (2.12), which then determines the coefficients, $c_{j,k}$.

The two types of *B*-splines above are defined on a partition of the respective domains, $0 \le l \le 2p$ and $0 \le q \le p$. The partition is created with a distribution of points, called knots that, in general, have multiplicity, especially at the ends of the total domain; that is, a single coordinate point may represent several knots. For the co-latitude domain, we define the knots, q_k, as shown in Figure 2a, with $K-n+2$ distinct knot points. Note that the geographic poles have multiple knots and that the knot interval, $q_{k+1}-q_k$, need not be a constant (when it is non-zero). A *B*-spline of order n is defined on any interval, (q_k, q_{k+n}), such that $q_k < q_{k+n}$. The following formulas yield (normalized) *B*-splines of order n:

$$N_k^1(q) = \begin{cases} 1, & q_k \le q < q_{k+1} \\ 0, & \text{otherwise} \end{cases}, \quad k=n,\ldots,K$$

$$N_{n+1-q}^q(q) = \begin{cases} \left(\dfrac{q_{n+1-q}-q}{q_{n+1}-q_{n+1-q}} \right)^{q-1}, & q_{n+1-q} \le q < q_{n+1} \\ 0, & \text{otherwise} \end{cases}, \quad q=2,\ldots,n \tag{3.2}$$

$$N_k^n(q) = \frac{q-q_k}{q_{k+n-1}-q_k} N_k^{n-1}(q) + \frac{q_{k+n}-q}{q_{k+n}-q_{k+1}} N_{k+1}^{n-1}(q), \quad n \ge 2 ,$$

where the recursion should only be applied when the splines on the right side exist (N_k^n is not defined if $q_k = q_{k+n}$). It can be shown (Schumaker, 1981, pp.121-124) that N_k^n is non-negative on the interval, (q_k, q_{k+n}) and vanishes outside it, which defines its support; and, that the *B*-splines, $\{N_1^n, \ldots, N_K^n\}$, are linearly independent on the interval $(q_n, q_{K+1}) = (0, p)$.

$$0 = q_1 = \ldots = q_n \;<\; q_{n+1} \;<\; q_{n+2} \;<\; \ldots \;<\; q_K \;<\; q_{K+1} = \ldots = q_{K+n} = p$$

Figure 2a: Partition of latitude and corresponding knots for the spline definition.

$$0 = l_m \;<\; l_{m+1} \;<\; l_{m+2} \;<\; \ldots \;<\; l_{J+m-2} \;<\; l_{J+m-1} \;<\; l_{J+m} = 2p$$

$$i = 1, \ldots, m-1: \quad l_i = l_{J+i} - 2p \qquad\qquad l_{J+m+i} = l_{m+i} + 2p$$

Figure 2b: Partition of longitude and corresponding knots for the spline definition.

Instead of the same type of polynomial *B*-splines for the longitude, as used, e.g. by Gmelig Meyling and Pfluger (1987), Schumaker and Traas (1991) made use of analogous (normalized) *trigonometric B*-splines (Schumaker, 1981, p.452), defined on the partition of longitude shown in Figure 2b, with *J* distinct knot points. In this way one is able to construct exact spline models that have continuous derivatives at the poles. In our application the knots for these splines have no multiplicity in the sense of the knots in latitude, but are defined periodically near the zero meridian. The following formulas sufficiently define the normalized trigonometric *B*-splines:

$$T_j^1(l) = \begin{cases} 1, & l_j \le l < l_{j+1} \\ 0, & \text{otherwise} \end{cases}, \quad j = 1, \ldots, J+m-1$$

$$T_j^m(l) = \frac{\sin\dfrac{l - l_j}{2}}{\sin\dfrac{l_{j+m-1} - l_j}{2}} T_j^{m-1}(l) + \frac{\sin\dfrac{l_{j+m} - l}{2}}{\sin\dfrac{l_{j+m} - l_{j+1}}{2}} T_{j+1}^{m-1}(l), \quad m \ge 2 \; . \tag{3.3}$$

As before, the j^{th} trigonometric *B*-spline of order *m* is non-negative on the interval, (l_j, l_{j+m}), and vanishes outside it, which defines its support. Figures 3a and 3b respectively show third-order polynomial and trigonometric *B*-splines for regular partitions in latitude and longitude.

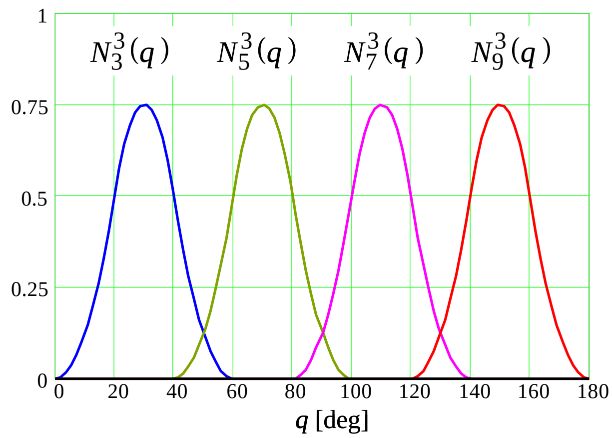

Figure 3a: Polynomial *B*-splines of third order on a partition of co-latitude, with $q_{k+1} - q_k = 20°$.

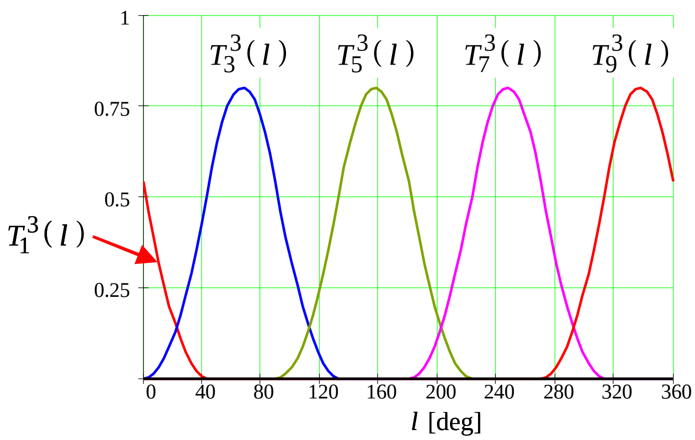

Figure 3b: Trigonometric *B*-splines of third order on a partition of longitude, with $l_{j+1} - l_j = 45°$.

The coefficients, $c_{j,k}$, in equation (3.1) can be determined by solving a linear system of equations:

$$\underset{V\times1}{y} = \underset{V\times K(J+m-1)}{A}\ \underset{K(J+m-1)\times1}{c}, \tag{3.4}$$

where $y = (y_1, \ldots, y_V)^{\mathrm{T}}$ (similarly c is the vector of coefficients) and the elements of matrix A are the values of the product of splines evaluated at the data points, h_i. This linear system is constrained by conditions that ensure the continuity of the spline, s_{TP}, and its first-order derivatives across the zero meridian and at the poles. Such conditions were given by Dierckx (1984) and Gmelig Meyling and Pfluger (1987) as follows. At the zero-meridian, there must be

$$
\begin{aligned}
&s_{TP}(0,q) = s_{TP}(2p,q), \quad 0 \le q \le p ; \\
&\frac{\partial}{\partial l} s_{TP}(0,q) = \frac{\partial}{\partial l} s_{TP}(2p,q), \quad 0 \le q \le p ;
\end{aligned}
\tag{3.5}
$$

and, at the poles we require that

$$
\begin{aligned}
&s_{TP}(l,0) = s_N , \quad s_{TP}(l,p) = s_S , \quad 0 \le l \le 2p ; \\
&\frac{\partial}{\partial q} s_{TP}(l,0) = A_N \cos l + B_N \sin l , \quad \frac{\partial}{\partial q} s_{TP}(l,p) = A_S \cos l + B_S \sin l , \quad 0 \le l \le 2p ;
\end{aligned}
\tag{3.6}
$$

where $s_N, s_S, A_N, B_N, A_S, B_S$ are constants. Condition (3.5) and the knot sequence defined in Figure 2b imply that the coefficients associated with the trigonometric *B*-splines, T_i^m, should be identified, respectively, with the coefficients associated with T_{J+i}^m:

$$
c_{i,k} = c_{J+i,k}, \quad i = 1,\ldots,m-1, \quad k = 1,\ldots,K . \tag{3.7}
$$

