REPORTS OF THE DEPARTMENT OF

GEODETIC SCIENCE AND SURVEYING

REPORT NO. 357

SPLINES: Their Equivalence to Collocation

by

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DEPARTMENT OF
GEODETIC SCIENCE AND SURVEYING
1958 NEIL AVENUE
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AUGUST 1984

ABSTRACT

The application of splines in geodesy was mainly restricted to the solution of one-dimensional problems like interpolation, differentiation, approximation, solution of differential equations, etc. Two-dimensional splines turned out to be an adequate tool for the representation of smooth surfaces based on grided data.

The purpose of the present paper is to present splines on the real line, on the circle, on the sphere, and on the two-dimensional plane in a common and simple framework: The Green's Function and the frequency domain method. Splines of arbitrary degree, no matter what their domain of definition is, are shown to be recursively related to each other by convolutions of Green's functions. The close relation (and little difference, if any) between spline and collocation solutions is demonstrated.

FOREWORD

This report was prepared by Dr. Hans Sünkel, Institute of Mathematical Geodesy, Technical University at Graz, Austria, under Air Force Contract No. F19628-82-K-0017, The Ohio State University Research Foundation, Project No. 714255, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory (AFGL), Hanscom Air Force Base, Massachusetts, with Christopher Jekeli, Contract Monitor.

The reproduction and distribution of this report was carried out through funds supplied by the Department of Geodetic Science and Surveying. This report was distributed by the Air Force Geophysics Laboratory, as document AFGL-TR-84-0215.

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1. INTRODUCTION

Various methods have been developed for the determination and representation of the gravity field (of the earth) and are more or less well established on the geodetic market. In particular least squares and/or least norm collocation is without doubt the market leader - it is the "big blue", using the present terminology in the microcomputer arena. Solutions which are not "collocation compatible" are generally not considered very seriously. Many attempts have been made to design so-called alternative methods. None of them has gotten as much attention as collocation did.

In the non-geodetic environment splines and finite elements are favorite tools and are used since many years. About one decade ago, splines appeared the first time on the geodetic horizon; after some people had realized the very pleasant properties of splines, they were and are frequently used, in particular for the purposes of graphical representation of curves and surfaces, for interpolation and differentiation, and for the numerical evaluation of several integral formulas. However, particularly in two-dimensional applications, only splines defined on regular grids have been known and used, until a few years ago when Freeden (1981) succeeded in constructing splines on the sphere by transferring the fundamental ideas of Schoenberg (1964) from the circle to the sphere. Unfortunately Schoenberg's and Freeden's work is little known by geodesists, probably due to the virtually inaccessible mathematics required.

In 1980 the author of this paper came across a paper by Schoenberg (1973) which focused on spline interpolation for regularly distributed data along the real line. A detailed study turned finally out to be worthwhile and resulted in a report which aimed both at a geodetically attractive presentation of splines and at an evaluation of its anticipated close relationship to least squares (or least norm) collocation (Sünkel, 1981). The natural explanation of some unexpected polynomial features of the collocation solution was one of the results of that investigation. About the same time Lelgemann (1980) independently looked into this problem on the sphere and obtained results which confirmed my work. He was probably the first one who has ever used analytical spline functions for the determination and representation of a local geoid (West Germany); "collocation-biased" Austria has used splines only for the representation of its geoid (Sünkel, 1983). Although much effort has been put into a sound establishment of spline methods in geodesy in general and physical geodesy in particular, it seems to be still a too remote subject for the geodetic community. Spline solutions are "best" in some sense and should therefore naturally attract the attention of the optimates" (I mean the geodesists, who love to optimize - not Cicero's party!). Practical applications will show if spline solutions are not only "best", but also "good".

It is the purpose of the present paper to present splines in a common simple framework and to provide the reader with a kind of plain mortal approach. We start with an easy going comprehensive presentation of splines defined on the real line (Chapter 2) and make the reader familiar with the fundamental concept of <u>Green's functions,</u> its central role they play on the spline stage and with its outstanding properties. The other highlights are the <u>frequency domain methods</u> which will demonstrate a tremendous power, particularly because of the convolution theorem. Supplementary information to

Chapter 2 can be found in (Sünkel, 1981). In Chapter 3 we investigate the large family of splines defined on the circle, and make frequent use of the basic investigation by Schoenberg (1964). Chapter 3 serves, so to say, as a jumping board to better understand Chapter 4, where splines on the sphere are derived. The key paper for Chapter 4 is (Freeden, 1981). In Chapter 5 we do a kind of interpreter work and formulate splines in the two-dimensional plane by analogy reasoning.

All our investigations stop as soon as the spline function has been obtained. The presentation of approximation error estimates, of minimal properties, though very important, interesting, demanding and enlightening, had to be postponed and will be dealt with in a coming report. The mathematics used here has been reduced to a bare minimum without completely giving up the mathematical rigor. Since the author is in favor of the "bottom-up" strategy rather than for "top-down" (which should be left for the labs), the presentation is entirely inductive.

2. SPLINES ON THE REAL LINE

Let an infinite number of data be given located at all integer numbers along the real line ..., -1, 0, 1, ... and corresponding function values ..., f_{-1}, f_0, f_1, ... ; then we have infinitely many methods to interpolate this data. Of particular importance are spline interpolations; therefore we shall investigate these functions in detail.

The simplest possible spline basis function for the space C^{-1} is the unit step function G_0 or B-spline of degree zero,

$$G_0(x) = \begin{cases} 1 & \text{for } |x| < \frac{1}{2} \\ \frac{1}{2} & \text{for } |x| = \frac{1}{2} \\ 0 & \text{else} \end{cases} . \qquad (2 - 1)$$

The spectrum of $G_0(x)$ will be denoted by $g_0(\kappa)$. It is defined in terms of the Fourier transform: The Fourier transform of a function F(x) is given by

$$f(\kappa) = \int_{-\infty}^{\infty} F(x)\, e^{-i2\pi\kappa x}\, dx \qquad (2 - 2a)$$

and its inverse (the transformation from the frequency domain back into the space domain) by

$$F(x) = \int_{-\infty}^{\infty} f(\kappa)\, e^{i2\pi\kappa x}\, d\kappa \ . \qquad (2 - 2b)$$

The Fourier transform of the step function $G_0(x)$ is therefore given by

$$g_0(\kappa) = \int_{-1/2}^{1/2} G_0(x)\, e^{-i2\pi\kappa x}\, dx = \int_{-1/2}^{1/2} e^{-i2\pi\kappa x}\, dx \ .$$

Since $G_0(x)$ is symmetric with respect to $x = 0$, $G_0(-x) = G_0(x)$, the transform reduces to a simple cosine transform and is equal to

$$g_0(\kappa) = \frac{\sin\pi\kappa}{\pi\kappa} \ . \qquad (2-3)$$

$g_0(\kappa)$ has zeroes at all integers $\kappa = \pm 1, \pm 2, \ldots$, and has the value 1 at the origin $\kappa = 0$.

Since the space-shifted B-splines of degree zero, $\{G_0(x-j)\}$, $j = \ldots, -1, 0, 1, \ldots$, are linearly independent, a function $s(x)$ can be assembled in terms of a linear combination of these B-splines,

$$s(x) = \sum_{j=-\infty}^{\infty} f_j \, G_0(x-j) \ . \qquad (2-4)$$

Note that the support of $G_0(x)$ is equal to 1, that its integral is equal to 1, and that the sum of its function values at all integers (knots) is also equal to 1. Although frequently used for numerical integration, it is not quite that what we understand by an "interpolation" function, because $s(x)$ defined by (2-4) is not even continuous. For later reference we mention, that $s(x)$ is obviously the solution of a homogeneous differential equation $D_x s(x) = 0$ on all open intervals $(j-1/2 < x < j+1/2)$, and the "boundary conditions" $s(j) = f_j$ for all $j = \ldots, -1, 0, 1, \ldots$. Note that the coefficients of the linear combination are just the function values at the knots.

Convolution is known to have smoothing properties; let us therefore try a convolution between $G_o(x)$ and $G_o(x)$ and denote the result $G_1(x)$:

$$G_1(x) = G_o(x) * G_o(x) = \int_{-\infty}^{\infty} G_o(x-x')\, G_o(x')dx' \qquad (2-5)$$

It is a piecewise linear and continuous function,

$$G_1(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1 \\ 0 & \end{cases}, \qquad (2-6)$$

is therefore an element of C^o, has a support of 2 units, an integral of 1 , and also the sum of all function values at the knots is equal to 1 .
It is the triangular-shaped roof function and is called B-spline of degree 1.
According to the convolution theorem, its spectrum is the square of the spectrum of $G_o(x)$,

$$g_1(\kappa) = g_o(\kappa)^2 = \left(\frac{\sin \pi\kappa}{\pi\kappa}\right)^2 . \qquad (2-7)$$

The space-shifted B-splines of degree one, $\{G_1(x-j)\}$, $j = \ldots, -1, 0, 1, \ldots$ are linearly independent and therefore, a function $s(x)$ can be assembled in terms of a linear combination of these functions,

$$s(x) = \sum_{j=-\infty}^{\infty} f_j \, G_1(x-j) . \qquad (2-8)$$

Note that also here the coefficientsof the linear combination are just the function values at the knots.

