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KINEMATICAL GEODESY II

by

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ABSTRACT

This report is a continuation of a previous OSU report, "Kinematical Geodesy" (1967). Part A gives basic principles and the theoretical foundations for the separation of gravitation and inertia by a combined accelerometer-gradiometer system, with applications to aerial gravimetry and to inertial navigation. In Part B, proposed methods for the geodetic use of second-order gradients are briefly described and evaluated. The new technique of least-squares collocation avoids the shortcomings of those methods. The application of this technique to the use of gradients for the determination of the gravity field and of spherical harmonics is investigated.

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FOREWORD

This report was prepared by Helmut Moritz, Professor, Technische Hochschule Graz, and Adjunct Professor, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. F19628-69-C-0127, OSURF Project No. 2758, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Air Force Systems Command, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen W. Williams and Mr. Bela Szabo, Project Scientists.

KINEMATICAL GEODESY II

Introduction

In an earlier report (Moritz, 1967) we have investigated theoretical questions related to the use of moving instruments for the measurement of elements of the gravitational field, such as airborne gravimeters or gradiometers. The main problem is the separation of gravitation and inertia, the extraction of the gravitational "signal" form the inertial "noise". We have seen that such a separation is rigorously possible for the second and higher derivatives of the potential, but not for the gravity vector itself. In the latter case, in aerial gravimetry, one is, therefore, obliged to reduce the effect of inertial noise as much as possible by using some statistical filtering. The fact that a statistical separation of gravitation and inertia can never be perfect, is mainly responsible for the fact that the accuracy of aerial gravimetry is considerably inferior to the sensitivity of the instruments themselves.

In Part A of the present report we shall describe the principles of a rigorous separation of gravitation and inertia for the gravity vector itself. This can be done by simultaneously measuring the first and second derivatives, that is, by combining a gravimeter, or accelerometer, with a gradiometer.

In (Moritz, 1967) we have also studied how measured second-order gradients can be used in geodesy. We have seen that geodetically important quantities, such as deflections of the vertical or geoidal heights, can be derived from these measurements either by line integration (somewhat similar as in astrogeodetic leveling) or by global integration, using formulas analogous to Stokes' integral.

Unfortunately, such integral formulas have severe shortcomings, which make their practical application hardly feasible:

1. Second-order gradients are much more irregular than gravity anomalies, so that interpolation is difficult.

2. For a Stokes'-type formula global coverage would be necessary.

3. The available information is very incompletely used, because only one of the five independent components of the second-order gradient tensor enters into a Stokes'-type integral formula.

4. It is impossible to combine second-order gradients with other data such as first-order gradients or gravity anomalies, in a simple and well defined way, and to adjust for measuring errors.

Since the first report on Kinematical Geodesy was written, however, a least-squares method for estimating the terrestrial gravity field (least-squares collocation) was developed (Krarup, 1968, 1969; Moritz, 1970a), which avoids these shortcomings and permits the optimal use of heterogeneous data for the determination of the earth's figure and its gravity field.

In Part B of the present report we shall first discuss proposed methods for the geodetic use of gradients and then apply least-squares collocation to the determination of the gravitational field from measured gradients. This includes also the determination of spherical harmonics by satellite gradiometry.

PART A

SEPARATION OF GRAVITATION AND INERTIA

1. First and Second Order Gradients

Let the gravitational potential of the earth be denoted by V. Then the first partial derivatives, or first-order gradients, $V_x = \partial V/\partial x$ etc. form a vector

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}, \tag{1-1}$$

which is the vector of gravitational force on a unit mass.

Adding to V the potential of centrifugal force of the earth's rotation, we get the gravity potential W. The vector

$$\begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} = \underline{g} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} + \text{centrifugal force} \tag{1-2}$$

is the gravity vector. Since the centrifugal force is given by a simple analytical expression and can be considered as known, the determination of gravitation (1-1) is equivalent to the determination of gravity (1-2).