Schumaker and Traas (1991) provided an additional set of $4J+2$ constraints on the coefficients to satisfy condition (3.6) for the case of third-order ($m=3$) trigonometric splines, of which 8 are duplicates of constraints (3.7). With $m=3$, the total number of conditions is $4J+2K-6$; and, in this case, we must solve

$$
\underset{V\times 1}{y} = \underset{V\times K(J+2)}{A} \; \underset{K(J+2)\times 1}{c} , \tag{3.8}
$$

subject to constraints

$$
\underset{(4J+2K-6)\times K(J+2)}{B} \; \underset{K(J+2)\times 1}{c} = \underset{(4J+2K-6)\times 1}{0} . \tag{3.9}
$$

For a least-squares solution for the coefficients in the case that the observed data contain errors, see Section 7.

Tensor-product splines are most suited for data on (co-)latitude/longitude grids (for randomly scattered data, see the following sections). Thus, we define the data points to be knots; and, we have

$$
\{h_i\}_{i=1,\ldots,V} = \{(l_j, q_k)\}_{j=m,\ldots,J+m-1;\,k=n,\ldots,K+1} , \quad \{y_i\}_{i=1,\ldots,V} = \{y_{j,k}\}_{j=m,\ldots,J+m-1;\,k=n,\ldots,K+1} . \tag{3.10}
$$

The spline fitting problem as such is a global problem, with large (albeit sparse) *A*-matrix. Schumaker and Traas (1991) offered an approximate method (introduced for planar tensor-product splines by Schumaker, 1976) to obtain the coefficients using local data. Figure 4 shows a local region of the data grid, where knot lines are interwoven with between-knot lines. At the knots we have data values, $y_{j,k}$; between the knots, they must be estimated by some standard procedure, such as by local biquadratic polynomial interpolation, or even by a simple average, for example:

$$y_{j+1/2,k+1/2} = \frac{1}{4}\left(y_{j,k} + y_{j+1,k} + y_{j,k+1} + y_{j+1,k+1} \right) . \tag{3.11}$$

Schumaker and Traas (1991) proved that for third-order splines, the following *quasi-interpolator* spline, $Qs_{TP}\left(l, q \right)$, satisfies the conditions (3.5) and (3.6), as well as

$$Qs_{TP}\left(l_j, q_k \right) \approx y_{j,k} , \quad j = m, \ldots, J + m - 1; \quad k = n, \ldots, K + 1 . \tag{3.12}$$

It can be shown (ibid.) that

$$Qs_{TP}\left(l, q \right) = \sum_{j=1}^{J+2} \sum_{k=1}^{K} \hat{c}_{j,k} T_j^m\left(l \right) N_k^n\left(q \right) , \tag{3.13}$$

where

$$\begin{aligned} \hat{c}_{j,k} &= \frac{1}{4} y_{j+1,k+1} - \frac{1+s_j}{2} y_{j+1.5,k+1} + \frac{1}{4} y_{j+2,k+1} \\ &\quad - y_{j+1,k+1.5} + 2\left(1+s_j\right) y_{j+1.5,k+1.5} - y_{j+2,k+1.5} \\ &\quad + \frac{1}{4} y_{j+1,k+2} - \frac{1+s_j}{2} y_{j+1.5,k+2} + \frac{1}{4} y_{j+2,k+2} ; \quad j = 1, \ldots, J+2 ; \quad k = 3, \ldots, K-2 \end{aligned} \tag{3.14}$$

$$\hat{c}_{j,1} = -\frac{1}{2} y_{j+1,1} + 2\left(1+s_j\right) y_{j+1.5,1} - \frac{1}{2} y_{j+2,1} ; \quad j = 1, \ldots, J+2 \tag{3.15}$$

$$\begin{aligned} \hat{c}_{j,2} &= \frac{q_4 - q_3}{4}\left(-y_{j+1,3}^{(q)} + 2\left(1+s_j\right) y_{j+1.5,3}^{(q)} - y_{j+2,3}^{(q)} \right) \\ &\quad + \frac{1}{2}\left(-y_{j+1,3} + 2\left(1+s_j\right) y_{j+1.5,3} - y_{j+2,3} \right) ; \quad j = 1, \ldots, J+2 \end{aligned} \tag{3.16}$$

$$\hat{c}_{j,K-1} = \frac{q_{K+1} - q_K}{4}\left(y^{(q)}_{j+1,K+1} - 2\left(1+s_j\right) y^{(q)}_{j+1.5,K+1} + y^{(q)}_{j+2,K+1}\right)$$
$$+\frac{1}{2}\left(-y_{j+1,K+1} + 2\left(1+s_j\right) y_{j+1.5,K+1} - y_{j+2,K+1}\right); \quad j = 1,\ldots,J+2 \tag{3.17}$$

$$\hat{c}_{j,K} = -\frac{1}{2} y_{j+1,K} + 2\left(1+s_j\right) y_{j+1.5,K} - \frac{1}{2} y_{j+2,K}\ ; \quad j = 1,\ldots,J+2 \tag{3.18}$$

where $y^{(q)}_{j,k} = \frac{\partial}{\partial q} f\left(l_j, q_k\right)$ and $s_j = \cos\frac{l_{j+2} - l_{j+1}}{2}$. Determination of the coefficients of the (approximate) spline model thus involves only the data values in the neighborhood of the support for the corresponding *B*-splines.

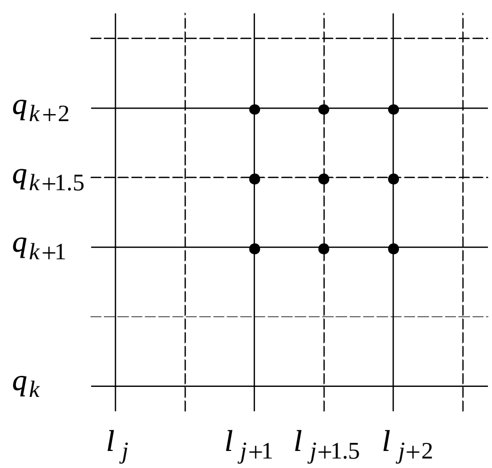

Figure 4: Grid for local quasi-interpolation.

4. Freeden / Schreiner Spherical Splines

For these and the following spherical splines, the data points need not be gridded in latitude and longitude – they may be distributed arbitrarily on the sphere. Freeden et al. (1998) developed spherical splines based on so-called *radial basis functions*. These functions depend only on the spherical distance from a given point (say, h_i) and thus can be represented as infinite series of Legendre polynomials (i.e., spherical harmonics of zero order):

$$K_i\left(x\right) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{A_\ell^2} P_\ell\left(x \cdot h_i\right) \ , \tag{4.1}$$

where the numbers, A_ℓ, are non-zero and are defined such that the series can be summed. (Freeden et al., 1998, also allowed a subset of the numbers, A_ℓ, to be zero, in which case the development becomes slightly more complicated, but for present purposes it is not necessary to consider this more general case.)

Consider continuous functions, *f*, defined on the sphere, W, whose spherical harmonic expansions are formulated, as usual:

$$f(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} F_{\ell m} \overline{Y}_{\ell m}(x) \ , \tag{4.2}$$

and whose harmonic coefficients satisfy:

$$\sum_{\ell=0}^{\infty} A_\ell^2 \sum_{m=-\ell}^{\ell} F_{\ell m}^2 < \infty \ , \tag{4.3}$$

for a given set of numbers, A_ℓ. Then, we can construct the following inner product for these functions:

$$\left\langle f, g \right\rangle_{H(A)} = \sum_{\ell=0}^{\infty} A_\ell^2 \sum_{m=-\ell}^{\ell} F_{\ell m} G_{\ell m} \ . \tag{4.4}$$

The square of the norm of a function is given by $\| f \|_{H(A)}^2 = \langle f, f \rangle_{H(A)}$. The subscript, $H(A)$, has been appended because the space of functions satisfying equation (4.3) and endowed with the inner product (4.4) is a *Hilbert space*, depending on the given set of numbers, $A = \{A_\ell\}$.