As a matter of fact, we are not limited with one convolution, actually, we may perform as many as we like and generate in this simple way B-splines of arbitrary degree n . The B-spline of degree n is an element of C^{n-1} (the space of continuous and (n-1)-times continuously differentiable functions), it has a support of $n+1$ units, its integral is equal to 1 (for all degrees n), and the sum of all its function values at the knots is also equal to 1 (for all degrees n). According to the convolution theorem the spectrum of the B-spline of degree n is equal to the spectrum of the zero degree B-spline to the power $n+1$.

$$G_n(x) = G_o(x) * \underbrace{G_o(x) * \ldots * G_o(x)}_{n\ -\ \text{times}} \qquad (2 - 9a)$$

$$G_n(x) = G_{n-1}(x) * G_o(x) \ldots \text{ recursion} \qquad (2 - 9b)$$

$$\left.\begin{aligned} g_n(\kappa) &= g_o(\kappa)^{n+1} \qquad (2 - 9c) \\ g_n(\kappa) &= \left(\frac{\sin\pi\kappa}{\pi\kappa}\right)^{n+1} \qquad (2 - 9d) \end{aligned}\right\} \ldots \text{ spectrum}$$

$$G_n(x) \in C^{n-1} \qquad \ldots \text{ smoothness} \qquad (2 - 9e)$$

$$\operatorname{supp}(G_n(x)) = n+1 \qquad \ldots \text{ localness} \qquad (2 - 9f)$$

$$\int_{-\infty}^{\infty} G_n(x)\, dx = 1 \qquad \ldots \text{ normalization} \qquad (2 - 9g)$$

$$\sum_{-\infty}^{\infty} G_n(j) = 1 \qquad \ldots \text{ normalization} \qquad (2 - 9h)$$

$$G_n(-x) = G_n(x) \qquad \ldots \text{ symmetry} \qquad (2 - 9i)$$

The set of all space-shifted B-splines of a certain degree n $\{G_n(x-j)\}$, $j = \ldots, -1, 0, 1, \ldots$, is linearly independent and can therefore be used to assemble a spline function of degree n,

$$s(x) = \sum_{j=-\infty}^{\infty} c_j \, G_n(x-j) \, . \qquad (2 - 10)$$

Unfortunately, the coefficients c_j of the linear combination are in general different from the function values at the knots. Only for degree $n = 0$ and $n = 1$ the coefficients c_j are equal to the function values f_j. The determination of the coefficients $\{c_j\}$ from the data set $\{f_j\}$ requires the solution of a linear system. If the data are regularly distributed, even an infinite system can be solved very easily as we shall see in the sequel.

Let us introduce "cardinal sampling splines" of degree n, $\Gamma_n(x)$ with the sampling property

$$\Gamma_n(j) = \delta_{j,0} \qquad (2 - 11a)$$

and let $\Gamma_n(x)$ be represented in terms of a linear combination of space-shifted B-splines $G_n(x-j)$,

$$\Gamma_n(x) = \sum_{-\infty}^{\infty} \alpha_k \, G_n(x - k) \, . \qquad (2 - 11b)$$

Then the infinite vector $\alpha := \{\alpha_k\}$ is determined from the sampling properties and the function values of the B-spline at the knots $G_n(j)$. The determination of the vector α requires the solution of a linear system

$$A \, \alpha = e \qquad (2 - 12a)$$

with

$$A = \{G_n(j - k)\} , \qquad (2 - 12b)$$

$$\alpha^T = [\ \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots] \qquad (2 - 12c)$$

$$e^T = [\ \ldots, 0, 1, 0, \ldots \ .\ ] \qquad (2 - 12d)$$

The matrix A has very special properties which are due to the properties of the basis function $G_n(x)$: it is an infinite symmetric matrix of band structure. A is a <u>Toeplitz</u> matrix and because of its infinite dimension, it is asymptotically equivalent to a circulant matrix.

The bandwidth N (= number of non-zero diagonals) depends only on the support of the basis function $G_n(x)$:

$$N = \begin{cases} n+1 & \text{if } n \text{ is even} , \\ n & \text{if } n \text{ is odd} . \end{cases}$$

E.g., for degree $n = 0$ and $n = 1$, A is a strict diagonal matrix, moreover, it is the unit matrix. Therefore, $\alpha = e$ and, as a consequence of equation (2-11b), the spline basis functions (= B-splines) $G_0(x)$, $G_1(x)$ are already cardinal sampling splines $\Gamma_0(x)$, $\Gamma_1(x)$.

And now we shall demonstrate how the infinite linear system (2-12a) is solved for degrees $n > 1$:
Denoting a row (or column) of A by a , $a := \{a_k\}$, the elements a_k are given by

$$a_k = G_n(k) . \qquad (2 - 13)$$

In order to solve for α , we need to know the inverse of A , which can be easily obtained by Fourier transform methods: denoting the discrete Fourier transform of a by $\bar{a}$,

$$\bar{a}(\mu) = \sum_{-\infty}^{\infty} a_k e^{-ik\mu}$$

$$= \sum_{-\infty}^{\infty} G_n(k) \cos k\mu$$

$$= G_n(0) + 2 \sum_{k=1}^{\mathrm{int}(\frac{n}{2})} G_n(k) \cos k\mu \ , \qquad (2 - 13)'$$

and by

$$\bar{e}(\mu) = \sum_{-\infty}^{\infty} e_k e^{-ik\mu} = 1 \ ,$$

the unknown coefficients can be found directly by an inverse Fourier transform,

$$\alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\bar{e}(\mu)}{\bar{a}\ \mu} e^{ik\mu} \, d\mu$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \frac{1}{\bar{a}(\mu)} \cos k\mu \, d\mu \ . \qquad (2 - 14)$$

This integral plays a central role within the framework of splines: if only 1 frequency is present in $\bar{a}(\mu)$ apart from the zero frequency, equation (2-14) admits a closed expression and the α_k can be calculated directly $(n \leq 3)$; if more than 1 frequency is present $(n > 3)$, we no longer have this simplicity.
With the coefficients α_k determined, the cardinal sampling

spline can be calculated by (2-11b). In contrast to the basis spline, the cardinal sampling spline of degree n, $\Gamma_n(x)$, has unlimited support for $n > 1$; the higher n, the weaker is its tendency to approach zero with increasing argument, and therefore, the weaker is its local behaviour.

The spectrum of all splines has the following common structure:

$$f_n(\kappa) = h_n(\kappa) \left(\frac{\sin\pi\kappa}{\pi\kappa}\right)^{n+1} \sum_{-\infty}^{\infty} f_j \, e^{-i2\pi\kappa j} \,, \qquad (2 - 15)$$

where $h_n(\kappa)$ is the characteristic of order n. For $n = 0$ and 1, $h_n(\kappa) = 1$, for $n = 2$, $h_n(\kappa)$ is given by

$$h_2(\kappa) = \frac{4}{3+\cos 2\pi\kappa} \,,$$

for $n = 3$, by

$$h_3(\kappa) = \frac{3}{2+\cos 2\pi\kappa} \,.$$

In general,

$$h_n(\kappa) = \sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\kappa k \qquad (2 - 16)$$

with $\{\alpha_k\}$ denoting the vector of coefficients for the degree n in consideration. These functions $h_n(\kappa)$ have the very interesting property of compensating the damping property of the function $(\sin\pi\kappa/\pi\kappa)^{n+1}$ better and better with increasing n.

Let us now put the obvious question: "What kind of function is the limit cardinal sampling spline if n goes to infinity ?"

The spline defined by

$$s_\infty(x) = \sum_{-\infty}^{\infty} f_j \, \Gamma_\infty(x-j) \qquad (2 - 17)$$

should obviously reproduce the data vector $\{f_j\}$, which is supposed to be an element of l^∞, the space of quadratically summable finite differences of all orders k,

$$l^\infty := \bigcup_{k=0}^{\infty} l^k .$$

The corresponding cardinal spline of infinite degree should be an element of K^∞, the space of quadratically integrable derivatives of all orders k,

$$K^\infty := \bigcup_{k=0}^{\infty} K^k .$$

There is a one-to-one correspondence between the spaces l^∞ and K^∞. It can be shown (Schoenberg, 1973), that the function $\sin\pi x/\pi x$ is the only element of K^∞ which interpolates the unit sequence. Therefore, we conclude

$$\lim_{n\to\infty} \Gamma_n(x) = \frac{\sin\pi x}{\pi x} . \qquad (2 - 18)$$

The corresponding limit interpolation spline is

$$s_\infty(x) = \sum_{-\infty}^{\infty} f_j \frac{\sin \pi(x-j)}{\pi(x-j)} . \qquad (2 - 19)$$

Γ_∞ can be said to be the cardinal sampling spline of highest possible degree, $s_\infty(x)$ the corresponding interpolating spline.

Its trend to approach zero with increasing argument is rather poor and, therefore, its behaviour can no longer be called local.

The spectrum of Γ_∞ is a unit step function, since the spectrum of the unit step function is exactly $\sin\pi\kappa/\pi\kappa$. Since $\gamma(\kappa) = 0$ for all $|\kappa| > 1/2$ (γ denotes the Fourier transform of Γ), we conclude that Γ_∞ is a low pass filter: the interpolation function does not contain frequencies higher than $|\kappa| > 1/2$. The relation between Γ_∞ and γ_0 is quite remarkable and deserves special attention:

The cardinal sampling spline of highest possible degree equals the spectrum of the cardinal spline of the lowest possible degree and vice versa.

$$\Gamma_\infty = \gamma_0 \,, \qquad \Gamma_0 = \gamma_\infty \,. \tag{2 - 20}$$

The Green's function approach

Let there be given an ordinary homogeneous differential equation of first order

$$D_x f(x) = \frac{df(x)}{dx} = 0 \,; \tag{2 - 21}$$

as a matter of fact we have the solution

$$f(x) = \text{const.}$$

If the domain of definition is the interval (a,b), then f will be constant on that particular interval.