We have

$$\begin{aligned} W_x &= g \cos \alpha , \\ W_y &= g \cos \beta , \\ W_z &= g \cos \gamma , \end{aligned} \tag{1-3}$$

where

$$g = \sqrt{W_x^2 + W_y^2 + W_z^2} \tag{1-4}$$

is gravity and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines defining the direction of the vertical or plumb line.

The measurement of the vector (1-2) thus consists in the determination of g and of the direction of the plumb line. Gravity g is measured by gravimetry. The terrestrial technique for determining the direction of the vertical is to define it by means of a spirit level and to refer it to a global rectangular coordinate system by means of astronomical observations (latitude and longitude).

If the z-axis is made to coincide with the normal to the reference ellipsoid and if the x and y axes are suitably oriented, then, obviously,

$$\xi = -\frac{W_x}{g}, \quad \eta = -\frac{W_y}{g} \tag{1-5}$$

are nothing else than the components of the deflection of the vertical. Such an orientation of the coordinate system can always be achieved by a coordinate transformation so that deflections of the vertical can be computed from first-order gradients.

In aerial measurements, the direction of the coordinate axes is maintained by gyroscopic stabilization, and the three components of the gravity vector can be measured by three accelerometers. The accelerometer output will be affected by inertial disturbances, which must be removed as much as possible by statistical filtering.

The best-known technique falling under this general principle is aerial gravimetry, where the vertical component of the gravity vector is measured by a gravimeter or a vertical accelerometer, which is basically the same; cf. (Szabo and Anthony, 1971). For a suggestion for determining deflections of the vertical by a similar principle cf. (Bradley, 1970).

The second-order derivatives, or second-order gradients, $V_{xx} = \partial^2 V/\partial x^2$ etc. form a symmetric matrix or tensor

$$\begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix} . \tag{1-6}$$

This tensor contains five independent quantities: because of symmetry we have

$$V_{yx} = V_{xy}, \qquad V_{zx} = V_{xz}, \qquad V_{zy} = V_{yz}, \tag{1-7}$$

and in free space, Laplace's equation

$$V_{xx} + V_{yy} + V_{zz} = 0 \tag{1-8}$$

holds, so that the 9 components of the tensor (1-6) must satisfy 4 conditions.

Instruments measuring second-order gradients are called gradiometers. A stationary instrument of this kind is the well-known torsion balance, cf. (Mueller, 1964); for recent developments in airborne and satellite-borne instruments cf. (Anthony, 1971), (Forward, 1971), (Savet, 1970), (Trageser, 1970).

One of the most interesting features of gradiometer measurements is that second-order gradients measured by moving instruments are purely gravitational, inertial disturbances having no effect on them provided the coordinate axes are gyroscopically stabilized. For an investigation of this problem, covering also the general-relativistic aspects, see (Moritz, 1967).

<u>Anomalous Gradients</u>. - It is convenient to approximate the gravity potential W by a given simple function U, called normal gravity potential and representing the gravity potential of an equipotential ellipsoid (Heiskanen and Moritz, 1967, sec. 2-13). The difference

$$T = W - U \tag{1-9}$$

is called disturbing potential, or anomalous potential. It is a harmonic function outside the earth, since the centrifugal part in W and U are equal and, therefore, cancel in (1-9). For the same reason we may also write

$$T = V - \overline{V} , \tag{1-10}$$

denoting by V the earth's gravitational potential and by $\overline{V}$ the normal gravitational potential, which is the normal gravity potential U minus the potential of the centrifugal force.

We may form first and second order gradients of the normal potential U (or $\overline{V}$, respectively), and subtract them from the measured gradients. In this way we obtain the anomalous gradients

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} . \tag{1-11}$$

If the position $P(x,y,z)$ of the measuring instrument is determined with respect to the center of the earth, then U and its derivatives can be determined at the observation point P itself, so that the quantities (1-11) can be directly formed. This is, at least approximately, the case in satellite gradiometry where the satellite position is determined by tracking, and in aerial gradiometry if position is determined by inertial navigation connected to points whose geocentric position is known.

