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A WORLD VERTICAL NETWORK

by

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Abstract

Gravimetric data, levelling, and precise position fixes using artificial satellites or the Moon could be combined to estimate the potential difference between benchmarks situated far apart to an accuracy of a few tenths of kgal m. This is substantially better than what can be obtained with tide gauges, which are affected by the stationary sea surface topography. A set of these benchmarks can link national and continental levelling nets into a unified World Vertical Network.

Foreword

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1. Introduction

Since spirit levelling cannot be used across the oceans, connecting continental vertical networks has long been a challenge for both oceanographers and geodesists. Among the former, Cartwright (1963) calculated a tie between the British and the European nets across the English channel. Because the oceanographic method requires a knowledge of currents that is not available for larger bodies of water, the geodesist Erik Tengstrom (1965) tried using gravimetry and deflections of the vertical to compute a geoidal profile through the Eastern Mediterranean, from Athens to Alexandria. His results suffered from lack of data. Lelgemann (1976) proposed unifying vertical datums by means of gravimetry, levelling, and very accurate position determinations, as those expected by the proponents of lunar laser ranging (Silverber et al., 1976). The late R. S. Mather (1978) considered the possibilities open for datum unification by constant improvements in gravity field models (notably Goddard's GEM series) and the large amounts of information provided by the altimeter satellites whose prototype has been GEOS-3.

Dynamic effects create a "sea surface topography", or departure of the mean sea surface from a true equipotential. This topography is not very well known at present, and has an estimated r.m.s. value of about 1 m. Tide gauges determine the position of mean sea level at their locations, so the error made by assuming that their mean sea level marks are on the same equipotential surface (geoid) is of the order of $\sqrt{2}$ x (r.m.s. of the stationary sea surface topography) $\simeq$ 1.5 kgal m. Without any additional work, Nature provides a world "levelling net" of 1.5 kgal m accuracy.

Would it be possible to obtain better transoceanic links using the various forms of geodetic data that are available at present, or are likely to become available in the near future? Could it be feasible, with such data, to establish benchmarks for levelling inside continents, rather like inland "tide gauges", whose potential differences are known so well that they can be used to constrain the adjustment of the net to reduce distortion? If so, in a more distant future, similar benchmarks could be used to survey other components of the Solar System, where only the Earth has any significant amount of free surface water.

In this work the reader will not find more than a passing reference to the geoid, a notion that appears inseparable from that of vertical height and of vertical datum. Since the geoid has been regarded as the natural universal datum by geodesists, a few words of explanation are due. The reason for its omission here is that, however useful otherwise, the geoid is not essential to the setting up of a vertical network, at least in theory. Such network, ultimately, is a set of potential differences estimated among the points that form the net, in particular the primary points or benchmarks. These potential differences do not convey information on the absolute potential of the gravity field, so they can be referred to any number of level surfaces, and not exclusively to one. For the same reason their meaning is not dependent on which surface is selected, and it remains intact even if no surface is selected. The reason why the geoid is so ubiquitous in the literature on levelling may be the complete reliance that levelling has had on tide gauges, idealized as points on the geoid. As long as gauges play a basic role, the concept of reference surface is relevant to levelling.

Here, instead of heights above an equipotential surface, we are going to consider distances to the center of mass of the Earth, or to a reference ellipsoidal centered on this point. Such approach is not unreasonable today, when new positioning techniques are being developed that promise accuracies far better than those available in the past. Methods based on the Global Positioning System satellites, on Lageos, and on portable interferometric and lunar laser ranging stations, are expected to achieve near decimeter accuracy in relative position, over continental distances.

In recent years, the use of artificial satellites has changed many aspects of geodesy. Space techniques for obtaining position fixes and models of the gravity field are in constant development, and both the quality and the quantity of the data provided by spacecraft are increasing. This work shall explore how these advances may affect levelling. Next paragraph, to begin with, introduces a basic idea: a World Vertical Network established without recourse to any reference surface, or geoid. In a way, such network is the datum.

1.1. Definition of World Vertical Network

The World Vertical Network (WVN) is a set of estimated potential differences among benchmarks situated in various continents.

A network of potential differences can be translated immediately into a variety of height systems such as those described by Krakiwsky and Mueller (1965). Potential differences are intrinsic to the set of benchmarks selected, and are independent on the choice of "geoid", or on the precise knowledge of the zero harmonic of the Earth's potential, present estimates of which have an uncertainty of some 3 kgal m. Existing regional networks can be tied to the closest benchmarks to create a dense, unified global levelling net. Any benchmark potential can be used to reference all points tied to the net. If so desired, the level surface through this arbitrary point may be regarded as a "geoid".

2. Method of Approach

If we had a perfect model of the gravity field and exact position fixes in geocentric coordinates at two points on the Earth's surface, then we could use this information to find the potential of each point and, from this, their potential difference. Repeating this process for all possible pairs of points out of a given set of benchmarks, the end result would be an exact WVN. Unfortunately, models and fixes are never perfect, so the potential differences must have some errors. To reduce these errors, we could combine the field model, which being finite cannot contain information above certain spatial frequencies, with additional data such as gravimetry, rich in high frequencies, particularly from the vicinity of the benchmarks.

2.1. Formulation of the Problem

If V is the gravitational potential due to the mass of the Earth and external to its surface, and U is a reference potential defined by a spherical harmonic's model:

$$U(\varphi,\lambda,r) = \frac{GM}{r}\left[1 + \sum_{n=2}^{N}(a/r)^n \bar{P}_{nm}(\sin\varphi)\{\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\}\right] \quad (2.1)$$

where: $\bar{P}_{nm}$ fully normalized Legendre function of the first kind, degree n and order m;

r,φ,λ geocentric distance, latitude, and longitude;

G universal gravitational constant;

M mass of the Earth;

a mean equatorial radius of the Earth;

$\bar{C}_{nm},\bar{S}_{nm}$ normalized spherical harmonic coefficients;

N maximum degree and order for terms present in the model;

then the disturbing potential T at a point P of geocentric coordinates r_P,φ_P,λ_P is

$$T(P) = V(P) - U(P) \quad (2.2)$$

The gravity potential of the Earth is

$$W(P) = V(P) + \varphi(P) \quad (2.3)$$

where $\varphi(P) = \frac{1}{2}\omega^2 r_P^2 \cos^2\varphi_P$ corresponds to the rotational potential, ω being the angular velocity of the planet about its spin axis. The potential difference between two points such as P and Q is, therefore,

$$\Delta W(P,Q) = U(P) + T(P) + \varphi(P) - U(Q) - T(Q) - \varphi(Q) \quad (2.4)$$

With both P and Q on the Earth's surface, the uncertainties in the calculated values of $\varphi(P)$ or $\varphi(Q)$, due to errors in the known positions of P and Q, will be thousands of times smaller than those arising in the determination of $U(P)$, $U(Q)$,

$T(P)$ and $T(Q)$. Consequently, only errors in the computed values in the right hand side of

$$\Delta W(P,Q) - [\phi(P) - \phi(Q)] = V(P) - V(Q)$$
$$= U(P) + T(P) - [U(Q) + T(Q)] \qquad (2.5)$$

shall be included in the error analysis that constitutes the major part of what follows.

2.2. Estimating T by Least Squares Collocation

The use of linear regression for predicting and filtering geodetic data appears to have been first proposed by Kaula (1959). Further developed by Moritz and others, this approach has become a familiar technique that has shown its value in many applications. Least squares collocation, as geodesists call it, provides a way of combining all relevant data into estimates of unobserved variables (minimum variance prediction), or into more reliable estimates of those actually observed (minimum variance filtering). For further information on this method, see Moritz (1972).

If T is to be estimated at point P, then the linear, unbiased, minimum variance estimator of $T(P)$ is

$$\hat{T}(P) = \underline{f}^T \underline{d} = f^T(\underline{z} + \underline{n}) \qquad (2.6)$$

where

$$\underline{f} = (C_{zz} + D)^{-1} C_{Tz}^T \qquad (2.7)$$

is the optimal estimator vector, and

$\hat{T}$	is the estimated disturbing potential, a scalar;
$\underline{d} = \underline{z} + \underline{n}$	is the N_d vector of measurements, or data vector;
$\underline{z}$	is the N_d vector of signal component in the measurements;
$\underline{n}$	is the N_d vector of the noise component in the measurements;
$C_{Tz} = M\{T\underline{z}^T\}$	is the $1 \times N_d$ covariance matrix (a row vector) of T and $\underline{z}$;
$C_{zz} = M\{\underline{z}\,\underline{z}^T\}$	is the $N_d \times N_d$ covariance matrix of the signal z;
$D = M\{\underline{n}\,\underline{n}^T\}$	is the $N_d \times N_d$ covariance matrix of the noise n.

The operator $M\{\ \}$ represents some kind of average. D is supposed to be diagonal, because the noise is not correlated from measurement to measurement (white noise). Furthermore, these assumptions apply:

and

$$M\{\underline{z}\} = M\{\underline{n}\} = M\{\underline{d}\} = \underline{0} \quad \text{(a null vector)}$$

$$M\{\underline{z}\,\underline{n}^T\}\ ,\quad M\{T\,\underline{n}^T\} \qquad \text{are both null matrices.}$$

More generally, we could be asked to obtain N_s estimates $\hat{\underline{s}}$ (N_s vector of estimates) from $\underline{d}$, using an estimator matrix F such that

$$\hat{\underline{s}} = F^T \underline{d} \tag{2.6)*}$$

minimizes the mean square values of the components of the error vector

$$\underline{e} = \underline{s} - \hat{\underline{s}}$$

(s is the N_s vector of true values of s). The variance-covariance matrix of these errors is

$$E = M\{\underline{e}\,\underline{e}^T\} = M\{(\underline{s} - F^T\underline{d})(\underline{s} - F^T\underline{d})^T\} = C_{ss} - F^T C_{sz}^T - C_{sz} F + F^T(C_{zz} + D) F \tag{2.8}$$

where $C_{ss} = M\{\underline{s}, \underline{s}^T\}$ is a $N_s \times N_s$ matrix, and $C_{sz} = M\{\underline{s}\ \underline{z}^T\}$ is a $N_s \times N_d$ matrix. Since the elements in the main diagonal of E are either positive or zero, minimizing each one of the mean square errors is the same as minimizing their sum, the trace of E ($tr(E)$). Accordingly (see for instance, (Rao, 1973)),

$$\frac{1}{2}\frac{\partial\, tr(E)}{\partial F} = -C_{sz}^T + (C_{zz} + D) F = \emptyset \quad \text{(a null matrix)}$$

or

$$(C_{zz} + D)\ F = C_{sz}^T \tag{2.9}$$

and, finally,

$$F = (C_{zz} + D)^{-1} C_{sz}^T \tag{2.10}$$

Replacing (2.10) in (2.8) we get

$$E = C_{ss} - C_{sz}(C_{zz} + D)^{-1} C_{sz}^T \tag{2.11}$$

In the special case where s is the scalar T, we get (2.7). The equations in the system (2.9) are known as "normal equations"; some people prefer to call them "Wiener-Hopf equations" because they bear a formal resemblance to the basic integral equation of linear, invariant, minimum variance filtering in the time domain.

In geodetic applications, $M\{\ \}$ is an average on rotations of some sort. If all possible rotations about the origin (center of mass) are included, then the covariance function c_{hh} of a function h of r, φ and λ

$$M\{h(P)\ h(Q)\} \equiv c_{hh}(P, Q) \tag{2.12}$$

depends only on the spherical distance

$$\psi_{PQ} = \cos^{-1}[\sin\varphi_P \sin\varphi_Q + \cos\varphi_P \cos\varphi_Q \cos(\lambda_P - \lambda_Q)]$$

and on the geocentric distances r_P and r_Q. If both P and Q are on the same sphere ($r_P = r_Q$), then c_{hh} depends on ψ_{PQ} alone. For this reason this type of covariance is known as isotropic, and the operator $M\{\ \}$ is then called the isotropic average operator[1]. The elements of F depend on those of matrices C_{sz} and C_{zz}, and these elements are, in turn, values of the covariance functions

$$c_{sz}(P,Q) = M\{s(P)\ z(Q)\} \text{ and } c_{zz}(P,Q) = M\{z(P)\ z(Q)\}.$$

A choice of $M\{\ \}$ determines those functions, their values, and, ultimately, the optimal estimator matrix F. Rummel and Schwarz (1977) have discussed different types of averages and covariance functions. From these considerations it is clear that the optimal estimator is not unique, but it depends on what average we choose. The "easiest" choice is the isotropic average, because of the simplicity of the corresponding covariance function. This function can be expanded as a series of Legendre polynomials

$$c_{uu}(P, Q) = \sum_{n=0}^{\infty} (2n+1) \left(\frac{a^2}{r_P\ r_Q}\right)^n \delta_{uu,n}\ P_n(\cos\psi_{PQ}) \tag{2.13}$$

where $\delta_{uu,n}$ is the nth degree variance of the spherical harmonic coefficients of $u(\varphi, \lambda, r)$:

$$\delta_{uu,n} = \sum_{m=0}^{n} (\overline{C}^2_{u,nm} + \overline{S}^2_{u,nm})(2n+1)^{-1} \tag{2.14}$$

The covariance between two functions u and v is

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[1] There is a small problem with $M\{\underline{n}\ \underline{n}^T\}$, because the measurements' "noise" is not an ordinary function of φ, λ and r, but a stochastic process. However, it can be manipulated as if it were such a function. For this, see the discussion by Balmino (1978).

$$c_{uv,n}(P,Q) = \sum_{n=0}^{\infty} (2n+1) \left(\frac{a^2}{r_P\, r_Q}\right)^n \delta_{uv,n}\; P_n(\cos \psi_{PQ}) \tag{2.15}$$

where

$$\delta_{uv,n} = \sum_{m=0}^{n} (\overline{C}_{u,nm}\, \overline{C}_{v,nm} + \overline{S}_{u,nm}\, \overline{S}_{v,nm})\, (2n+1)^{-1} \tag{2.16}$$

To understand in what sense the estimator is "optimal", imagine some pattern of measurement points and estimation points. The whole pattern is subject, in succession, to all possible rotations.[1] Before each rotation, measurements are taken and all estimates are made at their respective points, and the squares of the estimation errors are found, somehow. This is repeated over and over again, and running averages of the errors squared are kept. In the limit, these averages will tend to values that satisfy (2.8); if F is optimal, they will also satisfy (2.11) and will be smaller (or not worse) than for any other choice of F. Also, in the limit, we would have covered the whole Earth with estimates, which is why such mean squared values and their square roots (r.m.s. values) are called global. In practice we are always concerned with a finite, even a small number of estimates at isolated locations, and we are interested in the actual errors of those estimates, not "some global measure". The practical meaning of the latter is, therefore, a matter of interpretation. If signal and noise have near Gaussian distributions, then the errors (which, according to (2.6), (2.6)*, are linear transformations of both) will also be near Gaussian. In such cases the global values are related to the actual errors by the usual "one sigma" and "three sigma" rules, giving an indication of their likely sizes. Rapp (1978a) has shown that a world-wide data set of 38406 $1° \times 1°$ mean gravity anomalies, compiled at The Ohio State University, has a nearly Gaussian distribution. Gravity anomalies are the main type of data considered in this report for predicting T.

The probability distribution of the data does not characterize it enough, however, because all the large values could be concentrated in a few "rough" areas, the rest of the world being "smooth", with smaller values. The errors are likely to repeat this pattern (see Appendix A) so, if estimates are made in a "rough" region, the global r.m.s. may give an over-optimistic indication of the actual size of the errors. This quality of the data being "evenly behaved", so that there are no zones that are highly idiosyncratic, is known as stationarity. It is a rather elusive quality, but very important to the use of global, isotropic covariances. How stationary is the Earth? We know that trenches and ridges in the ocean floor produce strong localized features in the gravity field, set off against comparatively smooth surrounding areas. Mountainous regions in land also tend to be "rough"; however, there are very flat regions, such as the Nullarbor plain in S.W. Australia, where the field presents strong local anomalies.

[1] Not only rotations, but more generally all orthogonal transformations can be included (i.e., rotations, reflections and various symmetries), the result being precisely the same average values as with rotations alone.

Regardless of the significance of the results, getting them can present difficulties. We have to form and invert a matrix whose dimension is that of the data vector, so, if many measurements are involved, this two operations become quite large. Not only the computer time involved, but also the accumulation of rounding errors can escalate dramatically. Paradoxically, the more data are used, the better the results (in theory), but also the harder to get and the more unreliable. This is further complicated by the fact that, for close spacings of data, the normal equations can become very ill-conditioned. A special technique, presented in Section 4, has been developed by the author to overcome these problems in the case at hand.

On the positive side we must consider: the possibility of using mixed data sets, so $\underline{d}$ may consist of gravity measurements, satellite altimetry, deflections of the vertical, and even levelling; the ability to provide more than one optimal estimate at the same time; the simplicity and elegance of the theory. Another good aspect of collocation is that the covariance functions needed to set up C_{sz} and $(C_{zz} + D)$ do not have to be known with great accuracy. This is born out by the results presented in Section 5, where the same problem has been solved using somewhat different covariances. This is fortunate, as we can never gather sufficient data to obtain an exact empirical covariance, because to know such function is equivalent to knowing the whole field exactly (thus making estimation unnecessary).