The above differential equation could also be solved by the Green's function approach, although there is no obvious reason for it in this simple case:
We look for a solution of the differential equation in terms of an integral with an integral kernel to be determined; this integral kernel is called Green's function and will be denoted as usual by G ,

$$\int_a^b G(x,x') \, D_{x'} f(x') \, dx' = f(x) \, . \qquad (2 - 22)$$

If we had simply $f(x')$ instead of the derivative, then G would equal the Dirac δ - distribution; however, if we differentiate our equation with respect to x ,

$$\int_a^b D_x G(x,x') \, D_{x'} f(x') = D_x f(x) \, , \qquad (2 - 23)$$

we can take advantage of the reproducing property of the Dirac function and observe that

$$D_x G(x,x') = \delta(x,x') \, . \qquad (2 - 24)$$

This equation represents obviously a linear system (note that differential operators are linear) with the Green's function as input and the Dirac's function as output. The solution is obtained by a simple integration yielding a unit step function as result.

Let us now investigate a second order differential equation

$$D_x^2 f(x) = \frac{d^2 f(x)}{dx^2} = 0 \, ; \qquad (2 - 25)$$

Using again the concept of Green's function, we may represent the solution in terms of the integral transformation

$$\int_a^b G(x,x')\ D_{x'} f(x')\ dx' = f(x) \qquad (2 - 26)$$

with the two boundary conditions $f(a) = f(b) = 0$. If we differentiate equation (2-26) twice with respect to x, we obtain - taking again into account the reproducing property of the Dirac function - a linear system in terms of an inhomogeneous differential equation of second order with the Green's function as input and the Dirac's function as output. As before, the solution can be obtained by integrating the Dirac function twice with respect to x, yielding a roof function. If we center and normalize that roof function with respect to $a = -1$ and $b = 1$, we obtain the already familiar spline base function $G_1(x)$.
It is important to stress that the differential equation (2-25) can be interpreted as an Euler differential equation corresponding to the variational problem

$$\int [D_x f(x)]^2\ dx = \min. , \qquad (2 - 27)$$

the minimization of the integral of the squared first derivative of $f(x)$ over the domain of definition. This semi-norm is blind with respect to a constant function.
It should be obvious that the procedure described for the first and second derivative can be applied to higher order derivatives as well. Each differential equation of even order 2n can be interpreted as an Euler differential equation corresponding to a variational problem with squared n'th order derivatives. According to my opinion, it is very important to note that all Green's functions are related to each other through convo-

lutions and that, therefore, all Green's functions are, if centered and normalized (symmetric with respect to zero, integral equal to 1), base splines.

The Green's function method can be illustrated by a nice geodetic example: the solution of the Stokes' problem.

The differential operator B ,

$$B := -D_r - \frac{2}{r} , \qquad (2 - 28)$$

is applied to the disturbing potential T at the sphere S $(r=R)$, yielding the fundamental equation of physical geodesy $BT = \Delta g$, the boundary value problem. T must be harmonic for $r > R$ and should be represented in terms of an integral with the Green's function as integral kernel according to Green's integral theorem,

$$\int_S G(x,y)\, B_y T(y)\, dS(y) = T(x) - p_1(x) , \qquad (2 - 29)$$

where $p_1(x)$ is a harmonic function of first degree (a linear combination of the 3 solid spherical harmonics of first degree). Applying the operator B to (2-29) at x and observing that $p_1(x)$ vanishes under B , we obtain

$$\int_S B_x G(x,y)\, B_y T(y)\, dS(y) = B_x T(x) . \qquad (2 - 30)$$

Obviously BT is reproduced by this integral transformation; therefore, $B_x G(x,y)$ must be a Dirac function,

$$BG = \delta . \qquad (2 - 31)$$

The Dirac function on S has the series representation

$$\delta(x,y) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) P_n(xy) \qquad (2-32)$$

with the Legendre polynomials $P_n(xy)$ and $xy := x^T y = \cos(x,y)$. Representing G also in terms of a series of Legendre polynomials and observing that G must be harmonic both with respect to x and y,

$$G(x,y) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \left(\frac{R}{r}\right)^{n+1} g_n P_n(xy) , \quad |y| = R \qquad (2-33)$$

and applying the operator B_x to G on S, we obtain

$$B_x G(x,y) = \frac{1}{4\pi R} \sum_{n=0}^{\infty} (n-1)(2n+1) g_n P_n(xy) \qquad (2-34)$$

and comparing it to (2-32), we obtain the spectrum of the Green's function

$$g_n = \frac{R}{n-1} , \qquad (2-35)$$

and with (2-33) the series representation of Green's function restricted to S $(r=R)$,

$$G(x,y) = \frac{R}{4\pi} \sum_{\substack{n=0 \\ n\neq 1}}^{\infty} \frac{2n+1}{n-1} P_n(xy) , \qquad (2-36)$$

which is just the familiar Stokes function with zero degree included,

$$G(x,y) = S_o(x,y) . \qquad (2-37)$$

Basically the same principle will be repeatedly applied in the sequel.

3. SPLINES ON THE CIRCLE

Trigonometric splines on the circle will be investigated in order to find the transition to the spherical splines easier and because they share practically all properties with spherical splines. We shall primarily elaborate the ideas of Schoenberg (1964) and try to make it a bit more easy going for a mathematically limited geodesist's mind.
Let us start with an ordinary differential operator of first order on the unit circle

$$D_x f(x) = \frac{df(x)}{dx} , \qquad (3 - 1)$$

and apply the Green's function method for the solution. Then we obtain as in (2-22) an integral transformation with a Green's function G, which we shall here denote by $G(0;x,y)$ for reasons we shall see later,

$$\int_0^{2\pi} G(0;x,y)\, D_y f(y)\, dy = f(x) - p_0(x) \qquad (3 - 2)$$

with x and y points on the unit circle and $p_0(x)$ a trigonometric polynomial of degree zero. Differentiating equation (3-2) with respect to x yields the inhomogeneous linear differential equation in terms of a linear system with the Green's function as input and Dirac's function as output,

$$D_x G(0;x,y) = \delta(x,y) . \qquad (3 - 3)$$

The solution to (3-3) can readily be obtained by using the Fourier series representation of the Dirac function (Davis, 1965, p. 323)

$$\delta(x,y) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \cos k(x-y) \; . \qquad (3 - 4)$$

Integration of (3-4) yields, apart from the constant contribution of (3-4), the Green's function

$$G(0;x,y) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin k(x-y)}{k} \; . \qquad (3 - 5)$$

It can be verified immediately that equation (3-2) holds with (3-5) using a Fourier series representation of $f(x)$. For $G(0;x,y)$ we can even find a closed expression

$$G(0;x) = \frac{1}{2}\left(1 - \frac{x}{\pi}\right) \qquad (3 - 5)'$$

with $x := x - y$. Note that $G(0;x)$ is discontinuous at $x = 0$ (2π, resp.), therefore (3-5)' is only valid in the open interval $0 < x < 2\pi$. At $x = 0$ $G(0;x)$ has a jump of 1, $G(0;0_+) - G(0;2\pi_-) = 1$. $G(0;x)$ is an element of $C^{-1}_{2\pi}$, the space of discontinuous functions on the unit circle. This agrees with the situation on the real line.
Given J distinct knots on the circumference of the unit circle, $x_1, x_2, \ldots, x_J$, the Green's functions $\{G(0;x,x_j)\}$ $j = 1, \ldots, J$, are linearly independent and can therefore be used as base functions for interpolation, yielding a trigonometric spline of degree zero,

$$s(x) = \sum_{j=1}^{J} c_j \, G(0;x - x_j) + p_o(x) \qquad (3 - 6)$$

with $p_o(x)$ a trigonometric polynomial of degree 0 and $\{c_j\}$, $j = 1, \ldots, J$, the coefficients of the linear combination. Disregarding at the moment the discontinuity of $G(0;x,y)$ and postulating that $s(x)$ should reproduce the data $\{f_k\}$, $k = 1,\ldots, J$, we obtain a system of J equations for $J + 1$ unknowns $c_1, c_2, \ldots, c_J$, and p_o,

$$\sum_{j=1}^{J} c_j \, G(0;x_k,x_j) + p_o(x_k) = s(x_k) = f_k \; . \qquad (3 - 7)$$

In matrix notation (3-7) can be represented by

$$C\,c + A\,p = f \qquad (3 - 7)'$$

with

$$C := \begin{bmatrix} G(0;x_1,x_1) & G(0;x_1,x_2) & \ldots & G(0;x_1,x_J) \\ G(0;x_2,x_1) & G(0;x_2,x_2) & \ldots & G(0;x_2,x_J) \\ \vdots & & & \vdots \\ G(0;x_J,x_1) & G(0;x_J,x_2) & \ldots & G(0;x_J,x_J) \end{bmatrix} \qquad (3\text{-}7a)'$$

$$A^T := [\, 1, 1, \ldots, 1 \,] , \qquad (3\text{-}7b)'$$

$$c^T := [\, c_1, c_2, \ldots, c_J \,] , \qquad (3\text{-}7c)'$$

$$p := p_o , \qquad (3\text{-}7d)'$$

$$f^T := [\, f_1, f_2, \ldots, f_J \,] . \qquad (3\text{-}7e)'$$

Due to the orthogonality between G and p_o, the vector of coefficients c must be orthogonal to A,

$$A^T c = 0 \; , \tag{3 - 8}$$

or, explicitly, the sum of the coefficients must be equal to zero,

$$\sum_{j=1}^{J} c_j = 0 \; . \tag{3 - 8)'$$

This equation supplements the system (3-7)' to

$$\begin{bmatrix} C & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} c \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{3 - 9}$$

and has the solution

$$p = (A^T C^{-1} A)^{-1} A^T C^{-1} f \tag{3 - 10}$$

$$c = C^{-1}(f - Ap) \; .$$

The similarity with the collocation solution with parameters is striking - we shall discuss its relation to the collocation solution after we have investigated trigonometric splines of higher degree.