If, on the other hand, the vertical position of the aircraft is determined by measuring the height above ground, then it is more appropriate to consider the normal gradients as referring to a point Q which is situated on the plumb line of P and whose normal potential U is the same as the actual potential W of P, that is, $U_Q = W_P$. See (Heiskanen and Moritz, 1967, pp. 245-246, figure 6-3). Consider, for instance, the second vertical gradient W_{zz}, letting the z-axis coincide with the vertical through P. Then

$$T_{zz} = W_{zz}(P) - U_{zz}(P) , \tag{1-12}$$

whereas now we rather compute a quantity

$$T'_{zz} = W_{zz}(P) - U_{zz}(Q) . \tag{1-13}$$

The difference between the two quantities is

$$T'_{zz} - T_{zz} = U_{zz}(P) - U_{zz}(Q) \doteq U_{zzz} N_P \quad , \tag{1-14}$$

where

$$N_P = PQ$$

is the vertical separation of geop and spherop at P; cf. Fig. 6-3, loc. cit.

For an estimate, assume a spherical normal earth. Then

$$U = \frac{kM}{r} , \qquad U_r = - \frac{kM}{r^2} , \qquad U_{rrr} = - \frac{6kM}{r^4} ,$$

where k is the gravitational constant, M is mass of the earth, and r is the radius vector. Near the surface of the earth we have approximately

$$r = R = 6371 \text{ km},$$

$$\left| U_r \right| = \frac{kM}{R^2} = G = 980 \text{ gals} ,$$

$$\left| U_{zzz} \right| = \left| U_{rrr} \right| = \frac{6G}{R^2} \doteq 1.5 \times 10^{-7} \text{ mgal/m}^2 ,$$

so that, for $N_P = 100$ m ,

$$\left| T'_{zz} - T_{zz} \right| \doteq 1.5 \times 10^{-5} \text{ mgal/m} = 0.15 \text{ Eötvös}.$$

Since 100 m is a maximum value for N_P, this difference is well below the expected aerial measuring accuracy of about 1 Eötvös . Differences in other second-order gradients are even much less because of the near-spherical symmetry.

Thus we may probably safely put

$$T'_{xx} = T_{xx}, \; . \; . \; . \; , T'_{zz} = T_{zz} \tag{1-15}$$

in most cases.

As for the first-order gradients, the fact just mentioned affects vertical gradients and is responsible for the distinction between gravity anomaly and gravity disturbance and for the use of the former in aerial gravimetry. This, however, is well known and need not be discussed here, cf. (Heiskanen and Moritz, 1967, pp. 245-246).

Transformation of Gradients.- Let an orthogonal coordinate transformation between two rectangular coordinate systems xyz and $\xi\eta\zeta$ be given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} , \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = A^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} , \tag{1-16}$$

where A is an orthogonal matrix and A^T is its transpose. Then the first-order gradients transform like

$$\begin{bmatrix} T_\xi \\ T_\eta \\ T_\zeta \end{bmatrix} = A^T \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} , \tag{1-17}$$

and the second-order gradients transform like

$$\begin{bmatrix} T_{\xi\xi} & T_{\xi\eta} & T_{\xi\zeta} \\ T_{\eta\xi} & T_{\eta\eta} & T_{\eta\zeta} \\ T_{\zeta\xi} & T_{\zeta\eta} & T_{\zeta\zeta} \end{bmatrix} = A^T \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} A. \tag{1-18}$$

For a derivation cf. (Moritz, 1967 , sec. 1.5).

Let us now consider spherical polar coordinates r(radius vector), θ (polar distance), and λ (longitude). They are related to rectangular coordinates xyz by

$$
\begin{aligned}
x &= r \sin\theta \, \cos\lambda , \\
y &= r \sin\theta \, \sin\lambda , \\
z &= r \cos\theta .
\end{aligned}
\tag{1-19}
$$

Here the z-axis is the axis of rotation of the earth, and the x-axis passes through the Greenwich meridian; the origin is at the earth's center of mass.

Together with this global system xyz, let us introduce a local rectangular coordinate system $\xi\eta\zeta$ as follows. The origin is at the point P whose coordinates are x, y, z or r, θ, λ. The ζ-axis coincides with the radius vector, the ξ-axis points north, the η-axis points east.