A function of the form of equation (4.1) is now a reproducing kernel in this Hilbert space. Indeed, for the kernel,

$$K_A(x, h) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{A_\ell^2} P_\ell(x \cdot h) \ , \tag{4.5}$$

the reproducing property is readily verified:

$$\left\langle f, K(\cdot, h) \right\rangle_{H(A)} = \sum_{\ell=0}^{\infty} A_\ell^2 \sum_{m=-\ell}^{\ell} F_{\ell m} \left(\frac{\overline{Y}_{\ell m}(h)}{A_\ell^2} \right) = f(h) \ , \tag{4.6}$$

where the well known decomposition formula for Legendre polynomials (addition theorem for spherical harmonics) has been used to infer the harmonic coefficients of the kernel:

$$P_\ell(x \cdot h) = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(x) \bar{Y}_{\ell m}(h) \ . \tag{4.7}$$

The summability of the series (4.5) defining the reproducing kernel essentially depends on the set, *A*; that is, we require

$$\sum_{\ell=0}^{\infty} (2\ell+1) \frac{1}{A_\ell^2} < \infty \ . \tag{4.8}$$

One may impose even stricter summability conditions (Freeden et al., 1998, p.88). For example, the set *A* is "k -summable", $k \geq 0$, if

$$\sum_{\ell=0}^{\infty} (2\ell+1) \frac{\ell^{2k}}{A_\ell^2} < \infty \ . \tag{4.9}$$

Clearly, if inequality (4.9) holds, then so does (4.8). Consider the Laplace-Beltrami operator, D^*, which is just the familiar Laplacian operator restricted to the sphere; that is, a combination of second-order horizontal derivatives:

$$D^* = \frac{\partial^2}{\partial q^2} + \cot q \, \frac{\partial}{\partial q} + \frac{1}{\sin^2 q} \frac{\partial^2}{\partial l^2} \ . \tag{4.10}$$

From Heiskanen and Moritz (1967, p.20), we have

$$D^* \bar{Y}_{\ell m} = -\ell(\ell+1) \bar{Y}_{\ell m} \ , \tag{4.11}$$

and, consequently,

$$\left(-D^* + \frac{1}{4}\right)^{k/2} \bar{Y}_{\ell m} = (\ell+1)^k \, \bar{Y}_{\ell m} \ . \tag{4.12}$$

From the harmonic representation (4.2), we find immediately that the harmonic coefficients of the horizontal derivatives of order k increase essentially by the factor, $(\ell + 1/2)^k$:

$$\left(-D^* + \frac{1}{4}\right)^{k/2} f(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\ell + \frac{1}{2}\right)^k F_{\ell m} \bar{Y}_{\ell m}(x) \ . \tag{4.13}$$

By Schwartz's inequality, we have

$$\left|\left(-D^*+\frac{1}{4}\right)^{k/2} f(x)\right|^2 = \left|\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\ell+\frac{1}{2}\right)^k F_{\ell m}\bar{Y}_{\ell m}(x)\right|^2$$
$$\le \left(\sum_{\ell=0}^{\infty} A_\ell^2 \sum_{m=-\ell}^{\ell} F_{\ell m}^2\right)\left(\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)^{2k} A_\ell^{-2}\sum_{m=-\ell}^{\ell}\bar{Y}_{\ell m}^2(x)\right) < \infty , \qquad (4.14)$$

making use of the boundedness conditions (4.3) and (4.9). In other words, with k-summability for *A*, the functions in the consequent Hilbert space have continuous derivatives up to order k. This is an important result when we consider the definition of the Hilbert space that should contain the spline representation of our data.

Any function in the Hilbert space, $H(A)$, having the form

$$s_{FS}(x) = \sum_{i=1}^{V} c_i K_A^{(h_i)}(x) , \qquad (4.15)$$

where $K_A^{(h_i)}(x) = K_A(x, h_i)$, is defined to be a *spherical spline*. Here, as before, the points, h_i, on the unit sphere locate data values of some function. The coefficients, c_i, are determined on the basis of some constraints connected with the representation problem. The subscript, *FS*, is used to distinguish these splines as those developed primarily by the Kaiserslautern group, Freeden et al. (1998) and Schreiner (1997).

The interpolation (representation) problem is now formulated as follows. For a given Hilbert space, $H(A)$, find the function with minimum norm that interpolates the given data values. In fact, it is easily shown that the spline (4.15), subject to the constraint analogous to equation (2.12), is unique and is the function among all interpolants that has minimum norm (see Freeden et al., 1998, pp140-141). The linear matrix equation that must be solved is given by

$$\begin{pmatrix} y_1 \\ \vdots \\ y_V \end{pmatrix} = \begin{pmatrix} K_A^{(h_1)}(h_1) & \cdots & K_A^{(h_V)}(h_1) \\ \vdots & \ddots & \vdots \\ K_A^{(h_1)}(h_V) & \cdots & K_A^{(h_V)}(h_V) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_V \end{pmatrix} . \qquad (4.16)$$

A unique solution exists if the reproducing kernel is positive definite. Such is our present case, since the numbers, A_ℓ^2, are all positive.

The radial basis functions, $K_A^{(h_i)}(x)$, $i = 1, \ldots, V$, are localized around the data points, h_j (since $\arg\left(\max_y\left(P_n(\cos y)\right)\right) = 0$). Nevertheless, mathematically they still have global support. This means that the matrix in equation (4.16) is full, although it is mostly diagonally dominant. [Note, despite their name, these functions do not form a basis for the Hilbert space. An orthonormal Hilbert basis is given by the set $\left\{A_\ell^{-1}\bar{Y}_{lm}(x)\right\}$, $\ell = 0, 1, \ldots;\ m = -\ell, \ldots, \ell$].

Schreiner (1997) (see also Freeden et al. (1998)) provided truly locally supported kernel functions defined as follows. Consider the function (Figure 5) defined on the interval $-1 \le t \le 1$:

$$B_h^{(k)}(t) = \begin{Bmatrix} 0, & -1 \le t \le h \\ \left(\dfrac{t-h}{1-h}\right)^k, & h < t \le 1 \end{Bmatrix} = \sum_{\ell=0}^{\infty} (2\ell+1)\, b_\ell P_\ell(t) \;; \tag{4.17}$$

where $k \ge 1$; and, then define the kernel function as the convolution of $B_\ell^{(k)}$ with itself:

$$K_A^{(h_i)}(x, h_i) = B_h^{(k)}(x \cdot h_i) * B_h^{(k)}(x \cdot h_i) = \sum_{\ell=0}^{\infty} (2\ell+1)\, b_\ell^2 P_\ell(x \cdot h_i) \;, \tag{4.18}$$

where, by comparison to equation (4.5), we have

$$A_\ell = b_\ell^{-1} \,, \quad b_\ell \ne 0 \;. \tag{4.19}$$

Schreiner (1997) derived a recursion formula for b_ℓ and shows that

$$\left|b_\ell\right| = O\left(\ell^{-3/2}\right), \quad \ell \to \infty \;, \tag{4.20}$$

irrespective of the parameter *k*. He also showed that these coefficients, b_ℓ, that vanish are associated with the zeros of Gegenbauer polynomials. Thus, depending on the parameters, *k* and *h*, some spherical harmonics may be excluded from the Hilbert space (the corresponding A_ℓ is not defined). The local support for $K_A^{(h_i)}(x)$ (equation (4.18)) is defined by the parameter, *h*:

$$h_i = \sqrt{\frac{\cos y_i + 1}{2}} \;, \tag{4.21}$$

where y_i is the spherical radius of the support region (a spherical cap centered on h_i).. Figure 6 shows an example of kernel functions and their support regions for three neighboring points.