2.3. Data Arrangement

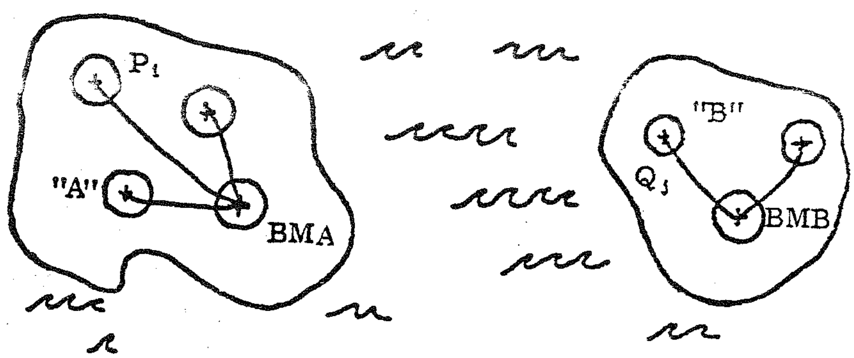


Figure 2.1. The circles represent spherical caps within which gravity anomalies with respect to a reference model have been measured. The wandering lines are levelling traverses. T is estimated at the center of each cap. The differences in the geocentric distances of these center points are known to decimeter accuracy. BMA and BMB are two benchmarks of the WVN.

Figure 2.1 shows the basic data arrangement to be studied. While for the simulations of Section 5 all caps are supposed to have the same size, to reduce computing, this is by no means essential. Other kinds of information (such as satellite altimetry) could be included among the data (see, for instance, paragraph 5.3), though only the types indicated in Figure 2.1 shall be considered here. Anomalies are determined at the Earth's surface, somewhat in the manner of Molodenskii. Details are given in Section 3.

2.4. Adjustment Theory

Consider a cap center P_i in zone "A" (Fig. 2.1) and another Q_j in zone "B". The potential difference between benchmarks BMA and BMB is

$$\Delta W(BMA, BMB) = U(P_i) + \hat{T}(P_i) + \Delta W_\ell(BMA, P_i) + \varphi(P_i) - U(Q_j) - \hat{T}(Q_j) - \Delta W_\ell(BMB, Q_j) - \varphi(Q_j) - V_{ij} \qquad (2.17)$$

where

$$V_{ij} = \epsilon\Delta W_\ell(BMA, P_i) - \epsilon\Delta W_\ell(BMB, Q_j) + \epsilon U(P_i) - \epsilon U(Q_j) + \epsilon\hat{T}(P_i) - \epsilon\hat{T}(Q_j)$$

is the sum of the error in levelling $\epsilon\Delta W_\ell$, the error in the potential according to the reference model ϵU, and the error in the estimated disturbing potential $\epsilon\hat{T}$.[1] ϵU is influenced both by errors in the model and errors in the coordinates of the P_i, Q_j. Expression (2.17) can be regarded as an observation equation with the potential difference ΔW as the only unknown. Each pair of caps (P_i, Q_j) provides an equation of this kind, so a redundant system can be set up

$$\underline{a}\,\Delta W(BMA, BMB) = \underline{p} + \underline{v} \qquad (2.18)$$

where $\underline{p}$ is the vector of "observed" potential differences, $\underline{v}$ is the vector or residuals, and $\underline{a}$ is a vector with all components equal to 1: the design "matrix" of this particular system. All three vectors have for dimension the number of equations. This number is restricted by the following considerations: the centers of the caps, paired in the same way as the caps, should not form a closed loop, as shown in Figure 2.2 by a broken line. Otherwise, because the "observed" values for the pairs in the loop always add up to zero, i.e., are linearly dependent, the a priori variance-covariance matrix of the "observed" values must be singular.

[1] As explained in Section 2.1, errors in $\varphi(P_i)$ and $\varphi(Q_j)$ due to errors in the coordinates of P_i, Q_j, and in the rotation rate ω, are considered to be negligible here.

The inverse of this matrix is needed for the adjustment of ΔW, so this limits the choice of pairs of caps to those not forming loops. The number of such pairs is one less than the number of caps, and this is also the maximum number of equations in system (2.18).

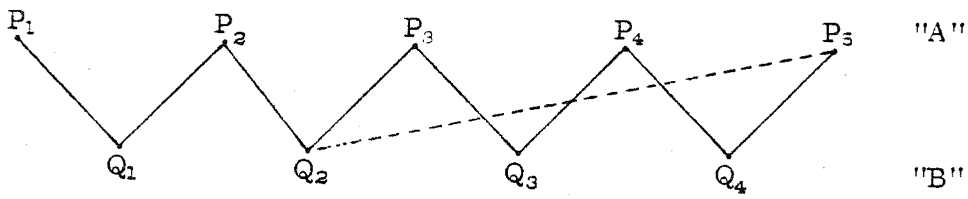


Figure 2.2. The solid and broken lines identify caps that have been "paired". The broken line shows a selection containing a loop (not permitted). The maximum number of equations (permitted) = number of caps-1.

The accuracy of $\Delta W(BMA, BMB)$ after the adjustment can be computed from the following formula

$$\sigma_{\Delta W(BMA,BMB)} = \left(\underline{a}^T V^{-1} \underline{a}\right)^{-\frac{1}{2}} = \left(\sum_{k}^{N_e} \sum_{\ell}^{N_e} (V^{-1})_{k\ell}\right)^{-\frac{1}{2}} \quad (N_e = \text{no. of equations} \le \text{no. of caps} - 1) \tag{2.19}$$

where V, the variance-covariance matrix of the data, is

$$V = V_{\epsilon\Delta W_\ell} + V_{\epsilon\Delta U} + V_{\epsilon\Delta\hat{T}} \tag{2.20}$$

In this expression,

$V_{\epsilon\Delta W_\ell}$ is the variance-covariance matrix of the levelling errors;
$V_{\epsilon\Delta U}$ is the variance-covariance matrix of the errors in $U(P_i) - U(Q_j)$;
$V_{\epsilon\Delta\hat{T}}$ is the variance-covariance matrix of the errors in $\hat{T}(P_i) - \hat{T}(Q_j)$.

The uncertainties in ΔW_ℓ and ΔU depend on those of the data that give them origin; $\epsilon\Delta\hat{T}$ depends also on the way T is estimated from the gravity anomalies.

3. Characteristics of the Data

In this section we shall consider those characteristics of the data that affect the accuracy of potential differences adjusted according to the method explained in the previous paragraph. The various types of data involved are: position fixes, the coefficients of a reference gravity field model, gravity anomalies, and levelling traverses.

3.1. Position Fixes

To compute the reference potential U at the center of every cap, it is necessary to know the coordinates of these points. The reference model contains terms below degree 20 or 30 (in this study) so the smallest detail it can show is of the order of 1000 km. U is, therefore, a smooth function of latitude and longitude, and cannot be substantially affected by horizontal position errors of the order of less than 2 m, which is the accuracy that can be obtained at present with satellite Doppler techniques. The reference potential, on the other hand, is quite sensitive to vertical (geocentric distance) errors, at the ratio of about 1 kgal m per meter of error. As we are concerned with finding potential differences, the most important errors are those in relative vertical position. To be precise, the vertical position of interest is the distance to the geocentre. There is little difference, however, between relative errors in ellipsoidal heights and relative errors in geocentric distance, and both can be regarded as equivalent here. The absolute vertical error might contain a nearly constant bias, due to the incorrect dimensions of the reference ellipsoid and to other systematic causes related to the positioning method. This error may be of several meters without any noticeable effect on the estimated potential differences, because it will nearly cancel-out when such differences are between points at the Earth's surface. Therefore, an error in the ellipsoid of the order of 2 m, which is the present level of accuracy, can be disregarded.

While the relative vertical position error is the one that matters, we still have to know the absolute geocentric distance to compute U. This can be done, essentially, in two ways:

a) find the absolute position of each point separately;

b) find the absolute position of one point, and then obtain the relative position of the other points with respect to this one.

In each case, the relative position errors will vary, the choice being always the alternative that gives the smallest errors.

This study is more concerned with forthcoming developments than with the present state of affairs: Anderle (1978) has estimated that the Global Positioning System currently being deployed could provide, when all the satellites are operational in the mid-Eighties, relative positions with errors of less than 0.1 m. This accuracy should be possible between stations thousands of kilometers apart, in all three coordinates, after less than one day of constant observation of the satellites. Estimates for absolute position determinations from lunar ranging stations made by Silverberg et al. (1977), using mobile stations supported by a network of a few fixed ones, are also in the decimeter range. In addition to these two, a variety of new positioning techniques based on satellites in high orbits that carry laser reflectors, mobile radiointerferometry, etc., being investigated at present, might provide even better accuracies in the coming decade. Present measurements from Doppler satellites have errors that are one order of magnitude worse. However, considering the progress made in this field over the past decade, and the new highly precise methods in the offing, it is probably not too optimistic to assume in this study relative accuracies as good as one decimeter in vertical position.

3.2. Reference Model of the Gravity Field

The coefficients C_{nm}, S_{nm} and the constant GM in (2.1) are not exact values, so the model does not represent to perfection the first N harmonic degrees of U or T. The effect of an incorrect GM is, at the present level of accuracy, equivalent to a bias of about ± 3 m in geocentric distance (Lerch et al., 1978). This error, being virtually constant, has a negligible effect on potential differences. The existence of coefficient errors

$$\begin{aligned} \epsilon \overline{C}_{nm} &= \overline{C}_{nm}\,(\text{true}) - \overline{C}_{nm}\,(\text{model}) \\ \epsilon \overline{S}_{nm} &= \overline{S}_{nm}\,(\text{true}) - \overline{S}_{nm}\,(\text{model}) \end{aligned} \qquad (3.1)$$

has to be considered when defining the disturbing potential[1]

[1] The 0 degree error $\epsilon GM/r$, being almost constant on the Earth's surface has no relevance to this work, and has been excluded from these formulas.

$$T(P) = V(P) - U(P) = \frac{GM}{r_P}\left\{\sum_{n=2}^{N}\left(\frac{a}{r_P}\right)^n \sum_{m=0}^{n} \overline{P}_{nm}(\sin\varphi_P)[\epsilon\overline{C}_{nm}\cos m\lambda_P + \epsilon\overline{S}_{nm}\sin m\lambda_P]\right.$$

$$\left. + \sum_{n=N+1}^{\infty}\left(\frac{a}{r_P}\right)^n \sum_{m=0}^{n} \overline{P}_{nm}(\sin\varphi_P)\;[\overline{C}_{nm}\cos m\lambda_P + \overline{S}_{nm}\sin m\lambda_P]\right\} \quad (3.2)$$

and the gravity anomaly

$$\Delta g(P) = -\frac{\partial T(P)}{\partial r} + \frac{\partial\gamma(P)}{\partial r}\frac{T(P)}{\gamma(P)} \simeq \frac{GM}{r_P^2}\left\{\sum_{n=2}^{N}\left(\frac{a}{r_P}\right)^n (n-1)\sum_{m=0}^{n}\overline{P}_{nm}(\sin\varphi_P)[\epsilon\overline{C}_{nm}\cos m\lambda_P + \epsilon\overline{S}_{nm}\sin m\lambda_P]\right.$$

$$\left. + \sum_{n=N+1}^{\infty}\left(\frac{a}{r_P}\right)^n (n-1)\sum_{m=0}^{n}\overline{P}_{nm}(\sin\varphi_P)\;[\overline{C}_{nm}\cos m\lambda_P + \overline{S}_{nm}\sin m\lambda_P]\right\} \quad (3.3)$$

The corresponding isotropic covariances, according to expression (2.13) are

$$c_{TT}(P,Q) = \frac{G^2M^2}{r_P\, r_Q}\left\{\sum_{n=2}^{N}(2n+1)\delta\epsilon_n P_n(\cos\psi_{PQ})\left(\frac{a^2}{r_P\, r_Q}\right)^n + \sum_{n=N+1}^{\infty}(2n+1)\delta_n P_n(\cos\psi_{PQ})\left(\frac{a^2}{r_P\, r_Q}\right)^n\right\} \quad (3.4)$$

$$c_{\Delta g\Delta g}(P,Q) = \frac{G^2M^2}{r_P^2 r_Q^2}\left\{\sum_{n=2}^{N}(2n+1)(n-1)^2\delta\epsilon_n P_n(\cos\psi_{PQ})\left(\frac{a^2}{r_P r_Q}\right)^n + \sum_{n=N+1}^{\infty}(2n+1)(n-1)^2\delta_n P_n(\cos\psi_{PQ})\left(\frac{a^2}{r_P r_Q}\right)^n\right\} \quad (3.5)$$

$$c_{T\Delta g}(P,Q) = \frac{G^2M^2}{r_P\, r_Q^2}\left\{\sum_{n=2}^{N}(2n+1)(n-1)\delta\epsilon_n P_n(\cos\psi_{PQ})\left(\frac{a^2}{r_P r_Q}\right)^n + \sum_{n=N+1}^{\infty}(2n+1)(n-1)\delta_n P_n(\cos\psi_{PQ})\left(\frac{a^2}{r_P r_Q}\right)^n\right\} \quad (3.6)$$

where

$$\delta\epsilon_n = \sum_{m=0}^{n}(\epsilon\overline{C}_{nm}^{\,2} + \epsilon\overline{S}_{nm}^{\,2})/(2n+1) \quad (3.7)$$

and $\delta_n \equiv \delta_{TT,n}$

In practice, neither $\delta\epsilon_n$ nor δn $(n > N)$ are known from direct measurements. The $\delta\epsilon_n$ can be approximated by the corresponding diagonal elements of the a posteriori variance-covariance matrix of the adjustment that produced the model. The δn are known to follow a more or less asymptotic decline law such as

$$\delta n \simeq \frac{10^{-10}}{n^4}$$

known as 'Kaula's rule", or such as

$$\delta_n = \left(\frac{GM}{a}\right)^2\left(\frac{\alpha_1 R_1^2}{(2n+1)(n-1)(n+A)}\left(\frac{R_1^2}{a^2}\right)^{n+1} + \frac{\alpha_2 R_2^2}{(2n+1)(n-1)(n-2)(n+B)}\left(\frac{R_2^2}{a^2}\right)^{n+1}\right) \quad (3.8)$$

This last expression, a "two terms law", has been investigated by Jekeli (1978), using the available information on the power spectrum of gravity anomalies, geoid undulations, and the horizontal gradient of gravity, to find the parameters A, B, α_1, α_2, R_1, R_2 that give the best fit to such information. This type of formulas has the advantage that the covariances (3.4), (3.5), and (3.6) can be computed using finite recursions.

3.3. Propagation of Position Errors through the Computed U

Because of the low degree of the terms in the expansion of U, horizontal errors in φ and λ are of little importance. The vertical errors ϵ_r, on the other hand, have a significant effect. If they are small, we can write

$$\epsilon\, \Delta U(P, Q) = \epsilon\, U(P) - \epsilon U(Q) \simeq \sum_{n=0}^{\infty}\left[\frac{\partial}{\partial r} U_n(P)\, \epsilon_{r_P} - \frac{\partial}{\partial r} U_n(Q)\, \epsilon_{r_Q}\right] \tag{3.9}$$

where U_n is the nth harmonic of U. Calling

$$\Delta U_n(P, Q) = U_n(P) - U_n(Q)$$

and

$$\epsilon_{r_{PQ}} = \epsilon_{r_P} - \epsilon_{r_Q}$$

we have

$$|\epsilon \Delta U_n| \leq 2(n+1) \max_{\partial \sigma(r,\underline{0})} \left\{\frac{|U_n|}{r}\right\} |\epsilon_{r_{PQ}}| \tag{3.10}$$

where $r = \min(r_P, r_Q)$ and $\partial\sigma(r, \underline{0})$ is the surface of the sphere $\sigma(r, \underline{0})$. Since

$$U_0 >> \max_{\partial\sigma(r,0)} \{|U_n|\}$$

for $n > 0$, it follows that

$$|\epsilon \Delta U_0| >> |\epsilon \Delta U_n|$$

for $n > 0$. Calling

$$\bar{r} = \tfrac{1}{2}(r_P + \epsilon_{r_P} + r_Q + \epsilon_{r_Q})$$

we get

$$|\epsilon \Delta U| \simeq |\epsilon \Delta U_0| \simeq \frac{GM}{\bar{r}^2} |\epsilon_{r_{PQ}}| \simeq \bar{\gamma}\, |\epsilon_{r_{PQ}}| \tag{3.11}$$

where $\bar{\gamma}$ is the mean value of gravity acceleration on the Earth's surface ($\bar{\gamma} \simeq 0.9798$ Kgal). The standard deviation of $|\epsilon \Delta U|$ is

$$\sigma_{\epsilon \Delta U} \simeq \bar{\gamma}\, \sigma_{\epsilon_{r_{PQ}}} \tag{3.12}$$

If the coordinates of points P and Q were determined separately, so ϵr_P and ϵr_Q could be considered uncorrelated, and if $\sigma\epsilon r_P = \sigma\epsilon r_Q = \sigma\epsilon_r$ then

$$\sigma\,\epsilon\Delta U \simeq \sqrt{2}\,\bar{\gamma}\,\sigma\,\epsilon_r \tag{3.13}$$

On the other hand, if the difference in vertical position is determined simultaneously for P and Q, then (3.12) applies. In any case, if all geocentric distances are computed with uncorrelated errors, except for some constant bias that does not affect the potential differences, then matrix $V_{\epsilon\Delta U}$ in (2.20) must be diagonal, each non-zero term being

$$v_{ii} = \sigma^2 \epsilon \Delta u_i \tag{3.14}$$

where Δu_i is the potential difference between the centers of the ith pair of caps.