Let us now introduce another differential operator of second degree, the oscillation operator $D^2 + m^2$,

$$D_x^2 + m^2 = \frac{d^2}{dx^2} + m^2 \; . \tag{3 - 11}$$

Then we search for a Green's function, such that

$$\int_0^{2\pi} G(m;x,y)\ (D_y^2 + m^2)\ f(y)\ dy = f(x) - p_m(x), \qquad (3 - 12)$$

where $p_m(x)$ denotes a trigonometric function which has only the frequency m. Let us now apply our differential operator onto the Green's function $G(m;x,y)$ with respect to x,

$$\int_0^{2\pi} (D_x^2 + m^2)\ G(m;x,y)\ (D_y^2 + m^2)\ f(y)\ dy = (D_x^2 + m^2) f(x)\ . \qquad (3 - 13)$$

Note that $p_m(x)$ vanishes under the differential operator $D_x^2 + m^2$, because the trigonometric functions cosmx and sinmx are eigenfunctions of that differential operator with the eigenvalue m^2,

$$(D_x^2 + m^2)\ \cos mx = 0 \qquad (3 - 14)$$

$$(D_x^2 + m^2)\ \sin mx = 0\ .$$

Again we observe the reproducing property

$$(D_x^2 + m^2)\ G(m;x,y) = \delta(x,y)\ . \qquad (3 - 15)$$

Using the Fourier series representation of the Dirac function (3-4), we see that the Fourier series of the Green's function is given by

$$G(m;x,y) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \cos k(x-y)\ , \quad m = 0\ ,$$

$$G(m;x,y) = \frac{1}{2\pi m^2} - \frac{1}{\pi} \sum_{\substack{k=1 \\ k \neq m}}^{\infty} \frac{1}{k^2 - m^2} \cos k(x-y)\ , \quad m > 0\ . \qquad (3 - 16)$$

(The reader is cordially invited to verify that (3-12) and (3-16) are indeed compatible.)

Also in this more complicated case closed formulas exist (Hansen, 1975, p. 239, No. 17.2.8 and p. 243, No. 17.3.11):

$$G(0;x) = \frac{x}{2}(1 - \frac{x}{2\pi}) - \frac{\pi}{6} \text{ , } m = 0 \text{ ,}$$

$$G(m;x) = \frac{1}{2}(1 - \frac{x}{\pi}) \frac{\sin mx}{m} - \frac{\cos mx}{4\pi m^2} \text{ , } m > 0 \text{ .} \qquad (3 - 16)'$$

It can be shown immediately that $G(m;x)$ is continuous for $m = 0, 1, \ldots$; its first derivative, however, is discontinuous at $x = 0$ (2π, resp.) with a jump of 1 . This discontinuity is due to the term $1/2(1 - x/\pi)$.

$G(0;x)$ has therefore very much the same properties as the spline basis function G_1 of Chapter 2, which is also continuous and has a discontinuous first derivative. Remember that G_1 was the result of a convolution between G_o and G_o, $G_1 = G_o * G_o$ this is also the case for $G(0;x)$:

$$G_1(0;x) = G_o(0;x) * G_o(0;x) \text{ ,} \qquad (3 - 17)$$

which can be easily proved as follows:

$$G_o * G_o = \frac{1}{\pi^2} \int_0^{2\pi} [\sum_{k=1}^{\infty} \frac{1}{k} \sin kx][\sum_{l=1}^{\infty} \frac{1}{l} \sin l(x-y)] \, dx \text{ ;}$$

representing $\sin l(x-y)$ in terms of its decomposition $\sin l(x-y) = \sin lx \cos ly - \cos lx \sin ly$ and observing the orthogonality relations for trigonometric functions, we readily obtain

$$G_o * G_o = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos kx ,$$

which is in perfect agreement with (3-16) .

Following this procedure, we can generate Green's functions corresponding to the differential operator D^n with arbitrary positive integer n and obtain recursion formulas, which are identical to those for the spline basis function on the real line:

$$G_n = G_{n-1} * G_o = G_o * \underbrace{G_o * \dots * G_o}_{n\text{ - times}} . \qquad (3 - 18)$$

We can write down the Fourier series for G_n immediately:

$$G_n(0;x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\sin kx , \quad n \text{ even} ,$$

$$(3 - 19)$$

$$G_n(0;x) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\cos kx , \quad n \text{ odd} .$$

Another recursion relation is evident, which expresses $G_{n-1}(0;x)$ in terms of the derivative of $G_n(0;x)$,

$$G_{n-1}(0;x) = - D_x G_n(0;x) . \qquad (3 - 20)$$

And last not least, the integral of all Green's functions $G_n(0;x)$ over the unit circle is equal to zero. (On the real line the integral of all spline basis functions is equal to one.) $G_n(0;x)$ is an element of $C_{2\pi}^{n-1}$.

From (3-20) we can deduce that $G_n(0;x)$ can be easily obtained from its predecessor $G_{n-1}(0;x)$ by simple integration with the integration constant determined from the condition that the integral of the Green's function has to be zero,

$$G_n(0;x) = - \int G_{n-1}(0;x)\, dx \ ,$$

$$\int_0^{2\pi} G_n(0;x)\, dx = 0 \ . \qquad (3 - 21)$$

In this way we can for example easily derive

$$G_2(0;x) = \frac{x}{12\pi}(x^2 - 3\pi x + 2\pi^2)$$

$$G_3(0;x) = - \frac{1}{720\pi}(8\pi^4 - 60\pi^2 x^2 + 60\pi x^3 - 15x^4)$$

$$\vdots$$

Let us put an obvious question: How does $G_n(0;x)$ look like if n goes to infinity? The simple answer is provided by (3-19)

$$\lim_{n\to\infty} G_n(0;x) = - \frac{1}{\pi} \cos x \quad (\text{or } \frac{1}{\pi} \sin x) \ , \qquad (3 - 22)$$

the trigonometric function (s) with the longest possible wavelength (disregarding a constant function) and therefore, with the least local and most global behaviour. Exactly the same has been observed for basis splines on the real line: if the degree of the basis spline increases, the spline looses its local behaviour and is getting smoother and smoother. The spline of infinite degree is the smoothest possible spline, it is infinitely often continuously differentiable ($G_\infty(x)$ and $G_\infty(0;x)$). And once more we observe the "balance of fright": the price for smoothness is paid by non-localness and vice versa, the price for localness is paid by non-smoothness. I wonder where in between 0 and ∞ the (so-called) optimum is located. Is it G_3 ? In terms of (elastic) energy minimization it is: We have already seen that G_1 was the Green's function corresponding to a variational problem aimed at the minimization of the integral of a function's squared first derivative (square of horizontal gradient),

$$G_1 \qquad \int [D_x f(x)]^2 \, dx = \min. \; .$$

Such a variational problem can be assigned to all Green's functions of odd degree:

$$G_{2n-1} \qquad \int [D_x^n f(x)]^2 \, dx = \min. \; .$$

Since $n = 2$ is closely related to elastic energy minimization, $G_{2n-1} = G_3$ might be a preferable compromise.

Considering only functions on the circle with a vanishing zero frequency component (mean value zero functions), then the factor $1/2\pi m^2$ can also be dropped from the Green's function $G_1(m;x,y)$ in (3-16),

$$G_1^*(m;x,y) := G_1(m;x,y) - \frac{1}{2\pi m^2} \; . \qquad (3 - 23)$$

It is obvious from the orthogonality relations of trigonometric functions, that the integral of $G_1^*(m;x)$ over the unit circle is also equal to zero.