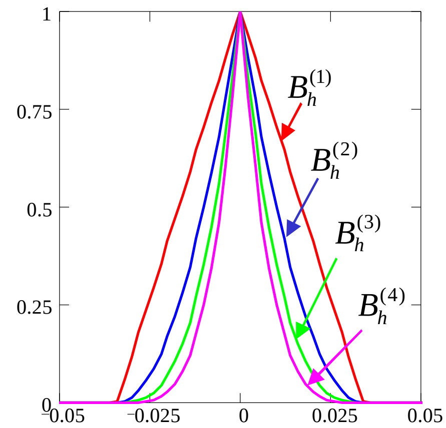

Figure 5: Generating functions, $B_h^{(k)}$, for locally supported reproducing kernels; $\psi_i = 15°$.

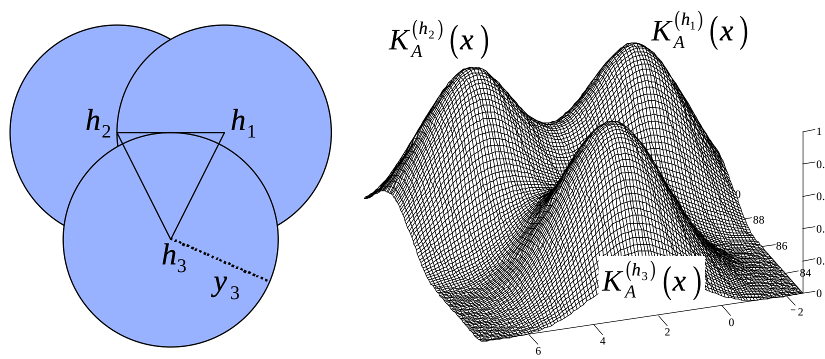

Figure 6: Locally supported kernels for three points of a triangle; support regions on right, kernels on left; $\psi_i = 5°$.

Because of the asymptotic decrease given by equation (4.20), the numbers, A_ℓ, are only "0-summable", and one cannot guarantee the continuity of first-order derivatives of the spline. Clearly, smoother splines are obtained by choosing higher-order convolutions of the functions, $B_\ell^{(k)}$ (this increases the support region for fixed h_i)

Writing equation (4.16) in matrix form and substituting it into equation (4.15), we find

$$s_{FS}(x) = \left[K_A^{(h_i)}(x)\right]\left[K_A^{(h_j)}(h_i)\right]^{-1}\left[y_j\right], \tag{4.22}$$

where the square-bracketed matrices have dimensions $1 \times V$, $V \times V$, and $V \times 1$, respectively. This is precisely the minimum-norm solution in collocation theory developed in physical geodesy (Moritz, 1980), except that the kernel, $K_A^{(h_i)}$, is not a covariance function of the gravity field (see also Sünkel, 1984). However, like the usually assumed covariance function it is positive definite and depends only on the spherical distance between observation data points. In the limit as the support region decreases, we may even compare the *FS* spline model to Bjerhammar's impulse function method (Bjerhammar, 1976, 1987), where Dirac delta functions replace the kernel functions and represent splines of zero support.

5. Alfeld / Neamtu / Schumaker Spherical Splines

The spherical splines developed by the Vanderbilt group are based directly on methods used in constructive approximation of surfaces with applications in computer-aided geometric design (CAGD; e.g., see Farin, 2002). We shall call these the *ANS* splines after the initials of the authors whose publications lay the necessary groundwork (Alfeld et al., 1996a,b,c).

We assume that for a given set of points, h_i, the sphere has been triangulated using well-known methods (e.g., the Delauney triangulation) yielding *N* triangles whose *E* edges are great circle arcs. The triangulation is further assumed to be total, meaning that any point on the sphere belongs to a spherical triangle (or an edge or a vertex). (Again, the triangulation may be generalized to any sphere-like surface, but this is beyond the present scope.) We assume that each triangle is non-degenerate (the three vertices do not lie on a plane that contains the center of the sphere). The number of (non-degenerate) triangles, *N*, formed on the sphere with *V* vertices is given by $N = 2V - 4$; and the corresponding number of edges is given by $E = 3V - 6$. For example, consider the simple "local triangulation" on the sphere formed by three nearby points. The 3 edges actually form 2 triangles, the obvious smaller triangle, as well as the large triangle comprising the spherical area exterior to it.

We introduce a local coordinate system for each triangle based on its vertices, (h_1, h_2, h_3), so that any point on the sphere can be represented by the combination:

$$x = b_1 h_1 + b_2 h_2 + b_3 h_3 \ , \tag{5.1}$$

where the coefficients, (b_1, b_2, b_3), are the *barycentric* coordinates of x. Writing h_i and x in terms of the usual Cartesian coordinates, e.g., $h_i = h_{ix} e_x + h_{iy} e_y + h_{iz} e_z$, we have immediately upon substituting these into equation (5.1):

$$\begin{pmatrix} x_x \\ x_y \\ x_z \end{pmatrix} = \begin{pmatrix} h_{1x} & h_{2x} & h_{3x} \\ h_{1y} & h_{2y} & h_{3y} \\ h_{1z} & h_{2z} & h_{3z} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} , \tag{5.2}$$

from which b_j, $j = 1, 2, 3$, can be determined:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} h_{1x} & h_{2x} & h_{3x} \\ h_{1y} & h_{2y} & h_{3y} \\ h_{1z} & h_{2z} & h_{3z} \end{pmatrix}^{-1} \begin{pmatrix} x_x \\ x_y \\ x_z \end{pmatrix} . \tag{5.3}$$

As such, these barycentric coordinates are *linear*, *homogeneous* functions of x :

$$b_j\left(a x\right) = a\, b_j\left(x\right) \ . \tag{5.4}$$

Note that if two of the points, h_i, are on the equator, 90º apart, with the third point at a geographic pole, then the barycentric coordinates are proportional to the usual Cartesian coordinates.

The following polynomials of degree *d*:

$$B_{pqr}^{(d)}\left(x\right) = \frac{d!}{p!q!r!} b_1\left(x\right)^p b_2\left(x\right)^q b_3\left(x\right)^r , \quad p + q + r = d \ \ , \tag{5.5}$$

are called *spherical Bernstein-Bézier* (BB) polynomials. They are *homogeneous* because of equation (5.4). The spherical BB polynomials are the restrictions to the sphere of the more general trivariate homogeneous Bernstein-Bézier polynomials for $v \in \mathbb{R}^3$ whose barycentric coordinates are defined similarly using three vectors in $\mathbb{R}^3$. The Bernstein-Bézier polynomials span the space of homogeneous polynomials of degree d in $\mathbb{R}^3$; and the spherical BB polynomials form a basis for homogeneous polynomials of degree *d* on the sphere, W. From the formula (5.3), it is immediately evident that $b_j\left(h_i\right) = d_{ji}$; hence also

$$\begin{aligned} & B_{d,0,0}^{(d)}\left(h_1\right) = d_{1,i}, \quad B_{0,d,0}^{(d)}\left(h_i\right) = d_{2,i}, \quad B_{0,0,d}^{(d)}\left(h_i\right) = d_{3,i}, \\ & B_{pqr}^{(d)}\left(h_i\right) = 0, \quad \text{for any two indices, } p, q, r, \text{ not equal to zero} \ . \end{aligned} \tag{5.6}$$

Figure 7 shows some of the spherical BB polynomials for $d = 5$ and an isosceles spherical triangle.