3.4. Gravity Anomalies

The theory used in this work assumes that the exterior potential of gravitation is harmonic. This is not strictly correct at the Earth's surface, because of the atmosphere above it. To avoid systematic errors this effect should be discounted from the measured gravity values, and put back on the estimated potential. These atmospheric corrections have been studied in detail by Christodoulidis (1976). Probably, the variation in gravity due to Earth tides and ocean loading should be corrected as well, in order to achieve the degree of accuracy required here. Another source of systematic error is the gravity net to which the measurements are "tied". As explained later, systematics of more than 0.1 mgal rms are undesirable, so the contribution from the net should be as small as possible. Master stations where absolute gravity is known to, say, 0.01 mgal would be quite adequate. Of course, a constant bias due to an error in the nets' datum has no effect on the estimated potential differences, and can be ignored.

A further cause of systematic errors in the gravity anomalies are the distortions in the levelling net to which the stations are tied. The influence of such distortions on estimates of the disturbing potential have been explained by Lelgemann (1976). We are going to study here a way of determining the anomalies that minimizes this influence.

Besides actual errors in levelling, the main reason for distortions in vertical nets is the use of tide gauges as benchmarks. Their mean sea level marks are supposed to be at the same potential, and the net is adjusted with this as a constraint. In reality, the stationary sea surface topography already mentioned is present, and the potential differences among gauges are not quite zero. These discrepancies propagate as errors throughout the adjusted net. The larger the network, the larger the distortions can be, and also the longer the distances over which they are correlated. To shorten this correlation length we shall break the existing net into smaller pieces, and to eliminate the effect of the sea surface we shall not use a net that is adjusted with constraints based on tide gauges. To achieve this we are going to use the center of each cap as the levelling datum for all the gravity stations inside that cap. The potential of each station is going to be referred, accordingly, to the center point. Since the caps considered here are small (5° and 10° semi-apertures) the levelling net for each cap will consist only of short traverses whose measurement errors can be ignored. If necessary, the net inside each cap can be adjusted, to filter out such errors. To use the estimated potential of each cap center in our vertical connections, we have to refer all of them to some common datum. To do this without reverting to the use of tide gauges for this purpose, we shall take advantage of the accurate position fixes taken at the cap centers, and find the reference potential U at each one of them. If we take the $U(P_i)$ for the true potentials, we make a mistake quite similar to that of assuming that all tide gauges are on the same level surface. The errors, however, are proportional to the $T(P_i)$, the quantities to be estimated. As shown below, this leads to equations that can be solved for these "errors", to obtain the desired $T(P_i)$ free from biases. In a way, it can be said that the centers of the caps are the equivalent of tide gauges in the adjustment of the World Vertical Network.

A gravity anomaly Δg, in terms of the reference model, is

$$\Delta g(Q) = g(Q) - \gamma(Q') \tag{3.15}$$

where $g(Q)$ is the acceleration of gravity measured at a point Q on the Earth's surface, and $\gamma(Q')$ is the model's acceleration at a point Q' such that $\lambda_Q = \lambda_{Q'}$ and $\varphi_Q = \varphi_{Q'}$, while $U(Q') + \phi(Q') = V(Q) + \phi(Q) = W(Q)$. The rotation potentials $\phi(Q)$ and $\phi(Q')$ are almost equal, because the difference $(r_Q - r_{Q'})$ is of the order of 3 m for a reference model up to degree and order 20. For the same reason, the linearized expression

$$\Delta g(Q) \simeq \frac{-\partial T(Q)}{\partial r} + \frac{\partial \gamma(Q)}{\partial r}\frac{T(Q)}{\gamma(Q)} \tag{3.16}$$

can be regarded as almost exact. These approximations hold better for this type of model than for the simpler, and traditional, ellipsoidal model. The gravity potential at the gravity station Q is

$$W(Q) = U(P_1) + T(P_1) + \Delta W(P_1, Q) + \varphi(P_1)$$

where $\Delta W(P_1, Q)$ is the (levelled) potential difference between the cap center P_1 and Q. Since $T(P_1)$ is not known, we can only measure

$$W(Q) - T(P_1) = U(P_1) + \Delta W(P_1, Q) + \varphi(P_1) \tag{3.17}$$

where $U(P_1)$ is obtained from the reference model and the coordinates of P_1 (precise fix). Therefore, instead of Δg we determine

$$\Delta g^* = g(Q) - \gamma(Q'') \tag{3.18}$$

Q'' is a point with the same φ and λ as Q and Q', and where

$$U(Q'') + \omega(Q'') = W(Q) - T(P_1)$$

The relationships between Q, Q', and Q'' are illustrated in Figure 3. The distance $\overline{Q''Q'}$ is

$$\overline{Q''Q'} = (U(Q') - U(Q''))\,\gamma(Q)^{-1} = T(P_1)\,\gamma(Q)^{-1}$$

and

$$\begin{aligned}\gamma(Q'') &\simeq \gamma(Q') + \frac{\partial\gamma}{\partial r}(Q')\,(\overline{Q''Q'}) \\ &\simeq \gamma(Q') + \frac{\partial\gamma}{\partial r}(Q)\,T(P_1)\,\gamma(Q)^{-1}\end{aligned}$$

So

$$\Delta g^* = g(Q) - \gamma(Q'') \simeq g(Q) - \gamma(Q') - \frac{\partial\gamma(Q)}{\partial r}\,T(P_1)\,\gamma(Q)^{-1}$$

or

$$\Delta g^* = \Delta g(Q) - \frac{\partial\gamma}{\partial r}(Q)\,T(P_1)\,\gamma(Q)^{-1} \simeq \Delta g(Q) + \frac{2}{r_o}\,T(P_1) \tag{3.19}$$

The measured Δg^* can be used in one of two ways: (a) as a true gravity anomaly corrupted by a "bias" $\dfrac{\partial\gamma}{\partial r}\dfrac{T(P_1)}{\gamma(Q)}$;

(b) as a function of the gravity field on its own right, defined by (3.19) as dependent on both $\Delta g(Q)$ and $T(P_1)$. Only (a) shall be considered here.

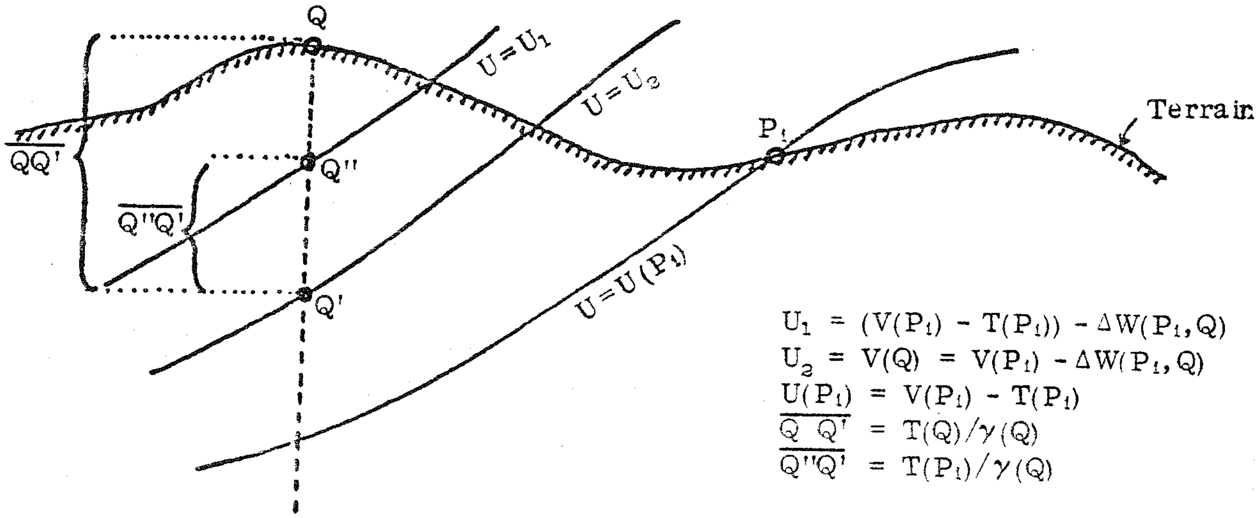


Figure 3.1. This picture illustrates the way in which the anomalies $\Delta \overset{*}{g}$ are defined at the Earth's surface, in terms of the available measurements, using the center of the cap as levelling datum. (See paragraph (2.1) for notation).

3.5. Propagation of Position Errors through Gravity Anomalies

To set up the optimal estimator (2.6) we have to know the correlations among the measurements and between the estimated variables and the data. For this we use the various correlation functions that are dependent on the position of the data and the estimates' points. These functions are usually quite smooth horizontally, and an error of up to a few seconds of arc in the geocentric angle ψ_{pq} is not likely to have any significant effect on them and, thus, on the estimator. The same functions are considerable more sensitive to a change in the radial positions of the points, so errors in r are more important than those in ψ. If we have the values of Δg (or $\Delta \overset{*}{g}$) at a point Q of coordinates φ, λ, r, but we assume, through incorrect position determination, that this point is $\hat{Q}$ or $\hat{\varphi}$, $\hat{\lambda}$, $\hat{r}$, then

$$\Delta g(\hat{\varphi},\hat{\lambda},\hat{r}) \simeq \Delta g(\varphi,\lambda,r) + \frac{\partial \Delta g}{\partial r}(\hat{\varphi},\hat{\lambda},\hat{r})\,\epsilon_r$$

where $\epsilon_r = \hat{r} - r$. The correction $\frac{\partial \Delta g}{\partial r}\epsilon_r$ cannot be computed, as ϵ_r is unknown, so we are confined to use $\Delta g(Q)$ as if it were $\Delta g(\hat{Q})$, thus having a position induced error $\frac{\partial \Delta g}{\partial r}\epsilon_r$ in the gravity anomaly. A constant position error will result in a constant bias in all gravity anomalies, as the variation of $\frac{\partial \Delta g}{\partial r}$ over the whole range of r on the Earth's surface (about ± 30 km) is very small. An error in Δg will, in turn, propagate into the estimated T's

$$\tilde{T} = \sum_{i=0}^{N_d} f_i (\Delta g + \epsilon \Delta g_i) = \hat{T} + \sum_i f_i \, \epsilon \Delta g_i = \hat{T} + O(\sigma_{\epsilon \Delta g_i} \sum_i f_i)$$

If the data arrangement is much the same for each cap, then the estimator "weights" f_i are also much the same in any cap, and the resulting error in each T from a bias in Δg is going to be nearly constant. These nearly equal errors in potential will very likely cancel-out when potential differences are computed. For the same reason, the effect of an erroneous GM on the mean value of the reference model's gravity will not have any appreciable consequence on the estimated potential differences. Thus we can have rather large biases in GM and on the set of station positions (error in the size of the reference ellipsoid). On the other hand, less correlated errors in r are not likely to cancel out. However, the rms value of $\frac{\partial \Delta g}{\partial r}$ at the Earth's surface is approximately only 3 μgals m^{-1}. From the results in section 4 it follows that errors up to 0.1 mgal in Δg, which are highly systematic inside a cap but vary randomly from cap to cap, can be tolerated. The same results show that several mgal in errors totally uncorrelated from station to station have very small effect on the accuracy of T. From all this, it can be concluded that: a) errors of several meters in r (at gravity stations) constant over the whole Earth, can be tolerated;

b) errors of several meters in r, constant inside each cap, but uncorrelated from cap to cap, are acceptable;

c) errors of several meters in r, uncorrelated from station to station have small effect.

The Global Positioning System, already mentioned, is expected to provide fixes accurate to 10 m in each coordinate after six seconds of receiving radio signals from four satellites visible at the same time (Anderle, 1978). After 15 minutes, this accuracy should improve to 1 m, plus a constant bias of no more than 3 m due to an error in the adopted ellipsoid. Thus, the contributions from (b) and (c) above should come to about 1 m, guaranteeing negligible position induced errors in Δg. By using a GPS receiver in conjunction with a gravimeter, one helicopter crew could collect all necessary information within a 5° cap (about 500 stations) in less than two months. Additional work would be needed to establish levelling ties between each station and the prediction point at the center, for which already existing traverses could be used, where available. In fact, even without GPS fixes, the positions of points like Q'' in paragraph (3.4), which are obtained using levelling and the reference model, can be used as station positions. The errors are going to be

$$\epsilon_{rQ} \simeq [V(Q) - U(Q'')]/\gamma \simeq (T(P_i) - T(Q))/GMa^{-2}$$

If a 20,20 model is used, the rms of T will be on the order of 3 kgal m, so $\epsilon_{r_Q} \approx \sqrt{2} \times 3$ kgal m. Because potential is a smooth function of distance, the correlation c_{TT} between $T(Q)$ and $T(P_i)$ will reduce the rms of the error

$$\sigma\epsilon_{r_Q} = \left(\sqrt{\sigma(T(Q))^2 + \sigma(T(P))^2 - 2\,c_{TT}(\psi_{PQ}, r_P, r_Q)}\right)\gamma^{-2}$$

over distances of the order of one cap radius (500 km), so levelling-determined positions might be sufficient for the gravity stations.[1]

3.6. Levelling

Levelling is needed to connect cap centers to benchmarks, and cap centers to gravity stations within their respective caps. Existing traverses should be used wherever possible, but only unadjusted values or values adjusted within relatively small regions must be used. Otherwise, results could be biased by the distortions accumulated in large adjusted nets. To keep errors small, the traverses should be as short as practicable. This goal is easily achieved for the levelling inside caps, as distances from center to rim range between 500 and 1000 km in the cases considered; the longer traverses from centers to benchmarks require a careful planning of the overall system. Caps should be placed, as much as possible, within flat areas with a smooth field of gravity anomalies, because estimates of T are likely to be better there than where both field and terrain change wildly, as suggested by the results in Appendix A. Traverses should be levelled across regions where the topography is gentle, to reduce their errors.

The simulations reported here have been done assuming that all levelling traverses run along arcs of maximum circles, that their errors are uncorrelated unless they overlap, and that the standard deviations of these errors obey the simple formula

$$\sigma_{\Delta W_\ell} = 0.1\sqrt{\ell}\ \text{kgal m} \tag{3.20}$$

where ℓ is the length of the traverse in thousands of kilometers. This formula represents a quality of measurement not much better than that of present day first order levelling (see, for instance, Lelgemann (1976)).

[1] An error ϵ_{r_P} will result in an error $\gamma\epsilon_{r_P}$ kgal m in $U(P)$; of about $0.3\,\epsilon_{r_P}$ mgal in the value of each $\Delta\overset{*}{g}$; and of approximately $0.3\,\epsilon_{r_P}\sum_{i=1}^{N_r} f_i$ kgal m in T_P. However, according to Tables 4.3 and 4.4 $\sum_{i=1}^{N_r} f_i \simeq 0.3$ for 5° and 10° caps, so ϵT_P due to ϵ_{r_P} is $O(0.1\,\epsilon_{r_P})$ kgal m and can be neglected here if $\epsilon_{r_P} < 0.5$ m.

4. Computing Disturbing Potentials

Paragraph (2.2) explains the choice of least squares collocation as the technique for predicting T at the cap centers. Several practical problems associated with this method must be considered before even relatively simple simulations can be carried out. A description of these problems and their treatment is given in this section.

4.1. Reducing the Dimension of the Data Covariance Matrix

Assuming that the gravity stations are spaced some 40 km from each other, then their number in each 5° cap will be close to 500, and to 2000 for a 10° cap. This means that $C_{zz} + D$ is either a 500 x 500 or a 2000 x 2000 matrix, respectively. Being symmetrical, it will have up to 2000000 different elements for a 10° cap, each one requiring one calculation of the covariance function $c_{\Delta g \Delta g}(P, Q)$. This can be a very costly process in terms of computer time, and inverting $C_{zz} + D$ can be costly as well. This is particularly true in the context of a study in which calculations may have to be repeated several times with slightly changed assumptions. For these reasons an arrangement of the data was chosen that gives the $C_{zz} + D$ matrix a strong structure. This, as explained in Colombo (1979), brings about great savings in both forming and inverting the matrix. To understand how this is possible in our case, consider the following argument. Imagine that the gravity measurements are arranged in concentric rows around the center point, which is also the estimation point, and that all stations are on the same geocentric sphere as the center. Instead of the individual anomalies, suppose that we use the row sums

$$\widetilde{\Delta g}_i = \sum_{n=1}^{N_i} \Delta g_{in}$$

as data. The number of $\widetilde{\Delta g}$'s is the same as that of the rows, N_r. The covariance function for the $\widetilde{\Delta g}$'s is

$$c_{\widetilde{\Delta g}\widetilde{\Delta g}}(i,j) = M\{\widetilde{\Delta g}_i \widetilde{\Delta g}_j\} = \sum_{n=1}^{N_i} \sum_{n=1}^{N_j} M\{\Delta g_{in} \Delta g_{jn}\} \tag{4.1}$$

(N_i, N_j are the numbers of stations in rows i, j) while that between $T(P)$ and $\widetilde{\Delta g}_i$ is

$$c_{T\widetilde{\Delta g}_i}(P, i) = M\{T(P) \sum_{n=1}^{N_i} \Delta g_{in}\} = N_i M\{T \Delta g_{i1}\} \tag{4.2}$$

as $M\{T\Delta g_{ij}\}$ is constant for all stations in the same row.