The first-order gradients in this system are given by

$$
\begin{aligned}
T_\xi &= -\frac{\partial T}{r\partial\theta} = -\frac{1}{r} T_\theta , \\
T_\eta &= \frac{\partial T}{r \sin\theta d\lambda} = \frac{1}{r\sin\theta} T_\lambda , \\
T_\zeta &= \frac{\partial T}{\partial r} = T_r .
\end{aligned}
\tag{1-20}
$$

The relation between the derivatives with respect to x,y,z and to r, θ,λ is found as follows. By the usual rules of partial differentiation we have

$$
T_r = T_x x_r + T_y y_r + T_z z_r
$$

and similar for T_θ and T_λ. The derivatives x_r etc. are obtained from (1-19), for instance

$$
x_r = \sin\theta \, \cos\lambda .
$$

In this way we find

$$
\begin{aligned}
T_r &= T_x \sin\theta\cos\lambda + T_y \sin\theta\sin\lambda + T_z \cos\theta, \\
T_\theta &= T_x r\cos\theta\cos\lambda + T_y r\cos\theta\sin\lambda - T_z r\sin\theta, \\
T_\lambda &= -T_x r\sin\theta\sin\lambda + T_y r\sin\theta\cos\lambda .
\end{aligned}
\tag{1-21}
$$

Substitution into (1-20) gives

$$
\begin{bmatrix} T_\xi \\ T_\eta \\ T_\zeta \end{bmatrix} = A^T \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}
$$

with

$$
A^T = \begin{bmatrix} -\cos\theta\cos\lambda & -\cos\theta\sin\lambda & \sin\theta \\ -\sin\lambda & \cos\lambda & 0 \\ \sin\theta\cos\lambda & \sin\theta\sin\lambda & \cos\theta \end{bmatrix}, \tag{1-22}
$$

which determines the transformation matrix A.

The first derivatives with respect to r, θ, λ are expressed in terms of rectangular gradients by

$$
\begin{aligned}
T_r &= T_\zeta , \\
T_\theta &= -r T_\xi , \\
T_\lambda &= r\sin\theta\, T_\eta .
\end{aligned}
\tag{1-23}
$$

For the second derivatives we have

$$
\begin{aligned}
T_{rr} &= T_{\zeta\zeta} , \\
T_{r\theta} &= -r T_{\xi\zeta} - T_{\xi} , \\
T_{r\lambda} &= r \sin\theta\, T_{\eta\zeta} + \sin\theta\, T_{\eta} , \\
T_{\theta\theta} &= r^2 T_{\xi\xi} - r T_{\zeta} , \\
T_{\theta\lambda} &= -r^2 \sin\theta\, T_{\xi\eta} + r\cos\theta\, T_{\eta} , \\
T_{\lambda\lambda} &= r^2 \sin^2\theta\, T_{\eta\eta} + r\sin\theta\cos\theta\, T_{\xi} - r\sin^2\theta\, T_{\zeta} .
\end{aligned}
\tag{1-24}
$$

These equations are readily derived by differentiating equations (1-21) with regard to r, θ, λ, using

$$T_{xr} = T_{xx} x_r + T_{xy} y_r + T_{xz} z_r , \quad \text{etc.},$$

and expressing the derivatives with respect to x, y, z by the derivatives with respect to ξ, η, ζ by means of (1-17) and (1-18) with (1-22).

Conversely we have

$$
\begin{aligned}
T_{\xi\xi} &= \frac{1}{r^2} T_{\theta\theta} + \frac{1}{r} T_r , \\
T_{\xi\eta} &= -\frac{1}{r^2 \sin\theta} T_{\theta\lambda} + \frac{\cos\theta}{r^2 \sin^2\theta} T_{\lambda} , \\
T_{\xi\zeta} &= -\frac{1}{r} T_{r\theta} + \frac{1}{r^2} T_{\theta} , \\
T_{\eta\eta} &= \frac{1}{r^2 \sin^2\theta} T_{\lambda\lambda} + \frac{1}{r} T_r + \frac{1}{r^2} \cot\theta\, T_{\theta} , \\
T_{\eta\zeta} &= \frac{1}{r\sin\theta} T_{r\lambda} - \frac{1}{r^2 \sin\theta} T_{\lambda} , \\
T_{\zeta\zeta} &= T_{rr} .
\end{aligned}
\tag{1-25}
$$

These relations are found by solving (1-24) with respect to $T_{\xi\xi}$ etc. and substituting (1-20).