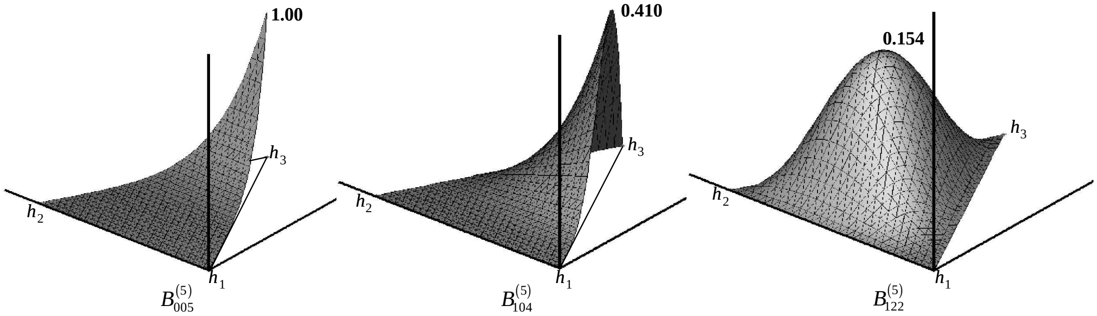

Figure 7: Three typical examples of the spherical BB polynomials for $d = 5$. Indicated values are the maxima attained on the triangle.

There is the remarkable result that any homogeneous polynomial can be decomposed into a sum of *harmonic* homogeneous polynomials of either even or odd degrees. If $H^{(d)}$ denotes the space of homogeneous polynomials of degree *d*, and $Y^{(\ell)}$ is the space of harmonic homogeneous polynomials of degree ℓ, then symbolically, we have

$$H^{(d)} = \begin{cases} Y^{(0)} \oplus Y^{(2)} \oplus Y^{(4)} \oplus \cdots \oplus Y^{(d)}, & \text{if } d \text{ is even} \\ Y^{(1)} \oplus Y^{(3)} \oplus Y^{(5)} \oplus \cdots \oplus Y^{(d)}, & \text{if } d \text{ is odd} \end{cases} \tag{5.7}$$

where $\oplus$ means "direct sum" of spaces. It can readily be verified that the dimension of either direct sum of harmonic spaces is precisely that of the homogeneous polynomials $\left((d+2)(d+1)/2\right.$; see equation (2.6)). This result, proved, for example in (Freeden et al., 1998, Section 2.2), holds both for points in $\mathbb{R}^3$ as well as for the restriction to the sphere, and it provides the relationship between the spherical Bernstein-Bézier polynomials and our familiar spherical harmonics. Specifically, we may write, if *d* is even,

$$B^{(d)}_{pqr}(x) = \sum_{\ell=0}^{d/2} \sum_{m=-\ell}^{\ell} b^{(2\ell,m)}_{pqr} \bar{Y}_{2\ell,m}(x), \quad d \text{ even} \ , \tag{5.8}$$

where the coefficients could be determined in the usual way through the orthogonality of the spherical harmonics:

$$b^{(2\ell,m)}_{pqr} = \frac{1}{4p} \iint_{s} B^{(d)}_{pqr}(x) \bar{Y}_{2\ell,m}(x) \, ds_x \ , \tag{5.9}$$

where s is the unit sphere. Here, the BB polynomials must be defined globally, but in the sequel we shall restrict their definition to one triangle of the triangulation of the sphere based on the data points, h_i.

That is, consider a spline defined by a patchwork of locally supported spherical Bernstein-Bézier (SBB) polynomials. Within this simple proposition lie non-trivial aspects. First, it is necessary to determine the dimension of the spline space so that the spline is properly specified. Since the local domains of the SBB polynomials (i.e., the triangles) will share vertices and edges, and since the spline should have a specified smoothness across the edges and vertices, determining the dimension of the spline space is non-trivial. Alfeld et al. (1996a) showed that the dimension of the space of splines with only *piecewise continuous*, homogeneous, SBB polynomials of degree d has dimension

$$M_0 = V + (d-1)E + \binom{d-1}{2} N = d^2 (V-2) + 2 \ . \tag{5.10}$$

If the spline should have smoothness k (that is continuous derivatives up to order k), then if $d \geq 3k+2$, Alfeld et al. (1996a) also proved that the dimension of the corresponding space is given by

$$M_k = (d-k)(d-2k)(V-2) + (k+1)(k+2) + m \ , \tag{5.11}$$

where

$$m = \sum_{i=1}^{V} m_i , \quad m_i = \sum_{m=1}^{d-k} (k+m+1-me_i)_+ \ , \tag{5.12}$$

and where e_i is the number of distinct (non-coplanar) edges at a vertex, and the summands of m_i are either zero or positive. Thus, a spline with continuous first derivatives ($k=1$) and comprising homogeneous polynomials of degree $d=5$ comes from a space of dimension $12V-18$, if all vertices have at least 3 distinct edges (in which case m_i is zero for all vertices).

We thus define a spherical *ANS* spline as follows:

$$s_{ANS}(x) = \sum_{p+q+r=d} c_{pqr}^{(d,n)} B_{pqr}^{(d,n)}(x), \quad x \in T_n \ , \tag{5.13}$$

where T_n is the n^{th} triangle of the triangulation, $n = 1, \ldots, N$; the subscript "*ANS*" is used to identify the principal authors. For each T_n, there is a new set of coefficients associated with the homogeneous polynomial defined using the barycentric coordinates of the vertices. Since there are $(d+2)(d+1)/2$ coefficients for each set of SBB's on a triangle, the total number of coefficients is $N(d+2)(d+1)/2 = (d+2)(d+1)(V-2)$, which is greater than the dimension of the space, M_k, given in equation (5.11). Clearly, the coefficients must be constrained somehow in order to be uniquely determined.

One may associate with each coefficient a domain point (analogous to the knots of *B*-splines; see Section 3). Alfeld et al. (1996a) defined these domain points for triangle, T_n, according to

$$y_{pqr}^{(n)} = \frac{1}{d}\left(p h_1^{(n)} + q h_2^{(n)} + r h_3^{(n)} \right), \quad p+q+r=d \ , \tag{5.14}$$

(see Figure 8). The vertices are also domain points:

$$y_{d,0,0}^{(n)} = h_1^{(n)}, \quad y_{0,d,0}^{(n)} = h_2^{(n)}, \quad y_{0,0,d}^{(n)} = h_3^{(n)} \ ; \tag{5.15}$$

and from equation (5.6), we have the following constraints:

$$c_{d,0,0}^{(n)} = y_1^{(n)}, \quad c_{0,d,0}^{(n)} = y_2^{(n)}, \quad c_{0,0,d}^{(n)} = y_3^{(n)} \ , \tag{5.16}$$

where for each *j*, $y_j^{(n)} \equiv y_i$, for some $i \in \{1,\ldots,V\}$.

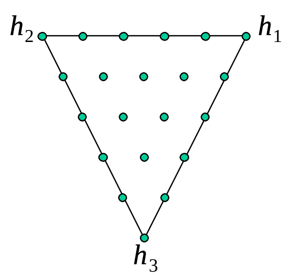

Figure 8: Domain points defined for a triangle on which spherical BB polynomials of degree, $d = 5$, are supported.