With these two functions we can set up C_{zz} and C_{sz}, while the noise matrix is

$$D = \begin{cases} d_{ij} = 0 & i \neq j \\ d_{ii} = \sum_{a=1}^{N_i} \sigma_{ia}^2 & \end{cases} \tag{4.3}$$

where σ_{ia} is the standard deviation of the mth measurement on the ith row. C_{zz} is now only $N_r \times N_r$: for a 5° with 40 km between rows, $N_r = 13$. This implies a reduction of more than one order of magnitude in the dimension of the data set and of several in computing time, because setting up the matrix is proportional to its (dimension) , while inverting it requires a time proportional to (dimension)^3. Moreover, as shown in the reference given above, if the points are equally spaced along each row, having the same number in each row, then estimating T from $\widetilde{\Delta g}$ or from Δg gives the same result. In other words: this arrangement largely improves computing efficiency, without changing the quality of the result.

A distribution of data with the same number of points in every row is not a good one, because if the maximum separation (in the outer row) is 40 km, then the stations will be very crowded at the center. If points are eliminated to thin out the central region, the exact equivalence to collocation using Δg will be lost. This will bring some deterioration in results, though not a very remarkable one. The actual accuracy can be found, as usual, using expression (2.11) and the matrices corresponding to $\widetilde{\Delta g}$'s. The unequal number of points will result in the outer rows' covariances being larger than the central ones', and this will probably worsen the condition number of the matrix. To avoid this problem, the row sums can be replaced by row averages.

$$\overline{\Delta g_i} = \frac{1}{N_i} \sum_{a=1}^{N_i} \Delta g_{ia} \tag{4.4}$$

The covariance functions for these are

$$c_{\overline{\Delta g}\,\overline{\Delta g}} = M\{\overline{\Delta g_i}\,\overline{\Delta g_j}\} = \frac{1}{N_i N_j} \sum_{a=1}^{N_i} \sum_{n=1}^{N_j} M\{\Delta g_{ia} \Delta g_{jn}\} \tag{4.5}$$

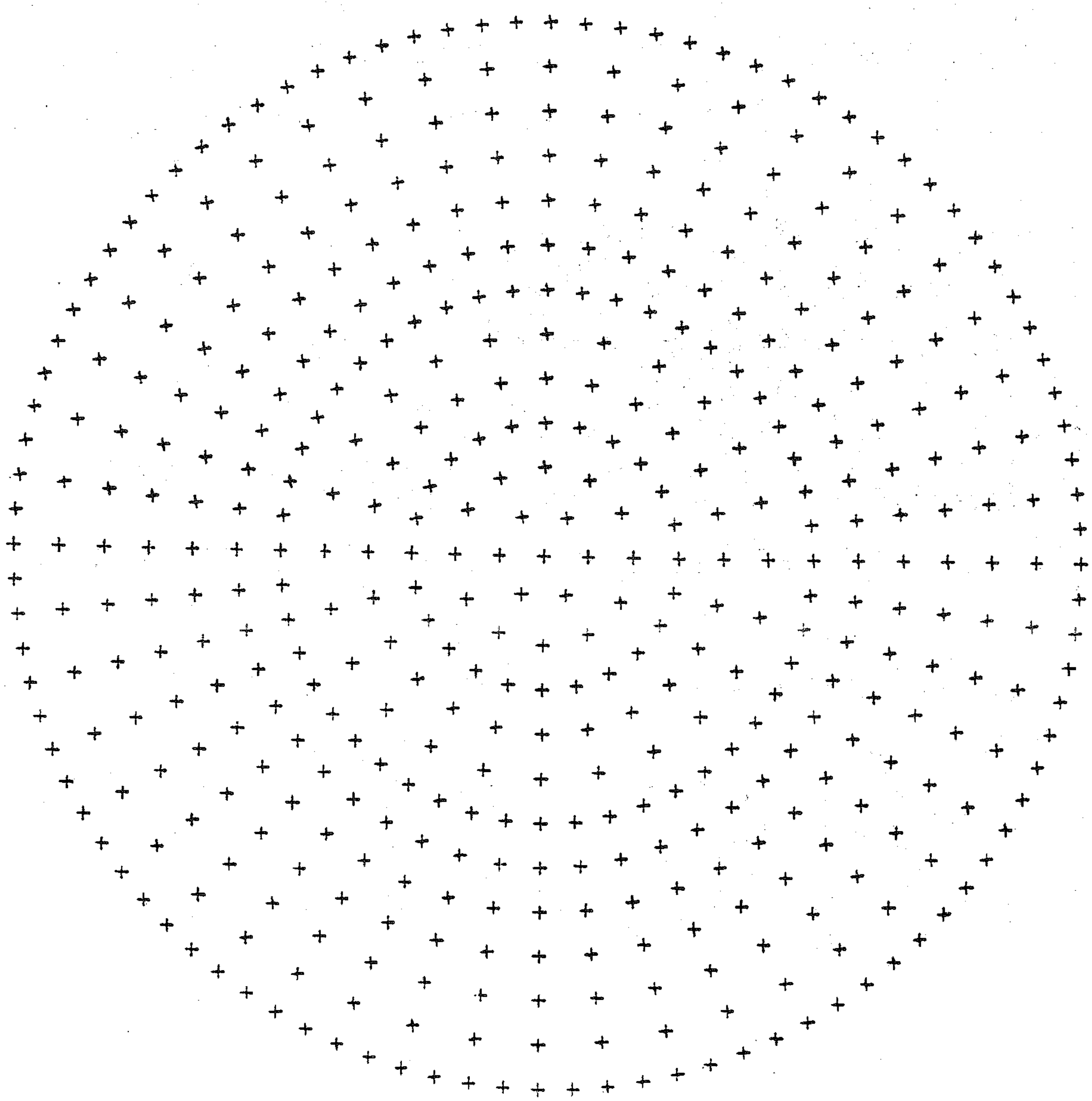
$$c_{T\overline{\Delta g}} = M\{T(P)\overline{\Delta g_i}\} = \frac{1}{N_i} \sum_{a=1}^{N_i} M\{T(P)\Delta g_{ia}\} = M\{T(P)\Delta g_{i1}\} \tag{4.6}$$

and

$$d_{ii} = \frac{1}{N_i^2} \sum_{a=1}^{N_i} \sigma_{ia}^2, \quad D \text{ being diagonal.} \tag{4.7}$$

The grids used in this study were constructed according to a simple pattern that keeps average distances between stations at about 40 km. There is one station at the very center, six in the first row, twelve in the second, twenty-four in the third; their number doubling from there on every time the diameter of the row doubles (at the 3rd, 6th, 12th, row, etc.), and staying constant otherwise. Figure 4.1 shows this scheme used on a 5° cap. In such a grid, the separation between rows is

Figure 4.1. Arrangement of Gravity Stations in the 5° Cap.



always almost 40 km, and the constant separation of stations in the same row varies, from row to row, from 30 km to 60 km. Furthermore, the values of the covariance between any point in row i and all points in row $j \geq i$ are repeated at all points in i. This greatly speeds up the creation of C_{zz}, as all terms such as

$$\sum_{n=1}^{N_J} M\{\Delta g_{in} \Delta g_{jn}\}$$

in (4.5) are equal regardless of m. Finally, computing time can be halved by taking advantage of the fact that if a radial line is drawn through any point in the figure its covariances with all points to the left of this line are the same as with those on the right.

It is most unlikely that all gravity stations will actually have the same geocentric distances, and there is no fundamental need for this, as collocation can be implemented with whatever coordinates the stations may have, although less efficiently, as long as they are known with reasonable certainty. However, if the computing savings mentioned above are to be realized, the data must first be reduced to an ideal grid by collocation on a spherical surface. Measurements in the vicinity of each node of the ideal grid can be used to interpolate a value on that node. Over a reasonably gentle terrain, the distance between it and the sphere is not likely to exceed 2 km within a 5° cap, and some of the nodes are going to be below, and some above the Earth's surface. Assuming that five gravity stations were used, one on the same vertical as the node but 2 km above (below) it, and four others forming cross with the first at the center and 20 km arms, also 2 km above (below) the sphere, the accuracy of a value collocated on the node is ± 0.8 mgal if the data has ± 0.5 mgal measurements' white noise. This was found using the same covariance functions employed in all the other simulations conducted during this study. Since collocation is a smoothing process, the rms of the estimation error is likely to be due, by and large, to the high frequency components of the data. As shown by the simulations in the next section, as much as 4 mgal of high frequency errors will have a negligible effect on the accuracy of the adjusted vertical connections.

4.2. Estimating T from $\Delta \overset{*}{g}$ instead of Δg

As already explained, the optimal estimator

$$\hat{T}(P_s) = \underline{f}^T \underline{d} = C_{sz} (C_{zz} + D)^{-1} \underline{d}$$

depends on the type of data chosen $\underline{d} = \underline{z} + \underline{n}$. Let C_{sz} and C_{zz} be covariance matrices for gravity anomalies Δg arranged inside a cap of center P_s, but suppose that the data available consists in values of $\Delta \overset{*}{g}$ with the same spatial arrangement. According to (3.19), if T is given in kgal m and Δg in mgal,

$$\Delta \overset{*}{g} = \Delta g + k\,T(P_a) \tag{4.8}$$

where $k \simeq \frac{2}{r_{P_a}} \times 10^6 \simeq 0.3$. Calling $\tilde{T}$ the estimate of $T(P_a)$ based on Δg, and $\hat{T}$ the estimate of $T(P_a)$ based on $\Delta \overset{*}{g}$, and using overbars to design row averages, as in the previous paragraph, then

$$\tilde{T}(P_a) = T(P_a) + \tilde{\epsilon}(P_a) = \sum_{i=1}^{N_r} f_i\, \overline{[\Delta g + k\,T(P_a) - k(T(P_a))]}$$

$$= \sum_{i=1}^{N_r} f_i\, \overline{\Delta \overset{*}{g}} - k\,T(P_a) \sum_{i=1}^{N_r} f_i$$

So

$$T(P_a)[1 + k \sum_{i=1}^{N_r} f_i] + \tilde{\epsilon}(P_a) = \sum_{i=1}^{N_r} f_i\, \overline{\Delta \overset{*}{g}} \tag{4.9}$$

and

$$\hat{T}(P_a) = T(P_a) + \hat{\epsilon}(P_a) = T(P_a) + \frac{\tilde{\epsilon}(P_a)}{[1 + k \sum_{i=1}^{N_r} f_i]} = \frac{\sum_{i=1}^{N_r} f_i\, \overline{\Delta \overset{*}{g}}}{[1 + k \sum_{i=1}^{N_r} f_i]} \tag{4.10}$$

Consequently,

$$\hat{\epsilon}(P_a) < \tilde{\epsilon}(P_a) \quad \text{if} \quad \sum_{i=1}^{N_r} f_i > 0 \tag{4.11-a}$$

$$\hat{\epsilon}(P_a) \geq \tilde{\epsilon}(P_a) \quad \text{if} \quad \sum_{i=1}^{N_r} f_i \leq 0 \tag{4.11-b}$$

If $\sum_{i=1}^{N_r} f_i > 0$ there is a <u>reduction</u> in the estimate's error, according to (4.11-a), and the opposite happens when $\sum_{i=1}^{N_r} f_i < 0$, according to (4.11-b).

4.3. Accuracies for $T(P_1)$ Estimated over 5° and 10° Caps

Tables 4.1 and 4.2 show the accuracy of T estimated under different conditions, using the theory explained in the two preceeding paragraphs. To obtain the theoretical accuracy, and for the reasons given in the previous paragraph, the square root of the value calculated according to (2.11) was corrected by the factor $(1 + k \sum_{i=1}^{N_T} f_i)^{-1}$. While varying slightly from case to case, this factor is always close to 0.9 for 5° caps, and to 0.8 for 10° caps.

The values of the "weights" f_i (i.e. the components of $\underline{f}$) do not change greatly, under varying circumstances, from the "typical" ones listed in Tables 4.3 and 4.4. This is particularly true of the largest "weights", from the center point to the 10th ring, which remain nearly constant.

In general we can say that, for a 20,20 model and up to several mgal rms of white noise in the gravity data, the accuracy of T estimated on a 5° cap is close to 0.4 kgal m, and for a 10° cap it is near 0.3 kgal m.

The "imperfect model" used for the results consists of the first N degree harmonics of a 180,180 model obtained by Rapp (1978b) as a combination of a world data set of 1°x 1° gravity anomalies with GEM-9. The standard deviations of the coefficients are listed up to degree N = 30 in Table 4.5. The "2L" and "2H" covariance models are based on formula (3.8) and have the following coefficients (Jekeli, 1978):

2L		2H	
A = 100	α_1 = 18.3906 mgal²	A = 140	α_1 = 14.0908 mgal²
B = 20	α_2 = 658.6132 mgal²	B = 10	α_2 = 160.6701 mgal²
s_1 = .9943667		s_1 = .9939083	
s_2 = .9048949		s_2 = .9997595	

where

$$s_1 = \left(\frac{R_1}{a}\right)^2 \quad \text{and} \quad s_2 = \left(\frac{R_2}{a}\right)^2 .$$

Covariances were computed with these coefficients and the closed expressions also given in the above reference. To simplify calculations, all measurements and estimations are supposed to be made on the same sphere of radius a = 6371000 m. To test the resulting accuracies, some numerical experiments were conducted, as reported in Appendix A.

Table 4.1.

Accuracy of Estimated Disturbing Potential (kgal m) for 5° Caps and "2 L" Covariances (some values obtained using "2 H" are in brackets).

Imperfect Model	Perfect Model	RMS of $\epsilon\Delta g^*$ in mgal	Maximum Degree, Order in Model
0.81	0.80	2	10
(0.37) 0.41	-	4	20
(0.36) 0.39	(0.23) 0.27	2	20
0.38	0.27	0	20
0.40	-	4	30
0.37	0.21	2	30

(Values estimated from Δg's are 1/0.9 of those given here.)

Table 4.2.

Accuracy of Estimated Disturbing Potential (kgal m) for 10° Caps and "2 L" Covariances

Imperfect Model	Perfect Model	RMS of $\epsilon\Delta g^*$ in mgal	Maximum Degree, Order in Model
0.29	-	4	20
0.27	-	2	20
0.24	0.19	0	20

(Values estimated from Δg's are 1/0.8 of those given here.)

Table 4.3.

Optimal Row "Weights" f_i (kgal m/mgal) for 5° Caps, Maximum Degree and Order in Model is 20, RMS of $\epsilon \Delta \overset{*}{g}$ is 2 mgal (all values multiplied by 0.1).

Row No.	Imperfect Model		Perfect Model	
	"2 L"	"2 H"	"2 L"	"2 H"
	f_i x 0.1	f_i x 0.1	f_i x 0.1	f_i x 0.1
0 (center)	0.225	0.225	0.225	0.224
1	0.438	0.438	0.436	0.435
2	0.408	0.409	0.404	0.402
3	0.388	0.391	0.383	0.380
4	0.351	0.354	0.344	0.340
5	0.301	0.304	0.292	0.287
6	0.301	0.307	0.290	0.285
7	0.255	0.260	0.242	0.237
8	0.228	0.234	0.214	0.208
9	0.198	0.204	0.183	0.176
10	0.176	0.182	0.158	0.150
11	0.122	0.129	0.109	0.104
12	0.234	0.243	0.189	0.168

From these values it is clear that a constant error in gravity ϵg_0 (mgal) will result in an estimation error (bias)

$$\epsilon T_0 = \epsilon g_0 \sum_{i=0}^{12} f_i \simeq 0.3 \, \epsilon g_0 \text{ kgal m.}$$

Table 4.4.

Optimal Row "Weights" f_i (kgal m/mgal) for 10° Caps, $\sigma \epsilon \Delta \overset{*}{g} = 2$ mgal, Imperfect Model to Degree and Order 20, "2 L" Covariance Function.

Row No.	f_i x 0.1	Row No.	f_i x 0.1	Row No.	f_i x 0.1
0	0.218				
1	0.434	9	0.294	17	0.153
2	0.419	10	0.276	18	0.138
3	0.413	11	0.241	19	0.123
4	0.392	12	0.252	20	0.110
5	0.352	13	0.218	21	0.097
6	0.368	14	0.204	22	0.085
7	0.331	15	0.186	23	0.073
8	0.316	16	0.169	24	0.059
				25	0.110

Table 4.5.

Accuracies of the Potential Coefficients in the Imperfect Model, by Degree.

n	$\delta\epsilon_n$	n	$\delta\epsilon_n$	n	$\delta\epsilon_n$
	$\times 10^{-12}$		$\times 10^{-12}$		$\times 10^{-12}$
1	—	11	4990.	21	2899.
2	2720.	12	4415.	22	2761.
3	6594.	13	4830.	23	2635.
4	4564.	14	4458.	24	2522.
5	7237.	15	4140.	25	2417.
6	5703.	16	3864.	26	2321.
7	6707.	17	3623.	27	2232.
8	5685.	18	3410.	28	2149.
9	5727.	19	3220.	29	2072.
10	5118.	20	3051.	30	2001.