The many useful relations we had derived for the Green's function with $m = 0$ are met for $m \neq 0$ as well. We can, for example, perform a convolution between $G_1^*(m;x)$ with $G_1^*(m;x)$ and obtain $G_3^*(m;x)$. From the convolution theorem and the above discussion we know already that a convolution in the space domain corresponds to a multiplication in the frequency domain (and vice versa). Therefore, the Fourier series of $G_3^*(m;x)$ must be given by

$$G_3^*(m;x) = -\frac{1}{\pi}\sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{1}{(k^2-m^2)^2}\cos kx \; , \qquad (3-24)$$

or, for an arbitrary number of convolutions,

$$G_{2n-1}^*(m;x) = -\frac{1}{\pi}\sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{1}{(k^2-m^2)^n}\cos kx \; . \qquad (3-25)$$

Closed expressions can be derived :

performing, e.g., the convolution between G_1^* and G_1^* with one G_1^* in the closed expression (3-16)' and the other G_1^* in the Fourier series representation (3-16), we notice that, due to the orthogonality of the trigonometric functions, only the term $x \sin mx$ is non-orthogonal to G_1^* , and therefore, the convolution reduces to

$$G_3^*(m;x) = G_1^*(m;x) * G_1^*(m;x)$$

$$= (-\frac{1}{2\pi m}\, x \sin mx) * G_1^*(m;x) \; ,$$

or, in general,

$$G^*_{2n+1}(m;x) = G^*_1(m;x) * G^*_{2n-1}(m;x)$$

$$= \left(-\frac{1}{2\pi m}\, x \sin mx\right) * G^*_{2n-1}(m;x) \,, \qquad (3-26)$$

for $n = 1, 2, \ldots$.
The recursion relation (3-20) , which is valid for $m = 0$, is governed by the differential operator $D^2 + m^2$ if $m > 0$.

$$G^*_{2n-1}(m;x) = (D^2_x + m^2)\, G^*_{2n+1}(m;x) \,. \qquad (3-27)$$

If we let (as above for $m = 0$) n go to infinity, the resulting Green's function, normalized to its lowest frequency component, becomes

$$\lim_{n\to\infty} G^*_{2n-1}(m;x) = \cos k_{min} x \,. \qquad (3-28)$$

Schoenberg (1964) has also considered quite general differential operators of the form

$$\Delta_M := D \prod_{m=1}^{M} (D^2 + m^2)$$

$$= D(D^2 + 1^2)(D^2 + 2^2) \ldots (D^2 + M^2) \qquad (3-29)$$

and also its square

$$\Delta^2_M = D^2 \prod_{m=1}^{M} (D^2 + m^2)^2 \,. \qquad (3-30)$$

Let us first investigate the operator (3-29) :
Since $(D^2 + m^2)$ annihilates cosmx and sinmx , since D annihilates a constant function, and because of the commutativity of the elements of Δ_M , it follows that Δ_M annihilates all trigonometric polynomials up to and including degree M . With other words, a Fourier series under the operator Δ_M has vanishing elements from m = 0 to m = M , a Fourier series under Δ_M starts at m = M + 1 .

What is Green's function then for the differential operator Δ_M ? We have seen before that to each individual element of Δ_M , $(D, (D^2 + 1^2), (D^2 + 2^2), \ldots)$, an individual Green's function can be derived and that a composite Green's function is obtained by a convolution of the individual functions. Let us demonstrate this procedure with a simple example:

Let M = 1 , then $\Delta_1 = D(D^2 + 1^2)$; Green's function for the operator D is given by equ. (3-5) , the one for the operator $(D^2 + 1^2)$ by equ. (3-16) with m = 1 . The convolution of $G_o(0;x)$ with $G_1(1;x)$ yields a composite Green's function, which we shall denote by $G_{\Delta 1}(x)$,

$$G_{\Delta 1}(x) := G_o(0;x) * G_1(1;x)$$

$$= \int_0^{2\pi} [\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin kx] [\frac{1}{2\pi} - \frac{1}{\pi} \sum_{l=2}^{\infty} \frac{1}{l^2-1} \cos l(x-y)] dx;$$

decomposing cos l(x-y) into cos lx cos ly + sin lx sin ly , and observing again the orthogonality relations between trigonometric functions, we obtain as result

$$G_{\Delta 1}(x) = -\frac{1}{\pi} \sum_{k=2}^{\infty} \frac{1}{k(k^2-1^2)} \sin kx , \qquad (3 - 31)$$

and if we apply this procedure M - times (for all m = 1, ..., M) we obtain Green's function for the differential operator Δ_M

$$G_{\Delta M}(x) = \frac{(-1)^M}{\pi} \sum_{k=M+1}^{\infty} k^{-1} \prod_{m=1}^{M} (k^2-m^2)^{-1} \sin kx \; . \qquad (3 - 32)$$

It is remarkable that closed expressions exist for (3-32): Schoenberg (1964) has proved that $G_{\Delta M}(x)$ can be represented in terms of

$$G_{\Delta M}(x) = (-1)^M \frac{2^M}{(2m)!} \frac{1}{2} (1 - \frac{x}{\pi})(1 - \cos x)^M + p_M(x), \qquad (3 - 33)$$

where $p_M(x)$ is a sine polynomial of degree M. (This can be easily shown by a Fourier analysis of equ. (3-33).)
It can also be verified that $G_{\Delta M}(x)$ is an element of $C_{2\pi}^{2M-1}$ and that its derivative of order $2M$ is discontinuous at $x = 0$ (2π, resp.) with a jump of 1,

$$D_x^{2M} \quad G_{\Delta M}(0_+) - G_{\Delta M}(2\pi_-) = 1 \; . \qquad (3 - 34)$$

Having analyzed the very principle behind the generation of Green's functions with respect to differential operators, we can even go a step further and consider the self-adjoint operator Δ_M^2 of equ. (3-30), which is obviously equal to

$$\Delta_M^2 = \Delta_M \; \Delta_M \; .$$

If $G_{\Delta M}$ is the Green's function corresponding to Δ_M, then $G_{\Delta M}^2$ with

$$\Delta_M G_{\Delta M}^2 = G_{\Delta M} \qquad (3 - 35)$$

is the Green's function for the operator Δ_M^2 . As a consequence of the convolution theorem, we obtain with (3-32) the Fourier series representation of $G_{\Delta M}^2$ by a convolution of $G_{\Delta M}$ with $G_{\Delta M}$,

$$G_{\Delta M}^2(x) = \frac{1}{\pi} \sum_{k=M+1}^{\infty} k^{-2} \prod_{m=1}^{M} (k^2 - m^2)^{-2} \cos kx \ . \qquad (3 - 36)$$

(As an instructive exercise the reader may try to verify that (3-35) holds.)

It is very remarkable that even for the series (3-36) a closed expression exists: it has been shown by Schoenberg (1964) that $G_{\Delta M}^2 (x)$ can be uniquely represented in terms of

$$G_{\Delta M}^2(x) = x(2 - \frac{x}{\pi})C_M(x) + (1 - \frac{x}{\pi})S_M(x) + p_M(x) \quad (3 - 37)$$

with $C_M(x)$ and $p_M(x)$ being cosine polynomials, and $S_M(x)$ a sine polynomial of degree M . The proof is very technical and will not be given here. In the above referenced paper it has also been proved that $G_{\Delta M}^2$ is an element of $C_{2\pi}^{4M}$ and that the usual jump relation exists at $x = 0$ (2π, resp.) ,

$$D_x^{4M+1}[G_{\Delta M}(0_+) - G_{\Delta M}(2\pi_-)] = 1 \ . \qquad (3 - 38)$$

We could go another step further and derive Green's functions with respect to a differential operator

$$\Delta_M^\mu : = D^{\mu_o} \prod_{m=1}^{M} (D^2+m^2)^{\mu_m} \ , \ \mu : = \sum_{m=0}^{M} \mu_m \qquad (3 - 39)$$

with non-negative integers μ_o , ..., μ_M . Then the Fourier series representation of Green's function corresponding to the operator (3-39) can be derived very quickly,

$$G^{\mu}_{\Delta M}(x) = \frac{1}{\pi}(-1)^{\mu - \mu_o} \sum_{k=M+1}^{\infty} k^{-\mu_o} \prod_{m=1}^{M} (k^2-m^2)^{-\mu_m} \begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix} \qquad (3 - 40)$$

with coskx if μ_o is even and sinkx if μ_o is odd.

Let us now proceed to the final step, the spline interpolation on the circle.

As before we assume that J distinct knots are given along the circumference of the unit circle $x_1, x_2, \ldots, x_J$. All Green's functions $\{G^{\mu}_{\Delta M}(x,x_j)\}$ for $j = 1, \ldots, J$ are linearly independent and can therefore be used as basis functions for an interpolation : The interpolating spline on the circle, corresponding to the differential operator Δ^{μ}_{M} , is represented in terms of a linear combination of those Green's functions,

$$s(x) = \sum_{j=1}^{J} c_j G^{\mu}_{\Delta M}(x,x_j) + p_M(x) \; . \qquad (3 - 41)$$

The unknown J coefficients $\{c_j\}$ of the linear combination of the Green's functions (evaluated at x_j) and the 2M + 1 unknown coefficients of the trigonometric polynomial of degree M , $p_M(x)$, can be determined by

a) the data reproducing (interpolating) conditions, providing J linear equations, and

b) the $2M + 1$ orthogonality conditions of the vector $c^T := [c_1, c_2, \ldots, c_J]$ with respect to all $2M+1$ trigonometric functions of $p_M(x)$ evaluated at the data points.

Using the notation (3-7)', we obtain again the system of equations (3-9) , where we have to replace in the matrix C $G(0;x_i,x_j)$ by $G^{\mu}_{\Delta M}(x_i,x_j)$; the scalar p of (3-7d)' becomes in general a (2M+1) - dimensional vector p ,

$$p^T = [\, p_0, p_1, \ldots, p_{2M+1} \,] , \qquad (3 - 42)$$

and the vector A becomes a matrix of dimension (J,2M+1) .

$$A = \begin{bmatrix} 1 & \cos x_1 & \sin x_1 & \ldots & \sin Mx_1 \\ 1 & \cos x_2 & \sin x_2 & \ldots & \sin Mx_2 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ 1 & \cos x_J & \sin x_J & \ldots & \sin Mx_J \end{bmatrix} \qquad (3 - 43)$$

The solution for c and p is given by (3-10) and is formally identical to the collocation solution with parameters; the matrix C , which in the spline case studied here, is the Gram matrix derived from the evaluation functionals applied to the Green's function at all data points, and corresponds to the covariance matrix and the Green's function to the covariance function; all the other vectors and matrices involved in the solution are anyway the same as in the collocation solution. The interesting point is here that, for all $\mu_m = 2$, we obtain a best solution with respect to a very well defined energy integral.