More generally, Alfeld et al. (1996b) showed that for two triangles, T_1 with vertices $\{h_1, h_2, h_3\}$ and T_2 with vertices $\{h_1, h_3, h_4\}$, that share an edge ($h_1 \leftrightarrow h_3$, in this case), the corresponding homogeneous polynomials of degree *d* have continuous derivatives up to order $k \geq 0$ across the edge if and only if

$$c_{pqr}^{(2)} = \sum_{m+n+r=r} c_{p+m,n,q+r}^{(1)} B_{mnr}^{(r,1)}\left(h_4\right), \quad \text{for all } r \leq k, \quad p+q+r=d \ , \tag{5.17}$$

where $B_{mnr}^{(r,1)}\left(h_4\right)$ is the BB basis polynomial of degree *r* for triangle T_1 but evaluated at the vertex, h_4, of triangle T_2. For example, if we wish *continuity* of the polynomials across the shared edge ($k = 0$), then we require ($r = 0$ in the formula above)

$$c_{pq0}^{(2)} = c_{p0q}^{(1)}, \quad p+q=d \ . \tag{5.18}$$

There are $d+1$ such conditions for each edge, corresponding to the $d+1$ domain points on the edge. For a total triangulation there are $(d+1)E$ conditions, but each vertex is thus counted twice. Hence, there are only $(d+1)E-V$ constraints of the type (5.18) on the coefficients. Therefore, of the total number of coefficients, $N\binom{d+2}{2}$, we can remove $(d+1)E-V$ and are left with

$$N\binom{d+2}{2} - (d+1)E + V = d^2(V-2)+2 \ , \tag{5.19}$$

precisely the dimension, M_0, of the space of *continuous* splines. For continuity of higher-order derivatives ($k>0$), there are additional constraints on the coefficients and the dimension of the space of corresponding splines reduces according to formula (5.11). The number of additional constraints per edge, L_k, according to equation (5.17), is given by

$$L_k = d + (d-1) + \cdots + (d-k+1) = \frac{k}{2}(2d-k+1) \ . \tag{5.20}$$

On the other hand, the total number of such constraints ($L_k E$) again includes some redundancies which were elaborated by Alfeld et al. (1996a).

Once the dimension, M_0, of the space of continuous splines is defined, and redundant coefficients (on shared edges) are removed, we wish to compute the remaining coefficients from the given data, subject to additional L_k smoothness constraints (redundant constraints also removed). However, to simplify the concepts, we retain all coefficients and deal with their redundancies separately. First, suppose the coefficients associated with all N triangles are renamed:

$$c_{pqr}^{(d,n)} \to c_m, \quad m = 1,\ldots,ND \ , \tag{5.21}$$

where $D=(d+1)(d+2)/2$ is the number of coefficients for one triangle. Then identify V of these with the domain points at the vertices. For some m_i we clearly have, in view of equation (5.6)

$$c_{m_i} = y_i, \quad i=1,\ldots,V \ . \tag{5.22}$$

We write this in terms of a matrix, A, of zeros and ones that selects the appropriate coefficients:

$$y = Ac \quad , \tag{5.23}$$

where $y = (y_1, \ldots, y_V)^{\mathrm{T}}$, $c = (c_1, \ldots, c_{ND})^{\mathrm{T}}$. The remaining coefficients are free, subject to the constraints (5.17), which we write concisely as

$$Gc = 0 \quad , \tag{5.24}$$

where G is a $L_k E \times ND$ matrix that is sparse if the order of smoothness is low, but formally, it also contains redundancies that make the matrix less than full rank. One can either remove the redundant constraints or use standard techniques to solve underdetermined (but consistent!) systems of equations. We now assume that the redundancies are removed so that G has full (row) rank.

These conditions still leave some coefficients free and they can be determined with additional constraints, such as minimum energy interpolation. Alfeld et al. (1996c) suggested to minimize a quadratic functional of the spline, which, if the minimization occurs independently over each triangle, is equivalent to

$$\left(c^{(n)}\right)^{\mathrm{T}} Q_n c^{(n)} \to \min \,, \quad n = 1, \ldots, N \quad , \tag{5.25}$$

where $c^{(n)}$ is the vector of D coefficients, $c_{pqr}^{(d,n)}$, for the n^{th} triangle, and Q_n is a $D \times D$ symmetric, positive definite matrix (see Alfeld et al., 1996c, for a discussion on the choice and evaluation of Q_n). In the end, one must solve for the coefficients according to the minimization (5.25) subject to constraints (5.22) and (5.24). Using the method of Lagrange multipliers , we set up the cost function with constraints:

$$f = \frac{1}{2} c^{\mathrm{T}} Q c - l_1^{\mathrm{T}} \left(y - Ac \right) + l_2^{\mathrm{T}} \left(Gc \right) \quad , \tag{5.26}$$

where Q is the block diagonal matrix, with blocks, Q_n ; and l_1 and l_2 are corresponding vectors of Lagrange multipliers. Setting partial derivatives of f with respect to c, l_1, and l_2, correspondingly, to zero, we find the following linear system to be solved:

$$\begin{pmatrix} Q & A^T & G^{\mathrm{T}} \\ A & 0 & 0 \\ G & 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} . \tag{5.27}$$

It can be shown that the solution for the coefficients is given by

$$c = \left(I - MG^{\mathrm{T}} \left(GMG^{\mathrm{T}} \right)^{-1} G \right) Q^{-1} A^{\mathrm{T}} N^{-1} y \quad , \tag{5.28}$$

where $M = Q^{-1}\left(I - A^{\mathrm{T}}N^{-1}AQ^{-1}\right)$ and $N = AQ^{-1}A^{\mathrm{T}}$. See Section 7 for a solution with data errors.

6. The Boundary-Value Problem in Geodesy

In general, considering the developments in Sections 3, 4, and 5, a spherical spline model of a function on the sphere has the following form as a sum of locally supported splines, $w_m(x)$:

$$s(x) = \sum_{m=1}^{M} c_m w_m(x) \ , \tag{6.1}$$

where the coefficients, c_m, are determined by solving a linear system of equations based on function values distributed on the sphere (and possibly a set of other constraints). In physical geodesy, the function to be modeled is usually a derivative (of some order) of the gravitational potential (more precisely, a residual or disturbing potential), and the function is defined in the free space above the Earth's surface according to potential theory. Formally, we set up a boundary value problem (BVP) for the disturbing potential, *T*, that in spherical approximation is expressed as:

$$\begin{aligned} &\text{solve } \nabla^2 T = 0 \text{ in the space exterior to } W_R \\ &\text{subject to } \left. LT \right|_{W_R} = f \ , \end{aligned} \tag{6.2}$$

where W_R is a sphere of radius *R*, *L* is a linear operator and *f* is supposed to be a given function on W_R. If $L = I$, the identity operator, the problem is known as the Dirichlet boundary-value problem which has the well known solution (if the boundary is a sphere) given by the Poisson integral:

$$T(x, r) = \frac{R\left(r^2 - R^2\right)}{4p} \iint_{W} \frac{f(x')}{r^3} dW \ , \quad r \geq R \ , \tag{6.3}$$

where r is the (straight-line) distance between the integration point, (x', R), and the evaluation point, (x, r):

$$r^2 = r^2 + R^2 - 2Rr x \cdot x' \ . \tag{6.4}$$

Equation (6.3) leads directly to the familiar spherical harmonic representation:

$$T\left(x,r\right)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{R}{r}\right)^{\ell+1} t_{\ell m}\bar{Y}_{\ell m}\left(x\right)\ , \tag{6.5}$$

where $t_{\ell m}$ is the geopotential harmonic coefficient, by considering that

$$\frac{1}{r}=\frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{R}{r}\right)^{\ell} P_{\ell}\left(x\cdot x'\right) \quad\Rightarrow\quad \frac{r^{2}-R^{2}}{r^{3}}=\frac{1}{r}\sum_{\ell=0}^{\infty}\left(2\ell+1\right)\left(\frac{R}{r}\right)^{\ell} P_{\ell}\left(x\cdot x'\right)\ , \tag{6.6}$$

and making use of the decomposition formula (4.7).

If $y_i=T\left(h_i,R\right)$, $i=1,\ldots,V$, is all that we have for boundary values, then the spline representation, equation (6.1), may be used as boundary function in Poisson's integral, yielding the following estimate for *T*:

$$\hat{T}\left(x,r\right)=\frac{R\left(r^{2}-R^{2}\right)}{4p}\sum_{m=1}^{M}c_m\iint_{W}\frac{w_m\left(x'\right)}{r^{3}}\,dW\ . \tag{6.7}$$

The integration region for the integrals is limited to the local support of the splines, $w_m\left(x\right)$.