4.4. Correlation Among Estimation Errors

The elements of matrix $V_{\epsilon\hat{\Delta}T}$ (expression (2.20)) are of the form[1]

$$v_{kk} = M\{(\epsilon\hat{T}(P_{ik}) - \epsilon\hat{T}(Q_{jk}))^2\} = \sigma^2\epsilon\hat{T}(P_{ik}) + \sigma^2\epsilon\hat{T}(Q_{jk}) - 2M\{\epsilon\hat{T}(P_{ik})\epsilon\hat{T}(Q_{jk})\} \quad (4.12)$$

if the element is on the main diagonal, or

$$v_{nm} = M\{(\epsilon T(P_{in}) - \epsilon T(Q_{jn}))(\epsilon T(P_{im}) - \epsilon T(Q_{jm}))\} \quad (4.13)$$

$$= M\{\epsilon T(P_{im})\epsilon T(P_{in})\} + M\{\epsilon T(Q_{jm})\epsilon T(Q_{jn})\} - M\{\epsilon T(P_{im})\epsilon T(Q_{jn})\} - M\{\epsilon T(P_{in})\epsilon T(Q_{jn})\}$$

if it is off-diagonal. To simplify matters, let us assume that all caps are of the same size and have the same data distribution inside. Then

$$\sigma_{\epsilon\hat{T}(P_i)} = \sigma$$

where σ is the global rms of the estimation errors, found in accordance to (2.11). Furthermore,

[1] The subscript k indicates the "caps pair" or observation equation number.

$$M\{\epsilon T(P_h)\,\epsilon T(Q_k)\} = M\{(T(P_h) - \underline{f}^T \underline{d}_h)(T(Q_k) - \underline{f}^T \underline{d}_k))\}$$

$$= c_{TT}(P_h, Q_k) - 2C_{T_h \underline{d}_k}\underline{f} + \underline{f}^T C_{\underline{d}_h \underline{d}_k}\underline{f} \qquad (4.14)$$

where $\underline{d}_h = [\Delta g_1, \Delta g, \ldots, \Delta g_i, \ldots, \Delta g_{N_r}]_h^T$ corresponds to the h th cap, and

$$C_{T_h \underline{d}_k} = M\{T(P_h)\,\underline{d}_k^T\}$$

is a $1 \times N_r$ row vector (N_r is the number of concentric rows in each cap)

$$C_{\underline{d}_h \underline{d}_k} = M\{\underline{d}_h \underline{d}_k^T\}$$

is a $N_r \times N_r$ matrix ($\underline{d}_h$ is the data in the h th cap).

The elements of these matrices are:

$$C_i = M\{T(P_h)\overline{\Delta g}_i(k)\} = \frac{1}{N_i} M\{T(P_h) \sum_{r=1}^{N_i} \Delta g_{ir}(k)\}$$

$$= \frac{1}{N_i} \sum_{r=1}^{N_i} M\{T(P_h)\Delta g_{ir}(k)\} \qquad (4.15)$$

for $C_{T_h \underline{d}_k}$

(where $\Delta g_{ir}(k)$ is the value of Δg at the rth gravity station on the ith ring centered at P_k), and

$$C_{ij} = M\{\overline{\Delta g}_i(k)\,\overline{\Delta g}_j(h)\} = \frac{1}{N_i N_j} M\{\sum_{r=1}^{N_i} \Delta g_{ir}(k) \sum_{s=1}^{N_j} \Delta g_{js}(h)\}$$

$$= \frac{1}{N_i N_j} \sum_{r=1}^{N_i} \sum_{s=1}^{N_j} M\{\Delta g_{ir}(k)\,\Delta g_{js}(h)\} \qquad (4.16)$$

for $C_{\underline{d}_h \underline{d}_k}$.

If the total number of caps is N_c, there are approximately $(1/4)\,N_c^2\,N_r^2$ different "ring covariances" (the general term in (4.16)), and each requires computing the covariance function of Δg $N_i \times N_j$ times. This can tax the largest computing budget; on the other hand $V_{\epsilon \hat{T}_\Delta}$ is only part of the a priori variance-covariance matrix used for adjusting the potential difference between benchmarks $\Delta W(BMA, BMB)$. Usually, a priori covariances are not needed to very great accuracy, because the adjusted values are not extremely sensitive to them, or should not be if the procedure has been designed properly. If approximate values are enough, then a most efficient way of obtaining them, correct to several significant figures, is to assume that the covariance of the "ring averages" $\overline{\Delta g}_i(h)\,\overline{\Delta g}_j(k)$ is

$$M\{\bar{\Delta g}_i(h)\,\bar{\Delta g}_j(k)\} \simeq M\left\{\frac{1}{2\pi a \sin\psi_i}\int_0^{2\pi}\Delta g(\psi_i,\alpha)\,d\alpha\,\frac{1}{2\pi a\sin\psi_j}\int_0^{2\pi}\Delta g(\psi_j,\beta)\,d\beta\right\}$$

$$= \frac{G^2M^2}{a^4}\Bigg[\sum_{n=2}^{N}(2n+1)(n-1)^2\,\delta\epsilon_n\,P_n(\psi_i)\,P_n(\psi_j)\,P_n(\psi_d)\;+$$

$$+\sum_{n=N+1}^{N_{max}}(2n+1)(n-1)^2\,\delta_n\,P_n(\psi_i)\,P_n(\psi_j)\,P_n(\psi_d)\Bigg] \qquad (4.17)$$

where ψ_d is the spherical distance between the centers of the caps where rings i and j are located; ψ_i and ψ_j are the sizes (semi-apertures) of rings i and j; while N, $\delta\epsilon_n$, δ_n are the same as in expression (3.4). Values computed as above are accurate to no less than three significant figures for $\psi_d > 10°$, ψ_i, $\psi_j < 5°$ and $N_{max} > 400$. Table 4.6 presents values of $M\{\epsilon T(P_k)\,\epsilon T(P_j)\}$ for different distances between cap centers: this illustrates the considerable independence of estimates separated by more than 2000 km. As the data are supposed to be $\overset{*}{\Delta g}$'s rather than Δg's, results have been divided by $1 + 0.3\sum_{i=1}^{N_4} f_i \simeq 0.9$

Table 4.6.

Correlation Between Disturbing Potential Estimates at Various Distances ($\sigma\,\epsilon\overset{*}{\Delta g}$ = 2 mgal, 5° Caps, Imperfect Model up to Degree and Order 20).

$M\{\epsilon T(P_i)\,\epsilon T(P_j)\}$ (kgal m)^2		Distance $\overline{P_i P_j}$
"2 H"	"2 L"	km
0.130	0.154	0
0.032	0.033	1150
0.027	0.028	1300
0.008	0.007	2000
0.002	0.002	2300
-0.002	0.002	2500
0.001	0.002	14000
-0.002	-0.002	17500

5. The Accuracy of the Adjusted Vertical Connection

This section presents the main results of this study: the theoretical accuracy of a World Vertical Network constructed along the lines already discussed. It also contains the theory of an optimal estimator for potential differences between centers of pairs of identical caps, and other ideas regarding possible uses of satellite and terrestrial data for setting up and strengthening levelling nets.

5.1. Transoceanic Connections Using Several 5° Caps

Two cases have been considered: in the first, five 5° caps were placed in North America (4 in the U.S. and 1 in Canada) and another four in Australia. The cap centers have been chosen to avoid overlaps and to ensure that almost all the area covered by the caps is land. The "benchmarks", two points whose coordinates are given below, are: BMA in Electra, Texas, and BMB in Wiluna, Western Australia. Cap centers in the same land mass are supposed to be joined to the corresponding benchmark by levelling traverses, as in Figure 2.1. To assign a standard deviation to the error of each traverse, formula (3.20) has been used, the length of the traverse being that of the maximum circle joining its endpoints. All data is supposed to have been measured (or reduced) to the same sphere of radius $a = 6371000$ m to simplify calculations. Tables 5.1 and 5.2 show the accuracies obtained using formula (2.19) with the "a priori" matrix V corresponding to various combinations of covariance function and reference model. The following is the list of cap centers, including their latitudes (the benchmarks are also cap centers). The difference between the North America/Australia and the U.S./Australia connections is the fact that cap number 9 in the former has been replaced by cap 9* in the latter.

Cap No.	Latitude	Longitude	Location	State
1 (BMB)	-27.5°	120.0°	WILUNA	W. Australia
2	-20.0°	130.0°	Tanami	N. Territory
3	-22.5°	142.0°	Middleton	Queensland
4	-32.5°	145.0°	Cobar	N. S. Wales
5	36.0°	-85.5°	Sparta	Tennessee
6 (BMA)	34.5°	-99.0°	ELECTRA	Texas
7	44.0°	-99.0°	Wessington	South Dakota
8	37.0°	-112.5°	Fredonia	Arizona
9	56.5°	-112.0°	Mc. Murray	Alberta
9*	65.0°	-150.0°	Manley	Alaska

Overall, the accuracies listed in Tables 5.1 and 5.2 can be separated into two groups: those based on the use of an imperfect reference model, and those obtained assuming a perfect model. Results within each group are much the same: approximately 0.3 kgal m accuracy with an imperfect model, about 0.2 kgal m with a perfect one. Changes of 100 in the accuracies of levelling or gravity measurements caused less than 10% variation in $\sigma \Delta W$ (the effect of levelling accuracy is shown in Table 5.2); while a change in covariance model had only a slight effect on ΔW, as suggested by the components of the respective pseudoinverse vectors $\underline{v}$ listed in Table 5.3 (where $\underline{v}^T \underline{p} = \Delta W$, see paragraph (2.4)), and as shown in Table 5.1.

In summary: <u>the accuracy of the reference model is the most important of the various sources of error included in this study</u>. Improvements in this model are likely to have a large effect on the quality of the resulting vertical network.

As the error correlations between caps, shown in Table 4.6, are very small compared to the autocorrelations (cap errors), we can ignore them in a first approximation, regarding all errors contributing to matrix V as <u>uncorrelated.</u> The standard deviation of the adjusted ΔW can then be guessed using the following formula, instead of the "exact" expression (2.19):

$$\sigma\Delta W(\mathrm{BMA}, \mathrm{BMB}) \simeq \left(2(\sigma^2\epsilon T + 0.01\, \ell_{max}) + \gamma^2\sigma^2\epsilon\,\Delta r\right)^{\frac{1}{2}} / Ne^{\frac{1}{2}}$$

where Ne is the number of independent equations (paragraph (2.4)), ℓ_{max} is the length of the longest traverse (< 5000 km), and $\sigma\epsilon\Delta r$ is the error in relative geocentric position between points. Assuming: eight equations, as in the two examples; $\sigma_{\Delta g}^{*} = 2$ mgal; and the imperfect model, then

$$\sigma\Delta W(\mathrm{BMA}, \mathrm{BMB}) \simeq (0.40 + 0.96\,\sigma^2\epsilon\Delta r)^{\frac{1}{2}}/\sqrt{8} \qquad \text{for "2 L"}$$

and

$$\sigma\Delta W(\mathrm{BMA}, \mathrm{BMB}) \simeq (0.36 + 0.96\,\sigma^2\epsilon\Delta r)^{\frac{1}{2}}/\sqrt{8} \qquad \text{for "2 H"}$$

Assuming, for instance, that $\sigma\epsilon\Delta r \approx 0.5$ m, the corresponding accuracies with these simplified expressions would be 0.28 kgal m and 0.27 kgal m, respectively. Now ± 0.5 m in relative geocentric position is within the reach of present-day Doppler satellite techniques. In any case, all the results given here are well below the ± 1.5 kgal m theoretical uncertainty for tidal gauges and spirit levelling alone.

Table 5.1.

Accuracy $\sigma\Delta W$ of the North America-Australia Vertical Connection (in kgal m)
$\sigma\epsilon\Delta r = 0.15$ m, $\sigma\epsilon\Delta g^{*} = 2$ mgal, $\sigma\epsilon\Delta W_{\ell} = 0.1\sqrt{\ell}$ kgal m

Imperfect Model (N = 20)		Perfect Model (N = 20)	
"2 L"	"2 H"	"2 L"	"2 H"
0.32	0.30	0.21	0.18

Table 5.2.

Accuracy $\sigma\Delta W$ of the U.S.A.-Australia Vertical Connection (in kgal m)
2 L covariance, $\sigma\epsilon\Delta\overset{*}{g} = 2$ mgal, $\sigma\epsilon\Delta r = 0.15$ m

Imperfect Model (N = 20)	Perfect Model (N = 20)	$\sigma\epsilon\Delta W_\ell$
0.32	0.21	$0.1\sqrt{\ell}$
0.32	0.21	$0.0\sqrt{\ell}$

Table 5.3.

Components of $\underline{v} = (\underline{a}^\tau V^{-1}\underline{a})^{-1}\underline{a}^\tau V^{-1}$ (Dimensionless)
North America-Australia Connection
$\sigma\epsilon\Delta\overset{*}{g} = 2$ mgal, $\sigma\epsilon\Delta r = 0.15$ m, $\sigma\epsilon\Delta W_\ell = 0.1\sqrt{\ell}$ kgal m

"Observation Equations" Potential Difference between caps number	Eqn. No.	Imperfect Model (N = 20)		Perfect Model (N = 20)	
		"2 L"	"2 H"	"2 L"	"2 H"
1 - 5	1	.216	.234	.207	.204
1 - 6	2	.067	.054	.048	.050
2 - 6	3	.109	.130	.204	.212
2 - 7	4	.147	.126	.098	.108
3 - 7	5	.016	-.059	.104	.085
3 - 8	6	.200	.218	.137	.155
4 - 8	7	.022	-.001	.072	.044
4 - 9	8	.224	.247	.129	.140

5.2. Optimal Estimator for the Potential Difference Between Two Caps

In addition to the more general configuration studied in the preceeding paragraphs, we shall consider a "minimal estimator" where only two caps are involved, each centered at a benchmark. As in paragraph 5.1, we shall restrict the estimator by assuming that the data $\Delta\overset{*}{g}$ has been converted to ring averages $\overline{\Delta g}$ and, furthermore, that both caps are identical in size and data arrangement. This limits somewhat the power of the estimator, in particular the use of ring averages makes it suboptimal by comparison to a "full" estimator based on point data. On the other hand, these constraints greatly expedite creation of the estimator, as in the case considered before.

The objective of the estimator

$$\hat{\Delta T}(P_1, P_2) = \underline{f}_1^T \underline{d}_1 - \underline{f}_2^T \underline{d}_2 = \sum_{i=1}^{N_r} f_{i1} \overline{\Delta g}_i(1) - \sum_{j=2}^{N_r} f_{j2} \overline{\Delta g}_j(2) \tag{5.1}$$

is to minimize the global mean square of the prediction error:

$$M\{\epsilon \hat{\Delta T}(P_1, P_2)^2\} = M\{(\Delta T(P_1, P_2) - (\underline{f}_1^T \underline{d}_1 - \underline{f}_2^T \underline{d}_2))^2\} \tag{5.2}$$

Because both caps are identical, and the average is <u>isotropic</u> (function of spherical distance and radial distances only) then, if both caps are on the same sphere, the optimal weights for each are the same:

$$\underline{f}_1 = \underline{f}_2 = \underline{f} \quad \text{or} \quad f_{i1} = f_{i2} = f_i \quad i = 1, 2, \ldots, N_r$$

Accordingly,

$$\begin{aligned} \tfrac{1}{2} M\{\epsilon \hat{\Delta T}(P_1, P_2)^2\} &= \tfrac{1}{2} M\{(T(P_1) - T(P_2) - \sum_{i=1}^{N_r} f_i(\overline{\Delta g}_i(1) - \overline{\Delta g}_i(2)))^2\} \\ &= \tfrac{1}{2} M\{(T(P_1) - T(P_2) - \underline{f}^T(\underline{d}_1 - \underline{d}_2))(T(P_1) - T(P_2) - (\underline{d}_1 - \underline{d}_2)^T \underline{f})\} \\ &= c_{TT}(0) - c_{TT}(\psi_{P_1 P_2}) + 2\underline{f}^T(C^T_{T_1 \underline{d}_2} - C^T_{T_1 \underline{d}_1}) + \underline{f}^T(C_{z_1 z_1} - C_{z_1 z_2} + D)\underline{f} \end{aligned}$$

where $C_{z_1 z_2}$ is the same as $C_{\underline{d}_h \underline{d}_k}$ in (4.14), and $C_{z_1 z_1}$, D, $C_{T_1 \underline{d}_1} \equiv C_{T_1 z_1}$ are as in (2.7). Thus,

$$\tfrac{1}{2} \frac{\partial}{\partial \underline{f}} M\{\epsilon \Delta T\} = C_{T_1 \underline{d}_2} - C_{T_1 \underline{d}_1} + (C_{z_1 z_1} - C_{z_1 z_2} \pm D)\underline{f} = 0 \tag{5.3}$$

or

$$\underline{f} = (C_{z_1 z_T} - C_{z_1 z_2} + D)^{-1} (C_{T_1 \underline{d}_2} - C_{T_1 \underline{d}_1}) \tag{5.4}$$

since the data consists in values of $\Delta \overset{*}{g}$ rather than Δg, a "correction factor" should be used, as before

$$\hat{\Delta T} \simeq \underline{f}^T(\underline{\Delta \overset{*}{\overline{g}}_1} - \underline{\Delta \overset{*}{\overline{g}}_2}) \times \frac{1}{1 + 0.3 \sum_{i=1}^{N_r} f_i} \tag{5.5}$$

where $\underline{\overline{\Delta \overset{*}{g}}} = [\overline{\Delta \overset{*}{g}}, \ldots, \overline{\Delta \overset{*}{g}}_{N_r}]^T$. Using the type of data grid shown in Figure 4.1, extended to a 10° cap, so that the average separation of the stations is 40 km, assuming the same reference gravity fields and covariance models (2 L and 2 H) used before, 0.1 m relative[1] position accuracy (cap centers) and 2 mgal accuracy for the gravity anomalies, the following global rms of the estimation errors were calculated for various separations ψ_d between the caps:

Table 5.4.