4. SPLINES ON THE SPHERE

With the experience we gained from the derivation of splines on the circle and from studying the outstanding contribution on spherical spline interpolation and approximation by Freeden (1981), the spherical approach should go very smoothly. But nevertheless, let us briefly summarize the steps to follow:

1) Choose differential operator(s) L_i.
2) Find the Green's function(s) G_i to L_i, observing that L_iG_i is the Dirac impulse, using frequency domain methods.
3) To any product L_iL_j of differential operators there exists also a Green's function G_{ij}, which is the convolution $G_i * G_j$. Employ the convolution theorem to obtain the spectrum of G_{ij} as the product of the spectrum of G_i and that of G_j. Find a closed expression for G_{ij}.
4) The interpolating spline function corresponding to the differential operator L_iL_j is a linear combination of G_{ij}'s superimposed by a polynomial; the polynomial's degree depends on the degree of L_iL_j.
   The coefficients of the linear combination plus the coefficients of the polynomial are obtained from the data reproducing and the orthogonality conditions.

Let us now derive spherical splines following closely these 4 steps.

STEP 1: Choose differential operator(s)

In chapter 3, when we discussed splines on the circle, we have chosen differential operators of the oscillation type $D_x^2 + m^2$; the trigonometric functions are eigenfunctions to this kind of operators with m^2 being the eigenvalue.

The differential operator on the (unit) sphere S, corresponding to D_x^2 on the circle, is the Laplace - Beltrami operator $\bar{\Delta}$(= Laplace operator restricted to S),

$$\bar{\Delta} := \frac{\partial^2 V}{\partial\theta^2} + \cot\theta \frac{\partial V}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 V}{\partial\lambda^2} \qquad (4 - 1)$$

(θ ... polar distance, λ ... longitude). It is well-known from the theory of harmonic functions, that the eigenfunctions to (4-1) are the (Laplace's) surface spherical harmonics $S_{nm}(\theta,\lambda)$ (or simply spherical harmonics), corresponding to the eigenvalue

$$\lambda_n := n(n+1) , \qquad (4 - 2)$$

$$(\bar{\Delta} + \lambda_n)S_{nm} = 0 \qquad (4 - 3)$$

(cf., e.g., Heiskanen & Moritz, 1967, p. 19 ff.).

STEP 2: <u>Find Green's function to $\bar{\Delta} + \lambda_n$</u>

The solution (integration) of the differential equation $(\bar{\Delta} + \lambda_n)$ f = ... is implied by Green's function $G_1(n;x,y)$ as integral kernel, where n stands for the eigenvalue λ_n

$$\int_S G_1(n;x,y) \, (\bar{\Delta}_y + \lambda_n) \, f(y) \, dS(y) = f(x) - p_n(x) , \qquad (4 - 4)$$

with $p_n(x)$ being a Laplace surface harmonic of degree n ,

$$p_n(x) := \sum_{m=-n}^{n} s_{nm} \, S_{nm}(x) . \qquad (4 - 5)$$

(Here and in the sequel x, y, z denote unit vectors corresponding to points on the unit sphere S ; the spherical harmonics S_{nm} are supposed to be fully normalized as defined in (Heiskanen & Moritz, 1967, p.31) .)

Applying the operator $\bar{\Delta}_x + \lambda_n$ to (4-4) at x , and observing that $p_n(x)$ is annihilated by this operator according to (4-3),

$$(\bar{\Delta}_x + \lambda_n)\, p_n(x) = 0 \ , \qquad (4 - 6)$$

we obtain

$$\int_S (\bar{\Delta}_x + \lambda_n)\, G_1(n;x,y)\, (\bar{\Delta}_y + \lambda_n) f(y)\, dS(y) = (\bar{\Delta}_x + \lambda_n) f(x) . \qquad (4 - 7)$$

$(\bar{\Delta} + \lambda_n)f$ is obviously reproduced by that integral transformation; therefore,

$$(\bar{\Delta}_x + \lambda_n) G_1(n;x,y) = \delta(x,y) \ . \qquad (4 - 8)$$

This differential equation represents a linear system with the Green's function as input and the Dirac function as output. We have to find the "inverse" of that system, which is a simple task following the frequency domain approach:

$\delta(x,y)$ is a homogeneous and isotropic kernel with reproducing property. According to the Funk-Hecke formula (Müller, 1966), a harmonic and isotropic integral kernel $K(x,y)$ on S has the system of spherical harmonics $\{S_{km}\}$ as eigenfunctions corresponding to the eigenvalues κ_k ,

$$\int_S K(x,y)\, S_{km}(y)\, dS(y) = \kappa_k\, S_{km}(x) \qquad (4 - 9a)$$

with κ_k being the projection of K onto the Legendre polynomial P_k ,

$$\kappa_k = 2\pi \int_{-1}^{1} K(t)\; P_k(t)\, dt \;,\; t := xy := x^T y = \cos(x,y), \qquad (4 - 9b)$$

and a series representation in terms of Legendre polynomials

$$K(x,y) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1)\, \kappa_k\, P_k(xy) \;. \qquad (4 - 9c)$$

If $K = \delta$, then all eigenvalues must be equal to 1 because of the reproducing property of the Dirac function. Since δ is homogeneous and isotropic, it can only depend on xy and should therefore have the following series representation :

$$\delta(x,y) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1)\, P_k(xy) \;. \qquad (4 - 10)$$

Since the right hand side of equ. (4-8) is homogeneous and isotropic, the left hand side must be homogeneous and isotropic as well, which requires Green's function to be homogeneous and isotropic. Therefore, $G_1(n;x,y)$ can be represented in terms of

$$G_1(n;x,y) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1)\, g_k\, P_k(xy) \qquad (4 - 11)$$

with eigenvalues g_k.

Due to the decomposition theorem for Legendre polynomials

$$P_k(xy) = \frac{1}{2k+1} \sum_{m=-k}^{k} S_{km}(x)\, S_{km}(y) \, , \qquad (4-12)$$

the differential equation (4-3) is fulfilled by the Legendre polynomials $P_k(xy)$ as well and we obtain

$$(\bar{\Delta}_x + \lambda_k)P_k(xy) = (\bar{\Delta}_y + \lambda_k)P_k(xy) = 0 \, . \qquad (4-13)$$

Using the series representations (4-10) and (4-11) in (4-8), we obtain

$$\sum_{k=0}^{\infty} (2k+1)\, g_k\, (\bar{\Delta}_x + \lambda_n)P_k(xy) = \sum_{k=0}^{\infty} (2k+1)P_k(xy) \, . \qquad (4-14)$$

Adding zero to (4-13) yields

$$(\bar{\Delta}_x + \lambda_n)P_k(xy) = (\bar{\Delta}_x + \lambda_k)P_k(xy) + (\lambda_n - \lambda_k)P_k(xy),$$

and due to (4-6)

$$(\bar{\Delta}_x + \lambda_n)P_k(xy) = (\lambda_n - \lambda_k)P_k(xy) \, . \qquad (4-13)'$$

Since $\{P_k\}$ is a basis, we can compare both sides of (4-14) by degree and obtain the eigenvalues g_k of Green's function,

$$g_k = \frac{1}{\lambda_n - \lambda_k} \, , \quad k \neq n \qquad (4-15)$$

and with (4-11) we have the series representation of Green's function corresponding to the differential operator $\bar{\Delta} + \lambda_n$,

$$G_1(n;x,y) = \frac{1}{4\pi} \sum_{k=0}^{\infty} \frac{2k+1}{\lambda_n - \lambda_k} P_k(xy) \ . \qquad (4 - 16)$$

Remember $\lambda_n = n(n+1)$, compare it to Green's function on the circle (equ. (3-16)), and notice the identical structure.

While the corresponding G_1 on the circle is continuous for $x = y$, this is not true on the sphere : consider (4-16) with $n = 0$; then it can be shown (Hansen, 1975, p. 301, No. 46.2.21) that

$$- \sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)} P_k(xy) = 1 + \log(\frac{1-xy}{2}) \qquad (4 - 17)$$

and as a consequence, $G_1(n;x,y)$ has a characteristic singularity at $x = y$; for all other points $x \neq y$, $G_1(n;x,y)$ is infinitely often continuously differentiable. How can we get rid of the singularity?

Very simple, by convolution !

From the discussion of the spline on the real line and on the circle we know already that convolution has smoothing effects.

STEP 3: <u>Derive composite Green's functions</u>

Following the procedure on the circle, we will now try to perform a convolution between $G_1(n;x,y)$ and $G_1(n;x,y)$ and call the result $G_2(n;x,y)$,

$$G_2(n;x,y) = G_1(n;x,y) * G_1(n;x,y) \ . \qquad (4 - 18)$$

With the series representation (4-16) and the decomposition formula (4-12), the composite Green's function is given by

$$G_2(n;x,y) = \int_S \left[\frac{1}{4\pi} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{1}{\lambda_n - \lambda_k} \sum_{r=-k}^{k} S_{kr}(x)\, S_{kr}(z) \right] \cdot$$

$$\left[\frac{1}{4\pi} \sum_{\substack{l=0 \\ l\neq n}}^{\infty} \frac{1}{\lambda_n - \lambda_l} \sum_{s=-l}^{l} S_{ls}(y) S_{ls}(z) \right] dS(z) \qquad (4 - 18)'$$

and due to the orthogonality relations between spherical harmonics,

$$\frac{1}{4\pi} \int_S S_{kr}(z)\, S_{ls}(z)\, dS(z) = \delta_{kl}\;\delta_{rs}\;, \qquad (4 - 19)$$

(δ_{ij} ... Kronecker symbol), the composite Green's function $G_2(n;x,y)$ follows,

$$G_2(n;x,y) = \frac{1}{4\pi} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{2k+1}{(\lambda_n - \lambda_k)^2} P_k(xy)\;. \qquad (4 - 18)''$$

It can be shown that $G_2(n;x,y)$ is continuous on S and has a logarithmic singularity under the operator $\bar{\Delta}_x + \lambda_n$ - try to verify it.