In fact, for the *FS* splines, the integral can be evaluated directly by substituting equations (4.5) and (6.6), and making use of equation (4.7) and the orthogonality of the spherical harmonics:

$$\frac{R\left(r^{2}-R^{2}\right)}{4p}\iint_{W}\frac{w_m\left(x'\right)}{r^{3}}\,dW=\sum_{\ell=0}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}\frac{2\ell+1}{A_{\ell}^{2}}P_{\ell}\left(x\cdot h_i\right)\ . \tag{6.8}$$

where the index, *m*, in this case corresponds to a particular data point, h_i. Denoting the right hand side as $K_A^{(h_i)}\left(x,r\right)$, we have, using the *FS* spline coefficients:

$$\hat{T}\left(x,r\right)=\sum_{i=1}^{V}c_iK_A^{(h_i)}\left(x,r\right)\ . \tag{6.9}$$

We note that the extended splines, $K_A^{(h_i)}\left(x,r\right)$, mathematically do not have local support. However, as seen in Figure 9, the support is practically local and for near-Earth radii, *r*, only a small number of splines need to be included in the summation of equation (6.9).

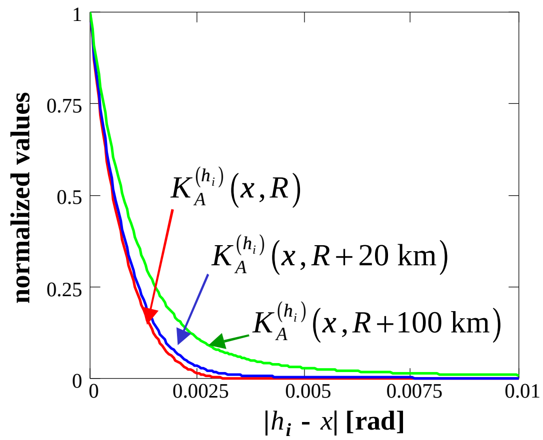

Figure 9: Extended kernels for *FS* splines; $y_i = 5°$.

Using the *ANS* spline representation as boundary function in problem (6.2) yields a computationally more complicated estimate for *T*. The integral in equation (6.7) is now given by substituting equation (5.13):

$$\frac{R\left(r^2-R^2\right)}{4p}\iint\limits_{W}\frac{w_m\left(x'\right)}{r^3}\,dW=\frac{R\left(r^2-R^2\right)}{4p}c_{pqr}^{(d,n)}\iint\limits_{T_n}\frac{B_{pqr}^{(d,n)}\left(x'\right)}{r^3}\,dW \quad , \tag{6.10}$$

where the index *m* corresponds to one of the $(d+1)(d+2)/2$ *ANS* splines for triangle, T_n. There should be no confusion in the notation between the triangle and the disturbing potential, nor between the radial coordinate and the index of the BB polynomial. Defining

$$B_{pqr}^{(d,n)}(x,r)=\frac{R\left(r^2-R^2\right)}{4p}\iint\limits_{T_n}\frac{B_{pqr}^{(d,n)}\left(x'\right)}{r^3}\,dW \quad , \tag{6.11}$$

the disturbing potential is given by

$$\hat{T}(x,r)=\sum_{n=1}^{N}\sum_{p+q+r=d}c_{pqr}^{(d,n)}B_{pqr}^{(d,n)}(x,r) \quad , \tag{6.12}$$

where the redundancies in the coefficients (see Section 5) correspondingly reduce the number of summands. Note that we cannot use the harmonic expansion of the *ANS* splines, equation (5.8), to take advantage of the orthogonality with respect to the expansion for $1/r^3$, as in the case of the *FS* splines, since the integration is truly local. Also, since $1/r^3$ does not have local support, more than one integral of the type (6.10) will contribute to the evaluation of $\hat{T}(x,r)$, if $r > R$.

Once the disturbing potential is estimated in external space, it is possible to apply any linear operator to express other useful gravitational quantities, such as the gravity disturbance, the gravity anomaly, deflections of the vertical, gravitational gradients, etc. For higher than first-order derivatives care must be exercised for points on the boundary since the curvature of the potential is discontinuous there. In the case of *FS* splines, the functional relationships between *T* and its vertical derivatives, for example, are familiar from the spherical harmonic spectral transforms. To illustrate, the gravity disturbance on the sphere of radius, *R*, is given by

$$d\,\hat{g}\left(x\right)=-\sum_{i=1}^{V}c_i\frac{\partial}{\partial r}K_A^{\left(h_i\right)}\left(x,r\right)\Big|_{r=R}\quad , \tag{6.13}$$

where

$$-\frac{\partial}{\partial r}K_A^{\left(h_i\right)}\left(x,r\right)\Big|_{r=R}=\sum_{\ell=0}^{\infty}\frac{\ell+1}{R}\frac{2\ell+1}{A_\ell^2}P_\ell\left(x\cdot h_i\right)\ . \tag{6.14}$$

Clearly, the coefficients, A_ℓ, of the *FS* splines must be such that the series in equation (6.14) is summable; i.e., $A_\ell^2=O\left(\ell^n\right),\ n>3$.

For the *ANS* splines, the gravity disturbance in free space follows from equation (6.4):

$$-\frac{\partial}{\partial r}B_{pqr}^{(d,n)}\left(x,r\right)=\frac{3}{4p}\iint_{T_n}\frac{B_{pqr}^{(d,n)}\left(x'\right)}{r^5}\left(r-Rx\cdot x'\right)dW\ . \tag{6.15}$$

In the limit, as $r\to R$, we encounter a strong singularity for the triangle containing x, which requires special care in numerical applications.

7. Data Errors

Up to this point, errors (e.g., observational errors) in the data values were excluded from consideration. When they are present (equation (2.3)), the spline representation should not necessarily reproduce the data values at the data points since data on neighboring points may contribute to reduce the error in the estimation. All the spline representations discussed in previous sections are linear models of the data and the standard linear least-squares estimation may be applied to obtain an optimal representation (generally, only optimal with respect to minimizing the errors, that is, the resulting residuals, at the data points). Let the observations at the data points, h_i, be given by

$$\tilde{y}_i=y_i+e_i\ , \tag{7.1}$$

where, as before, $y_i = f\left(h_i\right)$, and the covariance matrix of the random errors, e_i, is D_e.

For the tensor-product spline representation, equation (3.1), specialized to the case of third-order splines (see also equation (3.8)), the least-squares solution for the coefficients is obtained by minimizing (let *s* be the vector $s = \left(s\left(h_1\right),\ldots,s\left(h_V\right)\right)^{\mathrm{T}}$):

$$\left(s-\tilde{y}\right)^{\mathrm{T}} P\left(s-\tilde{y}\right) \rightarrow \min \ , \tag{7.2}$$

where

$$P = s^{\,2} D_{\mathrm{e}}^{-1} \tag{7.3}$$

is the weight matrix for the observations, based on the inverse covariance matrix of the errors and scaled by the variance of unit weight. The minimization is subject to the constraints formalized by equations (3.8) and (3.9); and then the estimate of the vector of coefficients is given by (e.g., Leick, 1990, p.100; see also below)

$$\hat{c} = \left(N^{-1} - N^{-1}B^T\left(BN^{-1}B^T\right)^{-1}BN^{-1}\right)A^T P\tilde{y} \ , \tag{7.4}$$

where $N = A^{\mathrm{T}}PA$.

For the quasi-interpolation scheme using tensor splines, the constraints are already incorporated and one could set up a least-squares estimation problem just to minimize the effect of the data errors. The estimate (7.4) can be adopted in this case with $B = 0$ and A defined by equations (3.14) through (3.18). However, this would not minimize the modeling errors associated with quasi-interpolation and some further development of this case appears warranted. We refer also to the numerical tests conducted by Schumaker and Traas (1991) that seem to indicate potential computational instability with the global solution, such as equation (7.4), although they did not consider data errors, and used data scattered on the sphere, rather than at the knots.