RMS of Error of Optimally Estimated Potential Difference Between the Centers of Two 10° Caps $\psi_d{}^\circ$ Apart. (Imperfect model to degree and order 20, $\sigma\epsilon\Delta\overset{*}{g}$ = 2 mgal, "2 L" covariance function.)

$\sigma\epsilon\Delta W$ (kgal m)	ψ_d (°)
0.35	10°
0.40	20°
0.42	30°

(RMS fluctuates between 0.41 and 0.42 kgal m for $30° < \psi_d \leq 180°$.)

If the position errors at each cap center are uncorrelated, and if $\sigma\epsilon r$ is their rms, then $\sigma\epsilon\Delta W$ should be corrected as follows: $\sigma\epsilon\Delta W' = \sqrt{(\sigma\epsilon\Delta W)^2 + 2(\sigma\epsilon r)^2}$. It is interesting to compare Table 5.4 to Table 4.2: for $\psi_d > 30°$. $\sqrt{M\{\epsilon\Delta\hat{T}^2\}}$ listed above clearly approaches $\sqrt{2M\{\epsilon\hat{T}^2\}}$ where $\sqrt{M\{\epsilon\hat{T}^2\}}$ is the "single cap" accuracy listed in Table 4.2. This also agrees with the increasing independence of estimates of T separated by more than a few thousand kilometers, pointed out in paragraph 4.4. Similarly, the values of the optimal "weights" f_i converge to those for a single 10° cap, with increasing ψ_d.

5.3. Height Differences Between Inaccessible Points

The technique described in this report, in essence, uses accurate position fixes and a good gravity field model to obtain an estimate of the potential difference between points, estimate that is then refined using data from the neighborhood of each point (gravity anomalies and levelling) to add high frequency information not contained in the model. So far we have assumed a good local coverage, easily accessible gravity stations and, generally, a most cooperative disposition from both Nature and men towards our project. If either, or both, were lacking, we would be left with the field model, plus some data in the perifery of the region of interest to refine the former, and perhaps not even a high accuracy relative position fix at each cap center. Through collocation, external data can be incorporated into the adjustment, though not as efficiently as in the scheme discussed

[1] The relevant error here is that in the measured difference of geocentric distances.

in paragraph 4.1. As the data is far from the point of interest, it is important to use information rich in long wavelengths signal. One possibility, if the inaccessible area is not extremely far from the sea, is the mean sea surface derived from satellite altimetry, regarded as a quasi-geoid with 1.5 m global rms of "noise" due to dynamic effects. Altimetry could be used on land as well, to provide fixes in radial position for the "cap centers". The surface of an inland sea or lake would be ideal as a target; other areas where consistent ellipsoidal heights could be obtained (i.e., independent of seasonal effects), such as salt-flats in desert areas, etc., could be used if systematic errors due to surface reflectivity were sufficiently understood. With very accurate gravity field models and continuous tracking of the altimeter satellite by another craft in a higher orbit it should be possible, eventually, to obtain computed orbits good to 0.1 m (rms), which, added to another 0.1 m (rms) error for the altimeter itself, would amount to near 0.15 m (rms) error in the ellipsoidal height fixes, or about 0.2 m relative geocentric height accuracy between benchmarks which would propagate as a 0.2 kgal m error in potential difference. Gravity field models have been improving at a fast rate in recent years; new and better tracking systems are being developed; full coverage of altimetry data over the oceans is now available from the Geos-3 and Seasat-I spacecrafts. These are three encouraging signs that, in the coming decade, there will be enough information to model the field up to degree 180 with a global residual rms of about 1 kgal m for the disturbing potential. Then, vertical connections good to at least $\sqrt{2 \times 1 + (0.2)^2} \simeq 1.5$ kgal m should become feasible. This is the same as the theoretical global accuracy of a system based on tide gauges: it should be much the same as having "tide gauges" inland.

5.4. Some Questions Regarding Accuracy Estimates

The accuracies listed in the various tables of this section depend, mostly, on the applicability of the theory of collocation to the real world. In particular, there are some aspects of collocation open to criticism that deserve a mention here. The first one is the unavoidable use of approximate covariance functions, as the true ones cannot be known exactly from finite amounts of data, basically because the expansion of the "true" gravity field in harmonic polynomials is infinite. Lauritzen (1973) has established the impossibility of obtaining the covariance of a random process on a sphere even from a complete data coverage, but it is not very clear why the gravity field should be treated as a random process in the first place. Be as it may, the available information is always going to be incomplete and inaccurate. Otherwise, there wouldn't be much point in using collocation, or any other form of interpolation and filtering, as there would be little to learn from it. This is why two different empirical covariance functions were used, "2 L" and "2 H", to find out how sensitive the adjusted potential differences were to the choice of function. The results, as shown in the various tables, were much the same with both.

Second, there is the question of how suitable is the spherical harmonic representation of the gravity field on which a good deal of the theory put forward here (as much of modern geodesy) rests. From the work of Petroskaya (1977) and Sjöberg (1978), who have proposed mathematical formulations for the field both inside and outside the Earth's surface, we learn that such expansions are not necessarily sums of solid spherical harmonics. On the other hand, it is well known that the gravity field of a homogeneous ellipsoid can be expressed in terms of such harmonics down to a sphere completely buried in the ellipsoid. The Earth being primarily ellipsoidal, we may expect it to have a field that does not behave too differently from that of the ellipsoid. However, there is no reason to expect that the harmonic series that describes the field exactly outside the bounding sphere does so also inside it and down to the Earth's surface. That it does not diverge too strongly is shown by the fact that low frequency models derived from satellites provide a fit to surface data that improves as more terms are used. That the model is not too inadequate follows from the usefulness of formulas such as Vening Meinesz' and Stokes', which are based on the assumption that the field can be expanded in solid spherical harmonics. But all this supporting evidence shows is that these ideas "work" enough to provide about one meter accuracy in computed undulations, seconds of arc in deflections of vertical, a few milligals in interpolated gravity. No evidence is available to suggest that they also "work" at the level of accuracy (approximately 0.3 to 0.5 m) expected of them here. Going back to the homogeneous ellipsoid, it is sufficient to add a most minute inhomogeneity, in the form of a tiny material sphere at any distance R from the origin, for the series of the composite body to fail to converge inside the sphere of radius R. The mass of the sphere (and therefore the difference between the pure ellipsoidal and the composite field) can be made as small as desired without the series ever converging again. This makes clear that, at least in some cases, a gravity field with a harmonic series that does not converge down to an internal sphere can be only slightly different (except at some isolated points) from one with a series that does. Krarup (1969) brought attention to this fact, and enquired whether this might not be always the case. He concluded that it is, furnishing proof of what he called a "Runge-type theorem", because of similar theorems for elliptical differential equations. Krarup's thesis is that any harmonic field can be approximated uniformly, together with all its derivatives, by sequences of series of spherical harmonics that converge to an arbitrarily small sphere centered at the origin, completely inside a surface (terrain) that separates the region where the field is harmonic from that where it is not. There are a few restrictions on the nature of this surface, but they are loose enough to ensure an adequate fit to the real topography. The uniform convergence takes place only down to that surface. Inside, the various approximations may differ increasingly from each other, and have no limit function. So, a spherical harmonic approximation to the exterior field can be quite irregular and "wild" in the interior, and this might be a cause for some concern here. The global rms, or accuracy obtained from (2.11), corresponds to the average of the square of the errors of all possible predictions made on the same sphere where the actual estimation point is situated. Such sphere is always partly inside the solid Earth, because of the equatorial bulge. If we imagine the covariance functions that we are using as corresponding to some spherical harmonic expansion that fits closely the field outside the terrain, they may also correspond to a function that has a markedly different character on that part of the sphere that is buried from that which is out in the open. In other words, the spherical harmonic approximations may not be sufficiently "stationary" for a meaningful application of collocation. It may help to understand this problem having some method to generate the sequences of convergent series Krarup has spoken of, in order to visualize their behavior. Such method, to this author's knowledge, has not been proposed yet.

The real question here is whether collocation, and the use of spherical harmonics theory, may not be a source of bias in the results. In practice there may be many sources of bias not treated here, such as systematic errors in gravity measurements, position fixes, etc. The only definitive way of knowing if such factors can be truly significant is to subject the whole idea to experimentation. If enough determinations of potential differences are carried out using this technique, at many places around the world, between points already connected by levelling traverses, then any large systematic discrepancies should become apparent. If they do not show up, then the experiments will provide supporting evidence for the use of the idea elsewhere.

6. Conclusions

This report has dealt with the concept of World Vertical Network: a set of benchmarks thousands of kilometers apart, the geocentric coordinates of, and the potential differences between which, are accurately known, so they can be used to connect separate levelling nets in a world-wide system.

The results of section 5 suggest that, according to theory, it might be possible to set up this network to a significantly better accuracy than that provided by tide gauges and spirit levelling alone. This could be done by using satellite and terrestrial data together, employing least squares collocation as the main mathematical tool for combining them.

Two cautionary notes are in order: first, there may be causes of error not considered in this study that turn out to be of practical significance: only actual experience can have the final word on this. Second, the estimates of T, crucial to the ideas in this work, are assumed made from values of $\Delta^* g$ measured, or interpolated, on spherical surfaces. Collocation can be used, in principle, on more general (terrain-like) surfaces, but the accuracy is not exactly the same as with the spherical arrangement of data points. "Common sense" suggests that, if the terrain is gentle, departing smoothly from the spherical shape, the accuracy should be much the same, but no evidence for this is given here. Numerical studies are far more difficult when the data is not on a simple spherical arrangement, and probably too expensive for an ordinary research project. Furthermore, whether on the terrain or on a partially buried sphere, the statistical relevance of the accuracies derived from collocation can be questioned on the grounds given at the end of paragraph (5.4). If this is a real problem, it is inherent to all applications of collocation theory to estimation of gravitational field variables inside the Earth's bounding sphere, and not just to the present ideas. This matter seems to have received little or no attention from workers in this area.

Depending primarily on the quality of the reference gravity field model, the accuracy of the vertical connections in the network could be between 0.2 and 0.3 kgal m, using the data arrangement described in sections 2 and 5. Such configuration has been chosen, mostly, to simplify this study, and is by no means the only possible one. With more data, including satellite altimetry, we could expect even better results, so those given here are to be regarded as upper limits to the ultimate quality of a global network.

The North America-Australia connection studied in section 5 requires relatively few data (about 4000 point gravity anomalies to 2 mgal accuracy, 8 accurate position fixes and a number of levelling traverses, plus a reference model to degree and order 20) and the quality of the measurements are almost entirely within present day limits. The exception, the position fixes, can be expected to become feasible within the coming decade. Much of this data can be obtained and used for other purposes, such as the study of polar motion and Earth rotation in the case of the accurate point coordinates. In this way, by sharing with other scientific enterprises, the establishment of the World Vertical Network could be made both cheaper and an integral part of the creation of a World Geodetic System for positions and heights.

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Appendix A

Verifying the Correctness of the Accuracies Given in Section 4

The covariance matrix for ring-averaged gravity anomalies is very poorly conditioned, with the arrangement of Figure 4.1, because of the closeness of the measurements. Evidence of this was observed while trying to use tables with equispaced entries and linear interpolation, as an inexpensive way of computing the covariance function values needed to set up the matrix. With spacings as small as 1 km, the small perturbation due to interpolation errors was enough to produce a matrix with negative eigenvalues, which the true covariance matrix can never have. On the light of this experience, it appeared reasonable to question whether any results dependent on such a ticklish matrix could be regarded as meaningful, including the accuracy estimates of Tables 5.1 and 5.2. To clarify this matter, a very high degree field model (up to $n = 1000$) consisting only of zonal terms was used

$$\approx{T}(\theta) = \frac{GM}{r} \sum_{n=2}^{1000} c_n P_n(\theta) \tag{A.1}$$

with the c_n selected so the disturbing potential $\overset{\approx}{T}$ would have the same spectrum/covariance function as the one assumed for the Earth. Using the appropriate expansion for the gravity anomaly of this model, $\overset{\approx}{\Delta g}$ was calculated at every point in the grid of Figure 4.1, and then the optimal "weights" from Table 4.4 were used to obtain the estimated $\hat{\overset{\approx}{T}}$. The error $\overset{\approx}{\epsilon}_T = \hat{\overset{\approx}{T}} - \overset{\approx}{T}$ was then calculated using once more the model, and the whole operation was repeated at 5° intervals, from pole to pole. The mean value of the squared error is, approximately

$$M\{\overset{\approx}{\epsilon}_T{}^2\} \simeq 2\pi a \sum_{\theta_i=0}^{\theta_i=\pi} \sin\theta_i\, \overset{\approx}{\epsilon}_T(\theta_i)^2 / (2\pi a \sum_{\theta_i=0}^{\theta_i=\pi} \sin\theta_i) \tag{A.2}$$

where advantage is taken of the fact that the error $\overset{\approx}{\epsilon}_T$ as well as $\overset{\approx}{T}$, $\hat{\overset{\approx}{T}}$ and $\overset{\approx}{\Delta g}$ have expansions consisting of zonal terms only. Assuming a perfect reference model up to degree and order 20, 2 mgal (rms) error in the gravity anomalies, the global rms of the prediction error (accuracy), estimated using (A.2) was 0.3 kgal m. This is about 30% below the value given in Table 4.1, but not unreasonably so: the sampling near the North pole, at 5° intervals, is probably too coarse to accurately cover the fast changes in the field there. The largest errors also occur near that pole, so a shorter interval is likely to pick up more of them, increasing the right hand side of (A.2).

Table A.1 shows $\overset{\approx}{T}$ and the errors $\overset{\approx}{\epsilon}_T$, $\overset{\approx}{\epsilon}_{\hat{\Delta g}}$ as functions of latitude (10° intervals). The largest errors, as already mentioned, coincide with the wild "spike" in the disturbing potential near the North pole. The error outside a 30°

polar cap is almost everywhere less than the global rms. This suggests that the worst predictions can occur where the field has large and fast variations. The percentage of error in the predicted value, however, is about 10% at the pole, so the relative goodness of the prediction is not particularly bad there. But we are interested in the absolute value of the error, so the pole is clearly a bad place, in this "artificial planet", for making estimates. Table A.1 shows that the behavior of $\tilde{\tilde{\Delta}}g$ closely resembles that of $\tilde{\tilde{T}}$. In this particular case, the prediction of $\tilde{\tilde{T}}$ is bad where that of $\tilde{\tilde{\Delta}}g$ is bad,[1] and vice versa, and this could be used as a practical criterion for selecting the locations of estimation points. Predicting Δg from gravimetry in a "candidate" region, at points in that region where Δg is already known from accurate measurements, we could compare the actual error in these estimates to the theoretical accuracy of expression (2.11). If the actual errors are close to their theoretical rms, the region is acceptable, and we can proceed to set up a cap like that one in Figure 4.1; if the errors are consistently larger, the region should be rejected.

Other points to consider when selecting a place for a cap are: lack of significant bodies of free water, because gravity cannot be measured as accurately on water as on land; smooth topography (with most places easily accessible) to permit accurate levelling and good coverage with gravity stations.

[1] Δg is estimated here using a "4 points' cross" pattern centered at the prediction point. The "arms" are 20 km wide.

Table A.1.

Prediction Errors in T and Δg in a Zonal Field. Imperfect Reference Field to Degree and Order 20, 5° Cap, $\sigma \epsilon \Delta g^* \begin{cases} 2 & \text{mgal (T)} \\ 0.5 & \text{mgal } (\Delta g) \end{cases}$

φ (°)	$\epsilon \hat{T}$ kgal m	$\epsilon \hat{\Delta g}$ mgal	T kgal m	Δg mgal
90	-72.88	3195.39	697.91	26095.57
80	.53	-1.24	-3.60	-14.36
70	.06	.20	- .50	- 2.14
60	-.17	.20	.77	4.15
50	-.31	.37	-1.26	- 3.20
40	.04	- .28	.78	2.41
30	-.13	- .07	- .60	- 1.81
20	-.01	- .04	.13	.45
10	-.02	.08	.01	.60
0	-.06	.14	- .33	- .89
-10	.06	- .12	.41	1.17
-20	-.02	- .03	- .34	- 1.34
-30	.07	- .07	.31	.80
-40	.04	.96	- .03	- .15
-50	.05	.10	- .03	- .15
-60	.07	- .09	.36	.97
-70	-.12	- .04	- .58	- 1.85
-80	-.11	- .20	.46	1.92
-90	-.69	3.63	-2.25	- .65

Appendix B

This Appendix contains listings of parts of the software developed for this project, and sample output. First there is a listing of "RINGS", a program that obtains the optimal components, or weights, of the estimator vector $\underline{f}$ (expression (2.6)). This program sets up a grid of the kind shown in Figure 4.1 for a cap of size CAPSIZ (in degrees). The number of rings in the cap is NP, so, with the origin, there are NP+1 weights to be found. The listing clearly shows the various constants used. REGPAR is a "regularization" parameter, chosen here very small. This number is added to the main diagonal of the normal matrix $C_{zz} + D$ to improve the stability of the solution. NMOD is the maximum degree and order in the reference model. To change it into a "perfect" model, a statement setting all values in DVAR to zero is added before the statement "55 CONTINUE". GNOISE is the standard deviation of Δg^* in mgals. The solution is obtained using the conjugate gradients procedure in subroutine CGRADS. The maximum number of iterations allowed, ITERMX, is the number of unknown (NP + 1) plus ten. However, if the improvement in the rms of the solution from iteration to iteration is less than one part in a million, the procedure terminates then. Program RINGS sets up the normal matrix taking advantage, as far as possible, of the various symmetries in the grid. After the solution has been found, it multiplies both rms and weights by the correction factor $1 + k \sum_{i=1}^{NP+1} f_i$ (paragraph (4.2)), and it also multiplies $(C_{zz} + D)\underline{f}$ = reconstructed right hand sides of normals, to compare them with the original rms's, showing whether the conjugate gradient procedure has, in fact, converged. The listing of RINGS is complemented by those of the subroutines it calls: CGRADS, MATVEC, LEGPOL, COVAR, and function F.