We can now proceed as we did with the construction of composite Green's functions on the circle and perform an arbitrary number p of convolutions by recursion

$$G_p(n;x,y) = G_{p-1}(n;x,y) * G_1(n;x,y)$$

$$= G_1(n;x,y) * \underbrace{G_1(n;x,y) * \ldots * G_1(n;x,y)}_{p\text{ - times}} , \qquad (4 - 20)$$

yielding the series representation

$$G_p(n;x,y) = \frac{1}{4\pi} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{2k+1}{(\lambda_n - \lambda_k)^p} P_k(xy) ; \qquad (4 - 21)$$

for $x \neq y$, $G_p(n;x,y)$ is infinitely often continuously differentiable, for $x = y$ $G_p(n;x,y)$ is continuous under the operator $(\bar{\Delta}_x + \lambda_n)^q$ for all $0 \leq q < p-1$ and has a logarithmic singularity for $q = p-1$ (try to verify it - it is very simple).

It should be pointed out, that in complete agreement with the two cases considered in chapters 2 and 3 (real line, circle), $G_p(n;x,y)$, with p even, can be considered as the Green's function corresponding to an Euler differential equation, which is due to a variational problem based on an energy minimization. Of particular importance is the case $p = 2$ with $n = 0$: in this case the differential operator is simply $\bar{\Delta}^2 = \bar{\Delta}\,\bar{\Delta}$, which is the Euler differential operator for the minimization of the energy integral

$$\int_S (\bar{\Delta} f)^2 \, dS = \min. ,$$

closely related to the minimization of the elastic energy.

As a matter of fact the convolutions of (4-20) are not restricted to Green's functions with one and the same eigenvalue and degree n, we can of course perform convolutions between $G(n;x,y)$ and $G(r;x,y)$ with arbitrary non-negative integers n and r, and, following the ideas of Schoenberg (1964) and Freeden (1981), define quite general differential operators (as we did already in chapter 3) of the form

$$\bar{\Delta}_N^{\mu} := \prod_{n=0}^{N} (\bar{\Delta} + \lambda_n)^{\mu_n}, \quad \mu := \sum_{n=0}^{N} \mu_n, \qquad (4 - 22)$$

observe the convolution theorem, and obtain a series representation for $G_{\bar{\Delta}N}^{(\mu)}$,

$$G_{\bar{\Delta}N}^{(\mu)}(x,y) = \frac{1}{4\pi} \sum_{k=N+1}^{\infty} (2k+1) \prod_{n=0}^{N} (\lambda_n - \lambda_k)^{-\mu_n} P_k(xy) . \qquad (4 - 23)$$

$G_{\bar{\Delta}N}^{(\mu)}(x,y)$ is continuous under the operator $\bar{\Delta}_N^q$ for $0 \le q < \mu-1$ at $x = y$ and has a logarithmic singularity for $q = \mu-1$ at $x = y$. $G_{\bar{\Delta}N}^{(\mu)}(x,y)$ comprises actually all possible Green's functions on the sphere discussed here.

This concept looks good; however, it should also be noticed that it can become very difficult (if not practically impossible) to find a closed expression for $G_{\bar{\Delta}N}^{(\mu)}(x,y)$ if N and/or μ is large; and moreover, it is not a good practice to use large N and/or μ because with increasing N and/or μ we loose what we actually wanted to achieve: a local behaviour of the interpolation function. Therefore, the user is strongly recommended to consider only moderate values of both

N and $\mu_o, \ldots, \mu_N$.

STEP 4: Calculate spline function

As in chapter 3 we assume J distinct knots to be given on the unit sphere S , $x_1, x_2, \ldots, x_J$. Then all Green's functions $\{G_{\bar{\Delta}N}^{(\mu)}(x,x_j)\}$ for $j = 1, \ldots, J$ are linearly independent and can therefore be used as a basis for interpolation.

The interpolating spline $s(x)$ on the sphere S , corrresponding to the differential operator $\bar{\Delta}_N^{\mu}$, is assembled by a linear combination of these Green's functions,

$$s(x) = \sum_{j=1}^{J} c_j \; G_{\bar{\Delta}N}^{(\mu)}(x,x_j) + p_N(x) \qquad (4 - 24)$$

where $p_N(x)$ is a polynomial of degree N - a series of spherical harmonics,

$$p_N(x) = \sum_{n=0}^{N} \sum_{m=-n}^{n} s_{nm} \, S_{nm}(x) \; . \qquad (4 - 25)$$

The unknown coefficients $\{c_j\}$ of the linear combination of the Green's functions (evaluated at x_j) and the $(N+1)^2$ coefficients $\{s_{nm}\}$, $n = 0, \ldots, N$, $m = -n, \ldots, n$, are as usual determined by

a) the data reproducing (interpolation) requirements, providing J linear equations, and

b) the $(N+1)^2$ orthogonality conditions of the vector $c := [c_1, \ldots, c_J]$ with respect to all $(N+1)^2$ harmonic functions of $p_N(x)$ evaluated at the data points.

Using the notation of chapter 3, we obtain once more the system of equations (3-9), where the elements of the matrix C are to be replaced by $G_{\bar{\Delta}N}^{(\mu)}(x_i, x_j)$; p is here a $(N+1)^2$ - dimensional vector and A a matrix of dimension $[J,(N+1)^2]$,

$$A = \{S_{nm}(x_j)\} , \quad \begin{aligned} j &= 1, \ldots, J, \\ n &= 0, \ldots, N, \\ m &= -n, \ldots, n . \end{aligned} \qquad (4-26)$$

The solution for the vector c and the parameter vector p is given by (3-10). The resemblence to the collocation solution has already been discussed at the end of chapter 3.

So far we have seen no conceptual difference between the spline and the collocation solution. It should also be mentioned that Green's functions can be extended to outer space just by integration of the upward continuation operator $r^{-(n+1)}$, $r > 1$; and in this way we could even use other functionals and data with $|x_i| > 1$. The only differences I can see between the two concepts are:

a) the singularity of the kernel (Green's function) is circumvented in the collocation concept by postulating $|x_j| > 1$ for all $j = 1, \ldots, J$, which is implicitly achieved by choosing the radius of the Bjerhammar sphere slightly smaller than 1, and

b) the kernel (the covariance function) is usually derived from real world data. This last property makes, as a matter of fact (as C.C. Tscherning would argue), collocation by far superior to any spline solution. But how far depends essentially on how well the spectrum of the Green's function, for a user-selected differential operator, matches the spectrum of the data-derived covariance function (the two solutions could even coincide). From a conceptual point of view, however, the two solutions are equivalent!

5. SPLINES IN THE PLANE

Splines in the plane can be constructed as simple products of splines along the real line, provided the data are located on a grid (with not necessarily constant grid spacing). In this case we could construct bilinear, bicubic, ... spline functions. For practical applications bilinear and bicubic turned out to be particularly useful. Detailed investigations and applications can be found in (Meissl, 1971; Sünkel, 1980; Sünkel, 1981). But remember, the data have to be regularly distributed. For an irregular data distribution the spline product concept is not adequate.

We assume an arbitrary data distribution in E_2 , the two-dimensional Euclidean plane, and translate the spherical procedure to the plane. For this purpose we shall establish a kind of dictionary. Our approach will be mathematically simple - minded and partly heuristic - but it works.

In chapter 4 we have used the Laplace-Beltrami differential operator $\bar{\Delta}$(equ. (4-1)). The Laplace operator in the plane is expressed in cartesian coordinates x_1, x_2

$$\Delta = D^2_{x_1} + D^2_{x_2} , \qquad (5 - 1)$$

or in polar coordinates s, α

$$\Delta = D^2_s + \frac{1}{s} D_s + \frac{1}{s^2} D^2_\alpha . \qquad (5 - 1)'$$

The discrete frequencies n (n ... integer) on the sphere are replaced by a continuous frequency η ; therefore, the index n on the sphere will be replaced by an argument η in the plane, and the operator $\bar{\Delta} + \lambda_n = \bar{\Delta} + n(n+1)$ by the operator $\Delta + \eta^2$.

From the considerations in the foregoing chapter we know that Green's function is the solution of the linear system

$$(\Delta + \eta^2)G_1 = \delta \; , \qquad (5 - 2)$$

with the Dirac function δ as output of that system. We have also seen that both δ and G_1 are homogeneous and isotropic; therefore, $D_\alpha^2 \, G_1 = 0$, and (5-2) reduces to

$$(D_s^2 + \frac{1}{s} D_s + \eta^2)G_1(\eta;s) = \delta(s) \; . \qquad (5 - 2)'$$

The differential operator on the left hand side is the well-known Bessel differential operator, and the associated ordinary homogeneous differential equation

$$(D_s^2 + \frac{1}{s} D_s + \eta^2)J_0(\eta s) = 0 \qquad (5 - 3)$$

has the Bessel function of order zero $J_0(\eta s)$ as solution.