Performing a least-squares adjustment of the observations in the case of the *FS* splines is straightforward since it is a special case of the collocation methods developed in physical geodesy. Assuming an admissible set of data points, we borrow from the theory of least-squares collocation (Moritz, 1980) and modify equation (4.22) as follows:

$$s_{FS}\left(x\right) = \left[K_A^{\left(h_i\right)}\left(x\right)\right]\left[K_A^{\left(h_j\right)}\left(h_i\right) + D_e\right]^{-1}\left[y_j\right] \ , \tag{7.5}$$

where the covariance matrix of the observation errors, likewise, has elements depending on their relative locations, $\left|h_i - h_j\right|$. Assuming that the observation errors are either uncorrelated or only correlated with their immediate neighbors, the sparseness of the matrix to be inverted is essentially preserved, as defined by the kernels, $K_A^{\left(h_i\right)}$.

In the case of the *ANS* splines, the minimization (5.25) is extended to include observation errors. The cost function to be minimized is then

$$f = \frac{1}{2} c^{\mathrm{T}} Q c + \frac{1}{2} v^{\mathrm{T}} P v - l_1^{\mathrm{T}} \left(y - Ac \right) + l_2^{\mathrm{T}} \left(Gc \right) \ , \tag{7.6}$$

where the residuals are defined as $v = y - \tilde{y}$ and P is the weight matrix for the observations, as in equation (7.3). Estimates for the coefficients are obtained by solving the system of equations for c:

$$\begin{pmatrix} P & 0 & -I & 0 \\ 0 & Q & A^T & G^{\mathrm{T}} \\ -I & A & 0 & 0 \\ 0 & G & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ c \\ l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tilde{y} \\ 0 \end{pmatrix} . \tag{7.7}$$

One can readily derive the following:

$$c = \left(Q + N \right)^{-1} \left(I - G^{\mathrm{T}} \left(G \left(Q + N \right)^{-1} G^{\mathrm{T}} \right)^{-1} G \left(Q + N \right)^{-1} \right) A^{\mathrm{T}} P \tilde{y} \ , \tag{7.8}$$

where $N = A^{\mathrm{T}} P A$, which is different here than in equation (5.28). If P and Q are diagonally dominant (as may be considered usual), the inverses above are easily computed.

8. Summary

Three types of spline models have been presented to represent functions whose values are known only at a finite number of discrete points on the sphere. In addition to a comparison of the splines, the main objective was to contrast these representations with the well known and conventionally used spherical harmonic models of functions. The essential difference between the spline models and the spherical harmonics is the nature of their support. Spherical harmonic basis functions have global support and the basis functions for the spline representations have local support defined in some sense by the resolution or distribution of the data points. This implies that the evaluation of the spline model is accomplished extremely rapidly, using just coefficients pertaining to the support region near the evaluation point, as opposed to the evaluation of the spherical harmonic model that requires processing all its coefficients. Moreover, if the data values are changed (e.g., because of a more accurate observation), then the coefficients of the spline models change only with respect to the support region of the changed data. Another salient feature of the spline representations is that the data points need not be regularly distributed on the sphere, as opposed to spherical harmonics where a regular latitude/longitude distribution is a practical requirement. There are, however, a number of distinguishing differences among the types of spline representations that also have important consequences for applications in physical geodesy.

The first spline model described above is the *tensor-product* of one-dimensional *B*-splines with knots on a grid of co-latitude and longitude lines. Special trigonometric *B*-splines in latitude were defined to enable smoothness of the model at the poles. These splines are easiest to visualize and manipulate, but tests have also shown (Schumaker and Traas, 1991) that a rigorous, global solution can be unstable. A quasi-interpolation procedure was developed (ibid.) that bypasses the global solution and determines with reasonable accuracy the coefficients of the spline model strictly from local data (some further tests of this method confirm the suitability of this procedure (Vangani, 2005)). One drawback of this type of representation of a function on the sphere is the requirement of a coordinate grid of knots. To increase the resolution of the model in some local region requires the addition of a *line* of latitude or longitude (or both). This automatically adds resolution globally along each added line, even where it is not warranted by the data. The use of the tensor-product spline model in the geodetic boundary-value problem was not specifically discussed, but is analogous to the case of the *ANS* spherical splines (below).

The second type of splines are the spherical splines developed by the Kaiserslautern group (e.g., Freeden et al., 1998) and named here the *FS* splines after the principal authors. The component splines are reproducing kernels in a Hilbert space of interpolating functions on the sphere. Each data point defines such a kernel and since the kernels are isotropic and depend only on the distance from the data point (they are called radial basis functions), the spline model is analogous to the collocation model, known well in physical geodesy. The model is identical to collocation if the kernel functions are the covariance functions of the signal being represented. Schreiner (1997) also developed strictly locally supported kernels in order to facilitate the determination of the corresponding coefficients of the model. Resembling closely the collocation model, which is constructed for the space of functions harmonic outside a sphere, the *FS* splines are readily extendable to three dimensions on the basis of potential theory. The data points for this spline model need not be on a coordinate grid and the addition of a data point does not affect the remote part of the model. However, interpolation is based on no particular characteristic of the function between the data points since the support regions of the kernels are chosen somewhat arbitrarily. Also, the kernel functions are expressed only as infinite series, which may be a computational concern.

The third type of splines is the spherical spline based on the homogeneous *Bernstein-Bézier polynomials*. These splines were developed for sphere-like surfaces in a series of papers by the Vanderbilt group (Alfeld et al., 1996a,b,c) and are named the *ANS* splines here in deference to their authors. The *ANS* spline representations of functions on a sphere have a strong mathematical link to the methods and theory of constructive approximation in CAGD. Given a set of data points on the sphere (not necessarily on any kind of regular grid), one first performs a triangulation of these points. The resulting triangles define the local support for the *ANS* splines. Additional domain points on the triangle (other than the triangle vertices) are used to impose a specified smoothness of the spline representation across triangle edges (analogous to the knots of the tensor-product splines). The coefficients of the spline model correspond to these domain points; and, therefore, the number of coefficients can be many times the number of data points (triangle vertices); e.g., see equation (5.19), depending on the square of the degree of the polynomials. On the other hand, the evaluation of the spline is extremely fast using algorithms developed for BB polynomials (the de Casteljau algorithm; Alfeld, 1996b). Also, the addition of one data point does not affect the model except in the immediate neighborhood of that point. The *ANS* spline representation is not as easily amenable to the boundary-value problems in physical geodesy (when the boundary is a sphere) as are the *FS* splines. Yet, it should be noted that, unlike the *FS* splines, the *ANS* splines can be generalized to data on any *sphere-like* surface because the BB polynomials are defined in terms of barycentric coordinates, rather than the usual spherical coordinates. Thus, one can construct a spline representation of data on the actual surface of the Earth, which could then be used in solutions to the boundary-value problem using boundary element methods (Klees, 1997).

A few other comments should be made concerning the spline representation. It is natural to express the global, longer-wavelength part of the geopotential in terms of (solid) spherical harmonics since they derive directly from potential theory. Therefore, it is equally natural to combine such a low-degree spherical harmonic model with a spline model that represents the finer, local structure of the field. The *FS* splines, in fact, were developed more generally in the context of such a combination (Freeden et al., 1998, Schreiner, 1997). On the other hand, that development is most useful if the spherical harmonic components are also estimated from the same data on the sphere. Today, spherical harmonic models are derived directly from satellite-borne instrumentation and the combination is not nearly as straightforward since heterogeneous data on different surfaces are involved. Any of the spline representations can be applied to residuals with respect to a given spherical harmonic model, but the optimal combination of all data (terrestrial and satellite) for a hybrid spherical harmonic/spline model remains a topic for future analysis and theoretical development. Finally, it is noted that the spline basis functions lend themselves to a multiresolution representation of the function on the sphere, analogous to the wavelets in Cartesian space. This has been developed for the tensor-product splines by Lyche and Schumaker (2000), and for the *FS* splines by Freeden et al., 1998; see also Fengler et al. (2004). For the *ANS* splines, a multiresolution representation has not been developed, but is possible (personal communication, L.L. Schumaker, 2004).

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