Finally, subroutine RINCOV, used to determine the covariance $M\{\epsilon \hat{T}(P_i) \epsilon \hat{T}(P_j)\}$ between prediction errors, based on (4.14) and the repeated use of expression (4,17), is listed as well. The main array and variables are given the same names as in RINGS and associated subroutines, with the exception of the optimal weights vector, here called "F".

SAMPLE OUTPUT OF "RINGS"

```
CAPSIZE, DELTA PSI (DEGS.), NO. OF RINGS :     5.00000          .416667       12


MAX. DEGREE IN MODEL :  20 REGUL. PARAM. :    .100000D-03   G. ANOM. "NOISE" S.D.      2.00000     MGAL
MODEL ERROR DEG. VARS.

   1      .0               2      .353858D-04     3      .116460D-02     4      .161398D-02     5      .881761D-02
   6      .101114D-01      7      .232365D-01     8      .257530D-01     9      .381513D-01    10      .426212D-01
  11      .547835D-01     12      .564038D-01    13      .867644D-01    14      .931721D-01    15      .996184D-01
  16      .106045         17      .112503        18      .118940        19      .125326        20      .131794

GRID SEGMENTS PER RING AND SIZE OF RING SEGMENTS (RADS)

   1    6     1.04720        2    12     .523599       3    24     .261799        4    24     .261799
   5   24      .261799       6    48     .130900       7    48     .130900        8    48     .130900
   9   48      .130900      10    48     .130900      11    48     .130900       12    96     .654499D-01

RIGHT HAND SIDES OF THE NORMAL EQUATIONS.

      62.17330325
      40.32108894
      26.85484644
      18.22412319
      12.33273725
      8.107965089
      4.970286909
      2.585955853
      .7515991462
     -.6630279520
     -1.745277855
     -2.556463448

COVARIANCE OF DIST. POTENTIAL :        8.827695581


ITERATION NO.     1
OBS. EQNS. : RMS OF ERRORS       1.364634983      PERCENT OF IMPROVEMENT IN RMS      54.07038276


ITERATION NO.     2
OBS. EQNS. : RMS OF ERRORS       .6400624445      PERCENT OF IMPROVEMENT IN RMS      53.09643586


ITERATION NO.     3
OBS. EQNS. : RMS OF ERRORS       .4765304807      PERCENT OF IMPROVEMENT IN RMS      25.54937650
```

ITERATION NO. 15

OBS. EQNS. : RMS OF ERRORS .4321263675 PERCENT OF IMPROVEMENT IN RMS .8463424608D-09

GLOBAL R.M.S. OF OPTIMIZED ERROR : .4321263675 KGAL.M

R.M.S./(1+BETA) = .3879760374

RECONSTRUCTED RIGHT HAND SIDES.

1	82.87210191
2	62.17330325
3	40.32108894
4	26.85484644
5	18.22412319
6	12.33273725
7	8.107965089
8	4.970286909
9	2.585955853
10	.7515991462
11	-.6630279519
12	-1.745277855
13	-2.556463448