Note that the Bessel differential equation is just the Laplace differential equation for isotropic functions in E_2 and corresponds to (4-3) .

We have noticed before that Green's function can be easily derived if we use spectral domain techniques. Let us apply this tool here too.

Since the system (5-2)' is isotropic, we take advantage of the Fourier transform for isotropic functions, which is known to be the Hankel transform. Let $F(s)$ be the isotropic function and $f(\kappa)$ its Hankel transform, then we have (Papoulis, 1968, p. 140 ff.)

$$f(\kappa) = \int_0^\infty J_o(\kappa s)\, F(s)\, s ds \tag{5 - 4a}$$

$$F(s) = \int_0^\infty J_o(\kappa s)\, f(\kappa)\, \kappa d\, \kappa . \tag{5 - 4b}$$

The first transformation is a Hankel transformation into the spectral domain and could be called Hankel (or Fourier) analysis, the second transform is the Hankel transformation from the spectral domain back into the space domain and could be called Hankel (or Fourier) synthesis. Both equations are denoted "Hankel transform pairs". Note the accordance with (4-9b,c). Comparing (5-4b) with e.g. equ. (4-11), we can establish the following dictionary for homogeneous and isotropic functions:

sphere	plane
k	κ
xy	$s := \|x-y\|$
$2k+1$	κ
$\sum_0^\infty$	$\int_0^\infty$

If we therefore denote Green's function for the plane with $G(\eta,s)$ and its spectrum (Hankel transform) by $g(\kappa)$, we obtain

$$G(\eta,s) = \int_0^\infty J_0(\kappa s)\, g(\kappa)\, \kappa\, d\kappa\ . \qquad (5-5)$$

The Dirac impulse is known to have a constant spectrum equal to 1; consequently, it should have formally a representation

$$\delta(s) = \int_0^\infty J_0(\kappa s)\, \kappa\, d\kappa\ . \qquad (5-6)$$

If we use these integral representations of δ and G in equation (5-2), we get

$$\int_0^\infty (\Delta + \eta^2) J_0(\kappa s)\, g(\kappa)\, \kappa\, d\kappa = \int_0^\infty J_0(\kappa s)\, \kappa\, d\kappa\ ; \qquad (5-7)$$

its counterpart on the sphere is equ. (4-14). Now we apply the same simple trick as in (4-13)' , add zero to the differential operator $\Delta + \eta^2$ and obtain, observing the Bessel differential equation (5-3) ,

$$(\Delta + \kappa^2 + \eta^2 - \kappa^2) J_0(\kappa s) = (\eta^2 - \kappa^2) J_0(\kappa s) \qquad (5-8)$$

(compare equ. (4-13)"). The equation

$$\int_0^\infty J_o(\kappa s)\ (\eta^2 - \kappa^2)\ g(\kappa)\ \kappa\ d\kappa = \int_0^\infty J_o(\kappa s)\ \kappa\ d\kappa$$

has to be fulfilled for all arguments s ; consequently, the spectrum of Green's function is given by

$$g(\kappa) = \frac{1}{\eta^2 - \kappa^2} \quad ; \qquad (5 - 9)$$

(compare it to (4-15) considering (4-2).)
With $g(\kappa)$ given, Green's function $G(\eta;s)$ is obtained by a Hankel transform (Fourier synthesis) of $g(\kappa)$, like (4-16) on the sphere,

$$G_1(\eta;s) = \int_0^\infty \frac{\kappa}{\eta^2 - \kappa^2} J_o(\kappa s)\ d\kappa\ . \qquad (5 - 10)$$

Also $G_1(\eta;s)$ in the plane has a logarithmic singularity at $s = 0$ in accordance with $G_1(\eta;x,y)$ on the sphere: with $J_o(0) = 1$ we obtain

$$G_1(\eta;0) = \int_0^{\infty\,o} \frac{\kappa}{\eta^2 - \kappa^2}\ d\kappa = -\frac{1}{2} \lim_{\kappa \to \infty} \log(1 - \frac{\kappa^2}{\eta^2}),\ \eta > 0$$

$^o \ldots\ |\kappa - \eta| \geq \varepsilon > 0$

The reader is invited to check the validity of this formula.
If $s \neq 0$, $G_1(\eta;s)$ is infinitely often continuously differentiable.

In order to eliminate the singularity of Green's function, we proceed as in the foregoing chapters and use convolutions between the Green's functions $G_1(\eta_1;s)$ and $G_1(\eta_1;s)$ with a fixed $\eta = \eta_1$, obtaining $G_2(\eta_1;s)$,

$$G_2(\eta_1;s) := G_1(\eta_1;s) * G_1(\eta_1;s) ,$$

which is according to the convolution theorem (Papoulis, 1968, p. 143) equal to

$$G_2(\eta_1;s) = 2\pi \int_0^\infty \frac{\kappa}{(\eta_1^2 - \kappa^2)^2} J_o(\kappa s)\, d\kappa . \qquad (5 - 11)$$

$G_2(\eta_1;s)$ is already continuous at $s = 0$ and has a logarithmic singularity at $s = 0$ under the differential operator $\Delta + \eta_1^2$. Recall that G_2 is Green's function corresponding to the Euler differential equation, which is derived from the minimization requirement of the energy integral

$$\int_{E_2} [(D^2_{x_1} + D^2_{x_2}) f(x_1,x_2)]^2\, dx_1 dx_2 = \min.$$

In accordance with the operations on Green's functions on the real line, on the circle, and on the sphere, we can define a quite general differential operator of the form

$$\Delta^\mu_N := \prod_{n=0}^{N} (\Delta + \eta_n^2)^{\mu_n} , \quad \mu := \sum_{n=0}^{N} \mu_n . \qquad (5 - 12)$$

Observing again the convolution theorem, the corresponding Green's function is given by

$$G_{\Delta N}^{(\mu)}(s) = (2\pi)^{\mu-1} \int_0^\infty \kappa \prod_{n=0}^{N} (\eta_n^2 - \kappa^2)^{-\mu_n} J_o(\kappa s)\, d\kappa \ . \tag{5 - 13}$$

The reader may show that $G_{\Delta N}^{(\mu)}(s)$ is continuous under the operator Δ_N^q for $0 \leq q < \mu-1$ at $s = 0$ and has a logarithmic singularity for $q = \mu-1$ at $s = 0$.

Calculation of the spline function

As in the foregoing chapters we will assume J distinct knots to be given in the Euclidean two-dimensional plane E_2, $x_1, x_2, \ldots, x_J$. Then all Green's functions $\{G_{\Delta N}^{(\mu)}(x_i,x_j)$ for $j = 1, \ldots, J$ are linearly independent and can therefore be used as a basis for interpolation (or even data combination, provided G is sufficiently smooth).

The interpolating spline $s(x)$ in E_2 (x denotes a point in E_2), corresponding to the differential operator Δ_N^μ, is assembled by a linear combination of these Green's functions,

$$s(x) = \sum_{j=1}^{J} c_j \, G_{\Delta N}^{(\mu)}(x,x_j) \ ; \tag{5 - 14}$$

the unknown coefficients $\{c_j\}$ are determined from the data reproducing (interpolation) conditions. As in the case of the real line, there is no polynomial explicitly involved, because there is simply no discrete frequency; however, the polynomial behaviour is contained in the Green's function (same as on the real line). Therefore, there is no matrix A and no parameter vector p ; the matrix C of equation (3-9) consists of the elements $\{G_{\Delta N}^{(\mu)}(x_i,x_j)\}$. The solution is provided by

$$c = C^{-1} f .$$

CONCLUSIONS

Spline functions can be considered as solutions of differential equations under certain boundary conditions. In particular, if the differential operator is self-adjoint, it can be considered as an Euler differential operator which is due to a variational problem, formulated in terms of the minimization of a certain energy integral.

The solution of the differential equation is implied by a Green's function. The set of Green's functions, referred to the data points, is a set of linearly independent functions and can therefore be used as a basis. Actually, the Green's functions are basis splines. The data reproducing spline can be assembled by a linear combination of these basis splines, in general superimposed by a low degree polynomial. The determination of the coefficients of the linear combination requires the solution of a linear system with a size equal to the number of data.

Composite Green's functions can be designed in terms of convolutions of simple Green's functions, corresponding to composite differential operators. This leads to simple recursion relations between Green's functions.

According to the convolution theorem, the spectrum of a composite Green's function is the product of the spectra of the individual Green's functions. This nice property suggests the extensive use of spectral techniques. In general, the degree of composition must be limited for the following reasons:

a) the number of data must exceed the degree of the differential operator in order to permit a unique solution;

b) a high degree of composition results in a Green's function (B-spline) with global rather than local character - a certainly unwanted property for local applications;

c) a closed expression for the composite Green's function must be available; to find it can become very tricky for high degrees, if not practically impossible.

B-splines and covariance (kernel) functions are close relatives: both are characterized by singularities. In collocation the singularities are surpassed by requiring all possible data to be located above the (dangerous) singularity level. This is implicitly achieved by dropping this level to a "save level ". This level defines at the same time the domain of harmonicity (if we are talking about applications in physical geodesy). If we do the same with splines, which is certainly legal, then both methods, spline and least norm collocation, are absolutely identical.

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