RING OPTIMIZED WEIGHTS

1	.2020043490D-01	2	.3934098640D-01	3	.3664451306D-01	4	.3487234093D-01	5	.3155308370D-
6	.2699597123D-01	7	.2705290460D-01	8	.2285855769D-01	9	.2049730602D-01	10	.1775837186D-
11	.1576596863D-01	12	.1091638800D-01	13	.2100555399D-01				

```
              C
              C            PROGRAM "RINGS" .
              C            PROGRAM TO ESTABLISH THE OPTIMIZED "RING WEIGHTS" FOR
              C            ESTIMATING THE DISTURBING POTENTIAL AT THE CENTER OF
              C                A CRIDED SPHERICAL CAP WITH POINT GRAVITY ANOMALIES.
              C
              C            OSCAR L. COLOMBO, GEODETIC SCIENCE, O.S.U. 1979 .
              C
0001                 IMPLICIT REAL*8 (A-H,O-Z)
0002                 REAL*8 NMIN,N2MIN
0003                 COMMON /COV/A1,A2,R12,R22,A,B,C2
0004                 COMMON/N/ DVAR(30),NMOD
0005                 COMMON/LEG/ POLS(30),NMIN(30),N2MIN(30)
0006                 DIMENSION RN(100),FVEC(100),PN(100),QPN(100),ALPHAN(1),CSD(100),
                    2 FON(1),DFIT(1), FN(1),IGRID(60),ALPHA(60),COSALF(200)
                    3,TABLE1(4008),TABLE2(4008),CPSI(60),SPSI(60)
                    4,CDD(4000)
0007                 DATA FVEC/100*0.D0/
0008                 DIMENSION F(30),FC(30)
              C
              C            BASIC CONSTANTS.
              C
0009                 PI = 3.141592654
0010                 TWOPI = PI*2.D0
0011                 PION2 = PI/2.D0
0012                 DRCONV = TWOPI/360.D0
0013                 RE = 6371000.D0
0014                 GAMMA = 0.978049D0
              C
              C            C. JEKELI'S GRAVITY COV. MODEL  "I2" (DOGS REPORT 275)
              C
0015                 A = 100.D0
0016                 B = 20.D0
0017                 A1 = 18.3906D0
0018                 A2 = 658.6132D0
0019                 S1 = 0.9943667D0
0020                 S2 = 0.9048949D0
0021                 R12 = S1*RE**2
0022                 R22 = S2*RE**2
0023                 C2 = 7.56D0
              C
              C            ARRAY  DVAR  CONTAINS THE STANDARD DEVIATIONS OF THE
              C            POTENTIAL COEFFICIENTS OF THE "IMPERFECT MODEL".
              C
0024                 DVAR(1) = 0.D0
0025                 DVAR(2) = 2720.D-12
0026                 DVAR(3) = 6594.D-12
0027                 DVAR(4) = 4564.D-12
0028                 DVAR(5) = 7237.D-12
0029                 DVAR(6) = 5703.D-12
0030                 DVAR(7) = 6707.D-12
0031                 DVAR(8) = 5685.D-12
0032                 DVAR(9) = 5727.D-12
0033                 DVAR(10) = 5118.D-12
0034                 DVAR(11) = 4990.D-12
0035                 DVAR(12) = 4415.D-12
0036                 DVAR(13) = 4830.D-12
0037                 DVAR(14) = 4458.D-12
```

```
      DVAR(15) = 4140.D-12
      DVAR(16) = 3864.D-12
      DVAR(17) = 3623.D-12
      DVAR(18) = 3410.D-12
      DVAR(19) = 3220.D-12
      DVAR(20) = 3051.D-12
      DVAR(21) = 2899.D-12
      DVAR(22) = 2761.D-12
      DVAR(23) = 2635.D-12
      DVAR(24) = 2522.D-12
      DVAR(25) = 2417.D-12
      DVAR(26) = 2321.D-12
      DVAR(27) = 2232.D-12
      DVAR(28) = 2149.D-12
      DVAR(29) = 2072.D-12
      DVAR(30) = 2001.D-12
C ************************************************************************
      CAPSIZ = 5.D0
      NP = 12
      NMOD = 20
      GNOISE = 2.D0
      REGPAR = 1.D-4
C ************************************************************************
      NP1 = NP+1
      ITERMX = NP1+10
      DPSI = CAPSIZ/NP
      RESQ = RE**2
      NM = (360.D0/DPSI*2.D0)+0.0000001D0
      WRITE(6,1) CAPSIZ,DPSI,NP
    1 FORMAT(//' CAPSIZE, DELTA PSI (DEGS.), NO. OF RINGS :',2G16.6,1X,
     2 I4//)
      CAPSIZ = CAPSIZ*DRCONV
      DPSI = DPSI*DRCONV
      DO 55      I = 1,NMOD
      DVAR(I) = DVAR(I)**2*(2*I+1)*(I-1)**2*(GAMMA*1.D6)**2
C ********************
   55 CONTINUE
      WRITE(6,2) NMOD,REGPAR,GNOISE,(I,DVAR(I),I=1,NMOD)
    2 FORMAT(//' MAX. DEGREE IN MODEL :',I4,' REGUL. PARAM. :',G15.6,'
     2 G. ANOM. "NOISE" S.D. ',G16.6,' MGAL '//' MODEL ERROR DEG. VARS.'
     3//5(I5,1X,G15.6,1X))
      GNOISE = GNOISE**2
C
C             PREPARE ARRAYS  NMIN  AND  N2MIN  FOR USE BY SUBROUTINE
C             "LEGPOL".
C
      DO 3          I = 1,NMOD
      NMIN(I) = -((I-1)*1.D0)/I
      N2MIN(I) = (2.D0*I-1.D0)/I
    3 CONTINUE
      COSMN = DCOS(2.01D0*CAPSIZ)
      COS1 = (1.D0-COSMN)/4001.D0
      COS1I = 1.D0/COS1
C
C             END INITIALIZATION. BEGIN CALCULATIONS.
C
C
C             FORMING GRID : RINGS ARE EQUISPACED, AND HAVE 90 DEGREES'
```

```
                C               SYMMETRY.
                C
                C               THIS GRID IS THE " 6, 12, 24, 48, 96, 192, " GRID.
                C
0081                  IGRID(1) = 6
0082                  DO 4       I = 2,NP
0083                4 IGRID(I) = 12
0084                  M = 3
0085                  DO 6     J = 1,4
0086                  DO 5       I = M,NP
0087                5 IGRID(I) = IGRID(I)*2
0088                  M = M*2
0089                6 CONTINUE
0090                  DO 7       I = 1,NP
0091                  ALPHA(I) = TWOPI/IGRID(I)
0092                7 CONTINUE
                C
                C               CREATE THE TABLES OF SINES AND COSINES TO COMPUTE COS(PSI).
                C
0093                  NMT = IGRID(NP)
0094                  N4 = NMT/4+1
0095                  AL = ALPHA(NP)
0096                  DO 8       I = 1,N4
0097                  COSALF(I) = DCOS(AL*(I-1))
0098                8 CONTINUE
0099                  COSALF(N4) = 0.D0
0100                  DO 9         I = 1,NP
0101                  CPSI(I) = DCOS(DPSI*I)
0102                  SPSI(I) = DSIN(DPSI*I)
0103                9 CONTINUE
0104              111 FORMAT(//' GRID SEGMENTS PER RING AND SIZE OF RING SEGMENTS (RADS)
                    2 '//,4(I5,1X,I5,1X,G15.6,1X))
0105                  IC = 1
0106                  HP = 0.D0
0107                  HQ = 0.D0
                C
                C               COMPUTE ALL RING COVARIANCES TO FORM THE "NORMAL" MATRIX  CDD.
                C
                C               FIRST COMPUTE THE FIRST ROW AND COLUMN (CENTRAL POINT) OF  CDD .
                C
0108                  COSPSI = 1.D0
0109                  CALL COVAR(COSPSI,HP,HQ,CPQ,IC)
0110                  CDD(1) = CPQ+REGPAR+GNOISE
0111                  DO 11      I = 1,NP
0112                  COSPSI = CPSI(I)
0113                  CALL COVAR(COSPSI,HP,HQ,CPQ,IC)
0114                  CDD(I+1) = CPQ
0115                  CDD(NP1*I+1) = CDD(I+1)
0116               11 CONTINUE
0117                  WRITE(6,111) (I,IGRID(I),ALPHA(I),I=1,NP)
                C
                C               NOW FORM THE REMAINDER OF  CDD .
                C
0118                  ALT = ALPHA(NP)
0119                  DO 20       I1 = 2,NP1
0120                  DO 18       J1 = I1,NP1
                C
                C               COMPUTING THE COVARIANCE BETWEEN THE ITH AND JTH "RING AVERAGES" .
```

```
      C
            CST = 0.D0
            J = J1-1
            I = I1-1
            CCP = CPSI(I)*CPSI(J)
            SSP = SPSI(I)*SPSI(J)
            AL = ALPHA(J)
            ALON = AL/ALT
            NN = IGRID(J)
            DO 15       K = 1,NN
            K1 = (K-1)*ALON+1.0001
            IF(K1.GT.N4) GO TO 19
            SA = SSP*COSALF(K1)
            COSPS = CCP+SA
            CALL COVAR(COSPS,HP,HQ,CG1,IC)
            COSPS = CCP-SA
            CALL COVAR(COSPS,HP,HQ,CG2,IC)
            CST = CST+(CG1+CG2)*2.D0
            IF(K.EQ.1.OR.K1.EQ.N4) CST = CST-(CG1+CG2)
         15 CONTINUE
         19 RIJ = 1.D0/IGRID(J)
            CST = CST*RIJ
            IF(I.EQ.J) CST = CST+REGPAR+GNOISE*RIJ
            CDD(NP1*I+J1) = CST
            CDD (NP1*J+I1) = CST
         18 CONTINUE
         20 CONTINUE
            IC = 4
      C
      C             CREATE THE VECTOR  CSD  , THE R.H.S. OF THE "NORMALS" .
      C             PRINT THEM OUT.
      C
            WRITE(6,776)
        776 FORMAT(//' RIGHT HAND SIDES OF THE NORMAL EQUATIONS.'//)
            COSPSI = 1.D0
            CALL COVAR(COSPSI,HP,HQ,CPQ,IC)
            CSD(1) = CPQ*GAMMA
            DO 25      I = 2,NP1
            COSPSI = CPSI(I-1)
            CALL COVAR(COSPSI,HP,HQ,CPQ,IC)
            CSD(I) = CPQ*GAMMA
            WRITE(6,777) CSD(I)
        777 FORMAT(1X,G20.10)
         25 CONTINUE
      C
      C                 COMPUTE COVARIANCE OF DISTURBING POTENTIAL .
      C
            IC = 2
            COSPSI = 1.D0
            CALL COVAR(COSPSI,HP,HQ,CPQ,IC)
            DFIT(1) = CPQ*GAMMA**2
            WRITE(6,36) DFIT(1)
         36 FORMAT(//' COVARIANCE OF DIST. POTENTIAL : ',G20.10//)
      C
      C            SOLVE THE NORMAL EQUATIONS USING CONJUGATE GRADIENTS.
      C
            DO 30       I = 1,NP1
            RN(I) = CSD(I)
```

```
0168          30 CONTINUE
0169             IO3 = 6
0170             NRHS = 1
0171             CALL CGRADS(FVEC,PN,QPN,RN,ALPHAN,ITER,ITERMX,NP1,NRHS,IO3,CSD,FON
                2,DFIT,FN,CDD)
0172             WRITE(6,40) FON(1)
0173          40 FORMAT(//' GLOBAL R.M.S. OF OPTIMIZED ERROR : ',G20.10,' KCAL.M'
                2//)
           C
           C      APPLY CORRECTION FACTOR  1/(1+SUM(FVEC(I)))
           C
0174             SUM = 0.D0
0175             DO 42        I = 1,NP1
0176          42 SUM = SUM+FVEC(I)
0177             F2 = 1.D0+2.D6*SUM/RE
0178             CORMS = FON(1)/F2
0179             WRITE(6,43) CORMS
0180          43 FORMAT(//' R.M.S./(1+BETA) = ',G20.10//)
           C             CALCULATE AND PRINT OUT "RECOVERED" RHS'S OBTAINED BY
           C             MULTIPLYING SOLUTION OF NORMALS BY NORMAL MATRIX,
           C             TO SEE HOW MANY SIGNIFICANT FIGURES CAN BE RECOVERED,
           C             IN THIS WAY CHECKING THE CONVERGENCE OF THE C. GRADS.
           C             SOLUTION.
           C
0181             CALL MATVEC(NP1,FVEC,RN,CDD)
0182             WRITE(6,1110)
0183        1110 FORMAT(//' RECONSTRUCTED RIGHT HAND SIDES.'//)
0184             WRITE(6,1111) (I,RN(I),I=1,NP1)
0185        1111 FORMAT(1X,I5,G20.10)
0186             DO 46        I = 1,NP1
0187          46 FVEC(I) = FVEC(I)/F2
           C
0188             WRITE(6,45) (I,FVEC(I),I=1,NP1)
0189          45 FORMAT(//' RING OPTIMIZED WEIGHTS '//,5(I4,1X,G20.10))
           C
           C             PUNCH OUT THE CDD MATRIX, THE CSD VECTOR, AND THE
           C             OPTIMAL RING "WEIGHTS" (FVEC).
           C
0190             NP12 = NP1**2
0191             WRITE(7,1000) (CDD(I),I=1,NP12),(CSD(K),K=1,NP1)
0192             WRITE(7,1000) (FVEC(I),I=1,NP1)
0193        1000 FORMAT(8F10.5)
0194             STOP
0195             END
```

```
      SUBROUTINE CGRADS(YN,PN,QPN,RN,ALPHAN,ITER,ITERMX,NCOL,NRHS,IO3,
     2R1,FON,DFIT,FN,CDD)
C
C                THIS SUBROUTINE PERFORMS THE CONJUGATE GRADIENTS
C                OPTIMIZATION PROCEDURE.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(1),PN(1),QPN(1),RN(1),ALPHAN(1),R1(1),FON(1),DFIT(1)
     2,FN(1),CDD(1)
C
C                  INITIALIZE BEFORE ITERATION NO. 1 .
C
      NDIM = NRHS*NCOL
      TEN = 10.D0
      ITER = 0
      DO 3     J = 1,NRHS
      FON(J) = DSQRT(DABS(1.D0*DFIT(J)))
    3 CONTINUE
      PNQPN = 0.D0
      DO 1     I = 1,NCOL
      PN(I) = 0.D0
      QPN(I) = 0.D0
    1 CONTINUE
      IP = -NCOL
      DO 5       I = 1,NRHS
      IP = IP+NCOL
      DO 5       J = 1,NCOL
      YN(IP+J) = 0.D0
    5 CONTINUE
      IB = -NCOL
C
C              BEGINNING OF MAIN LOOP .
C
 1000 ITER = ITER+1
      DSUM = 0.D0
      DO 20      I = 1,NCOL
      DSUM = DSUM+RN(I)*QPN(I)
   20 CONTINUE
      IF(ITER.EQ.1) GO TO 21
      DFACT = DSUM/PNQPN
      GOTO 22
   21 DFACT = 1.D0
   22 DO 25        I = 1,NCOL
      PN(I) = RN(I)-DFACT*PN(I)
   25 CONTINUE
      NN = 1
      CALL MATVEC(NCOL,PN,QPN,CDD)
      PNQPN = 0.D0
      DO 30      I = 1,NCOL
      PNQPN = PNQPN+PN(I)*QPN(I)
   30 CONTINUE
C
C              UPDATE ALL YN, RIGHT HAND SIDE AFTER RIGHT HAND SIDE.
C
   35 ISM = -NCOL
      DO 100       NS = 1,NRHS
      ISM = ISM+NCOL
C
```

```
C       COMPUTE ALPHA FOR THE NS RIGHT HAND SIDE .
C
        DALF = 0.D0
        DO 40       I = 1,NCOL
        DALF = DALF+PN(I)*RN(ISM+I)
     40 CONTINUE
        ALPHAN(NS) = DALF/PNQPN
C
C               UPDATE THE GRADIENT OF EACH RIGHT HAND SIDE .
C
        DO 45       I = 1,NCOL
        RN(ISM+I) = RN(ISM+I)-ALPHAN(NS)*QPN(I)
     45 CONTINUE
C
C               UPDATE YN FOR EACH RIGHT HAND SIDE.
C
        DO 50       I = 1,NCOL
        YN(ISM+I) = YN(ISM+I)+ALPHAN(NS)*PN(I)
     50 CONTINUE
C
C               CALCULATE THE R.M.S.'S OF THE ERRORS.
C
        FUN = 0.D0
        DO 60        I = 1,NCOL
        FUN = FUN-YN(ISM+I)*(RN(ISM+I)+RI(ISM+I))
     60 CONTINUE
        FNEW = DFIT(NS)+FUN
        DRMS = DSQRT(1.D0*FNEW)
        DIFF = (FON(NS)-DRMS)/FON(NS)*100.D0
        FON(NS) = DRMS
C
C               PRINT OUT INFORMATION RELEVANT TO THE PRESENT ITERATION .
C
        IF(DIFF.GE.TEN) WRITE(IO3,66) ITER,DRMS,DIFF
     66 FORMAT(///' ITERATION NO. ',I5//,' OBS. EQNS. : RMS OF ERRORS',
      2 1X,G20.10,' PERCENT OF IMPROVEMENT IN RMS',G20.10/)
    100 CONTINUE
        IF(ITER.LT.ITERMX.AND.DIFF.GT.1.D-6) GO TO 1000
C
C               END OF MAIN LOOP .
C
        IF(DIFF.LT.TEN) WRITE(IO3,66) ITER,DRMS,DIFF
        RETURN
        END
```

```
      SUBROUTINE MATVEC(NP1,VEC,VM,CDD)
C
C      THIS SUBROUTINE MULTIPLIES A VECTOR BY A MATRIX.
C
       IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION VM(1),VEC(1),CDD(1)
      INC = -NP1
      DO 20      J = 1,NP1
      VM(J) = 0.D0
      INC = INC+NP1
      DO 15      I = 1,NP1
      VM(J) = VM(J)+CDD(INC+I)*VEC(I)
   15 CONTINUE
   20 CONTINUE
      RETURN
      END
```

```
0001           SUBROUTINE COVAR(PSI,HP,HQ,CPQ,IC)
          C                                                               00004500
          C       THIS SUBROUTINE AND FUNCTION  "F"  COMPUTE C. JEKELI'S
          C       TWO TERM COVARIANCE FUNCTIONS.
          C
0002           IMPLICIT REAL*8(A-H,O-Z)
0003           REAL*8 L1,L2,M1,M2,N1,N2,F,LPQ,L13,LL1,LL2                 00004600
0004            REAL*8 NMIN,N2MIN                                         00004700
0005           COMMON/N/ DVAR(30),NMOD
0006           COMMON/LEG/ POLS(30),NMIN(30),N2MIN(30)
0007           COMMON/LMN/P2,T /COV/A1,A2,R12,R22,A,B,C2
0008           DATA RE/6371000.D0/,GM/398601.D9/                          00004800
0009           CC1=0.                                                     00004900
0010           T = PSI                                                    00005000
0011           P2=1.5D0*T*T-.5D0
0012           RP = RE+HP                                                 00005200
0013           RQ=RE+HQ
0014           RPQ=RP*RQ                                                  00005400
0015           SIG=RE*RE/RPQ                                              00005500
0016           EFC2=C2*SIG**4*P2                                          00005600
0017           IF (A1 .EQ. 0.D0) GO TO 40                                 00005700
0018           S1=R12/RPQ                                                 00005800
0019           S12=S1*S1                                                  00005900
0020           L1=DSQRT(1.-2.*S1*T+S12)                                   00006000
0021           NA=A                                                       00006100
0022           FA=F(NA,S1)                                                00006200
0023           TERMA=FA-S1*(1./A+S1*(T/(A+1)+S1*P2/(A+2)))                00006300
0024           TERM1=S1*(1./L1-1.-S1*(T+S1*P2))                           00006400
0025           M1=1.-L1-S1*T                                              00006500
0026           GO TO (10,20,30,35),IC                                     00006600
0027        10 CC1=S1*(TERM1-(A+1.)*TERMA)                                00006700
0028           GO TO 40                                                   00006800
0029        20 N1=M1+2.*L1                                                00006900
0030           F1A=S1*(M1+T*S1*DLOG(2./N1))                               00007000
0031           CC1=R12*(F1A-S12*S1*P2-TERMA)/(A+1.)                       00007100
0032           GO TO 40                                                   00007200
0033        30 L13=L1**3                                                  00007300
0034           DFDS1=(M1+L1)/L13                                          00007400
0035           D2FDS1=2.*T/L13-3.*S1*(1.-T*T)/L13/L1/L1                   00007500
0036           CC1=S12*(TERM1*(A*(A-2.)-2.)+S1*(2.-A)*(DFDS1-(1.+2.*S1*T+3.*S12* 00007600
              *   P2))+S12*(D2FDS1-(2.*T+6.*S1*P2))-(A+1.)*(A-2.)**2*TERMA)/R12  00007700
0037           GO TO 40                                                   00007800
0038        35 CC1=R12*TERMA                                              00007900
0039        40 S2=R22/RPQ                                                 00008000
0040           S22=S2*S2                                                  00008100
0041           L2=DSQRT(1.-2.*S2*T+S22)                                   00008200
0042           M2=1.-L2-S2*T                                              00008300
0043           N2=M2+2.*L2                                                00008400
0044           NB=B                                                       00008500
0045           FB=F(NB,S2)                                                00008600
0046           F2B=S2*(M2*(3.*T*S2+1.)/2.+S22*(P2*DLOG(2./N2)+(1.-T*T)/4.)) 00008700
0047           TERMB=FB-S2*(1./B+S2*(T/(B+1)+S2*P2/(B+2)))                00008800
0048           GO TO (50,60,70,75),IC                                     00008900
0049        50 CC2=(F2B+(B+1.)*TERMB)*S2/(B+2.)                           00009000
0050           SCALE=1.D0                                                 00009100
0051           ADD=EFC2                                                   00009200
0052           GO TO 80                                                   00009300
0053        60 F1B=S2*(M2+T*S2*DLOG(2./N2))                               00009400
                                                                          00009500
```

```
      CC2=R22*(F2B/(B+2.)-(F1B-S2*S22*P2)/(B+1.)+TERMB/(B+1.)/(B+2.))
      GAM2=GM*GM/RPQ/RPQ
      ADD=RPQ*EFC2
      SCALE=(1.D-10)/GAM2
      GO TO 80
   70 TERM2=S2*(1./L2-1.-S2*(T+S2*P2))
      DFDS2=(M2+L2)/L2**3
      CC2=S22*(S2*(DFDS2-(1.+2.*S2*T+3.*S22*P2))+(4.-B)*TERM2+16.*F2B/
     *   (B+2.)+(B+1.)*(B-2.)**2*TERMB/(B+2.))/R22
      ADD=16.D0*EFC2/RPQ
      SCALE=1.D+8
      GO TO 80
   75 CC2=R22*(F2B/(B+2.)-TERMB/(B+2.))
      ADD=RPQ*EFC2
      SD=RP*GM/RQ/RQ
      SCALE=1.D-5/SD
   80 CPQ = (A1*CC1+A2*CC2)*SCALE
      CALL LEGPOL(NMOD,T)
      SS = 0.D0
      SI1 = S1**4
      SI2 = S2**4
      FACT = 1.D0
      DO 100      I = 3,NMOD
      IF(IC.EQ.2) FACT = RPQ/(I-1)**2
      IF(IC.EQ.4) FACT = RPQ/(I-1)
      SI1 = SI1*S1
      SI2 = SI2*S2
      SS = SS+((A1*(I-1)/(I+A)*SI1+A2*(I-1)/((I+B)*(I-2.D0))*SI2)
     2 -DVAR(I))*POLS(I)*FACT
  100 CONTINUE
      FOC = RPQ
      IF(IC.EQ.1) FOC = 1.D0
      SS = (SS-DVAR(2)*FOC*P2)*SCALE
      CPQ = CPQ-SS
      RETURN
      END
```

```
      FUNCTION F(I,S)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 L,F
      COMMON /LMN/P2,T
      L=DSQRT(1.-2.*S*T+S*S)
      F1=DLOG(1.+2.*S/(1.-S+L))
      F2=(L-1.+T*F1)/S
      IF (I .EQ. 1) F=F1
      IF (I .EQ. 2) F=F2
      IF (I .LE. 2) RETURN
      SIN = 1.D0/S
      DO 10 J=3,I
      F3 = (L+(2.D0*J-3.D0)*T*F2-(J-2.D0)*F1*SIN)/(S*(J-1.D0))
      F1=F2
   10 F2=F3
      F=F3
      RETURN
      END
```

```
0001          SUBROUTINE LEGPOL(NM,T)
        C
        C            THIS SUBROUTINE COMPUTES THE FIRTS NM LEGENDRE POLYNOMIALS.
        C
0002          IMPLICIT REAL*8 (A-H,O-Z)
0003           REAL*8 NM1N,N2M1N
0004          COMMON/LEG/ POLS(30),NM1N(30),N2M1N(30)
0005          PNM1 = 1.D0
0006          PN = T
0007          POLS(1) = T
0008          DO 10     N = 2,NM
0009          POLS(N) = NM1N(N)*PNM1+N2M1N(N)*PN*T
0010          PNM1 = PN
0011          PN = POLS(N)
0012       10 CONTINUE
0013          RETURN
0014          END
```

```
      SUBROUTINE RINCOV(CPSI,COV12)
C
C            SOUBROUTINE FOR COMPUTING THE COVARIANCE OF THE ERRORS OF
C            DISTURBING POTENTIALS PREDICTED AT THE CENTER OF TWO "CAPS"
C            PSI  DEGREEES APPART .
C
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 NMIN,N2MIN
      COMMON /LEG/ POLS(1000),NMIN(1000),N2MIN(1000)
      COMMON /C10/ F(30),CSD12(30),CDD12(30,30)
      COMMON /N/ DVAR(30),NMOD
      COMMON /COV/ A1,A2,R12,R22,A,B,C2
      DIMENSION FN(27000),CN(1000),R(1000),G(1000),GF(30)
C
C            INITIALIZING WHEN SUBR. IS CALLED THE FIRST TIME .
C
      DATA IHX/0/
      IF(IXH.NE.0) GO TO 1000
      PI = 3.141592654D0
      DRCONV = PI/180.D0
      RE = 6371000.D0
      RED = RE*1.D-6
      GAMMA = 0.978049D0
C
C           C. JEKELI'S "2L" COVARIANCE MODEL.
C
      A1 = 18.3906D0
      A2 = 658.6132D0
      S1 = 0.9943667D0
      S2 = 0.9048949D0
      A = 100.D0
      B = 20.D0
      C2 = 7.6D0
      R12 = RE**2*S1
      R22 = RE**2*S2
      DVAR(2) = 2720.D-12
      DVAR(3) = 6594.D-12
      DVAR(4) = 4564.D-12
      DVAR(5) = 7237.D-12
      DVAR(6) = 5703.D-12
      DVAR(7) = 6707.D-12
      DVAR(8) = 5685.D-12
      DVAR(9) = 5727.D-12
      DVAR(10) = 5118.D-12
      DVAR(11) = 4990.D-12
      DVAR(12) = 4415.D-12
      DVAR(13) = 4830.D-12
      DVAR(14) = 4458.D-12
      DVAR(15) = 4140.D-12
      DVAR(16) = 3864.D-12
      DVAR(17) = 3623.D-12
      DVAR(18) = 3410.D-12
      DVAR(19) = 3220.D-12
      DVAR(20) = 3051.D-12
C ***********************************************************************
      CAPSIZ = 10.D0
      NP = 25
      NMOD = 20
```

```
      NPOL = 1000
C ***************************************************************************
      NP1 = NP+1
      DVAR(1) = 0.D0
      DO 1      I = 2,NMOD
      DVAR(I) = DVAR(I)**2*(2*I+1)*(I-1)**2*(GAMMA*1.D6)**2
C ***************************************************************************
    1 CONTINUE
      CAPSIZ = CAPSIZ*DRCONV
      DZETA = CAPSIZ/NP
      DO 2      I = 1,NPOL
      NM1N(I) = -((I-1)*1.D0)/I
      N2M1N(I) = (2.D0*I-1.D0)/I
    2 CONTINUE
      CN(2) = DVAR(2)
      DO 4      I = 3,NPOL
      E = I+2.D0
      S1NP2 = S1**E
      S2NP2 = S2**E
      CN(I) = (A1*S1NP2*(I-1)/(I+A)+A2*S2NP2*(I-1)/((I-2)*(I+B)))
      IF(I.LE.NMOD) CN(I) = DVAR(I)
    4 CONTINUE
C
C            CREATING "RINGS ARRAY" OF LEG. POLS.
C
      INC = -NPOL
      ZETA = 0.D0
      DO 10      J = 1,NP
      INC = INC+NPOL
      ZETA = ZETA+DZETA
      T = DCOS(ZETA)
      CALL LEGPOL(NPOL,T)
      DO 5      I = 1,NPOL
    5 FN(INC+I) = POLS(I)
   10 CONTINUE
      IXH = -1
C
C            BEGIN CALCULATIONS .
C
 1000 CALL LEGPOL(NPOL,CPSI)
      DO 15      I = 1,NPOL
      G(I) = CN(I)*POLS(I)
   15 CONTINUE
C
C      COMPUTING THE ELEMENTS OF VECTOR  CSD12 .
C
      C0 = 0.D0
      R(1) = 0.D0
      DO 20      I = 2,NPOL
      R(I) = G(I)*RED/(I-1)
      C0 = C0+R(I)
   20 CONTINUE
      CSD12(1) = C0
      IN = -NPOL
      DO 25      J = 2,NP1
      IN = IN+NPOL
      S = 0.D0
      DO 24      I = 1,NPOL
```

```
               S = S+R(I)*FN(IN+I)
            24 CONTINUE
               CSD12(J) = S
            25 CONTINUE
      C
      C             NOW COMPUTE THE MATRIX  CDD12 .
      C
      C             FIRST ROW AND FIRST COLUMN FIRST .
      C
               C00 = 0.D0
               DO 30       I = 1,NPOL
            30 C00 = C00+G(I)
               CDD12(1,1) = C00
               IC = -NPOL
               DO 35       J = 2,NP1
               IC = IC+NPOL
               S = 0.D0
               DO 34       I = 1,NPOL
               S = S+G(I)*FN(IC+I)
            34 CONTINUE
               CDD12(1,J) = S
               CDD12(J,1) = S
            35 CONTINUE
      C
      C             NOW COMPUTE THE REMAINDER OF  CDD12  .
      C
               DO 50       J = 2,NP1
               IXX = (J-2)*NPOL
               DO 38         II = 1,NPOL
            38 R(II) = G(II)*FN(IXX+II)
               IX = IXX-NPOL
               DO 45         I = J,NP1
               IX = IX+NPOL
               SST = 0.D0
               DO 40             K = 1,NPOL
               SST = SST+R(K)*FN(IX+K)
            40 CONTINUE
               CDD12(J,I) = SST
               CDD12(I,J) = SST
            45 CONTINUE
            50 CONTINUE
               RETURN
               END
```