The dominant terms in (1-24) and (1-25) are the first terms on the right-hand side. The following terms, representing effects of first-order gradients, are usually below 1 Eötvös. For instance, take the second term on the right-hand side of the first equation of (1-25). For $r = 6371$ km and $T_r = 100$ mgals it amounts to

$$\frac{1}{r} T_r = \frac{100 \text{ mgal}}{6371 \text{ km}} \doteq 0.016 \text{ mgal/km}$$

$$= 0.16 \text{ Eötvös} .$$

We have given these formulas for later reference. The meaning of the coordinate systems introduced is as follows. The system xyz is customarily used as global coordinate system in geodesy. The gradients referred to this system can be measured if the inertial platform carrying the accelerometer or gradiometer is made to maintain a fixed orientation in inertial space, the direction of the instrument axes coinciding with the directions of the xyz axes.

Usually, however, the instrument axes are made to slowly rotate in such a way as to always coincide with the normal to the reference ellipsoid, the tangent to the ellipsoidal parallel, respectively. For the quantities of the anomalous gravity field, such as the disturbing potential, gravity anomalies, deflections of the vertical, or anomalous gradients, the spherical approximation may be used, which consists in neglecting the flattening in ellipsoidal formulas so that formally spherical formulas are obtained. Loosely speaking we may say that the reference ellipsoid is formally replaced by a sphere (Heiskanen and Moritz, 1967, sec. 2-14). Then spherical coordinates r, θ, λ may be conveniently used; and the instrument axes coincide with the axes ξ, η, ζ as defined above.

For a summary of relevant aspects of inertial navigation cf. (Schultz and Winokur, 1969, pp. 4892-5).

Relations Between First and Second Order Gradients. - From the second derivatives it is possible to obtain the first derivatives by an integration along the flight path. For instance,

$$V_x = (V_x)_o + \int_{P_o}^{P} (V_{xx} dx + V_{xy} dy + V_{xz} dz) . \tag{1-26}$$

Here $(V_x)_o$ refers to an initial point along the flight path, and V_x refers to a current point P. If the flight path is given as a function of the time t:

$$x = x(t) , \quad y = y(t) , \quad z = z(t), \tag{1-27}$$

then

$$dx = \dot{x} dt , \quad \text{etc.,}$$

the dot denoting differentiation with respect to time, and we obtain

$$\begin{aligned} V_x &= (V_x)_o + \int_{P_o}^{P} (V_{xx} \dot{x} + V_{xy} \dot{y} + V_{xz} \dot{z}) dt , \\ V_y &= (V_y)_o + \int_{P_o}^{P} (V_{yx} \dot{x} + V_{yy} \dot{y} + V_{yz} \dot{z}) dt , \\ V_z &= (V_z)_o + \int_{P_o}^{P} (V_{zx} \dot{x} + V_{zy} \dot{y} + V_{zz} \dot{z}) dt . \end{aligned} \tag{1-28}$$

In order to apply these formulas, it is necessary to know the flight path as a function of time. In the next section we shall see how this can be achieved using a combined accelerometer-gradiometer system.

2. Separation of Gravitation and Inertia by Using a Combined Accelerometer-Gradiometer System

Assume that all first-order and all second-order gradients are simultanously measured in an airplane, the instrument axes, now denoted by x_1, x_2, x_3, being inertially stabilized in such a way as to maintain a fixed orientation in space.

In this case, the second-order gradient tensor measured is purely gravitational, the inertial part being zero (Moritz, 1967, p. 28); it is, therefore, given by (1-6). The measured first-order gradient vector, however, is not (1-1) because it is affected by inertial disturbances.

By equation (67) of (Moritz, 1967) we have

$$\frac{\partial V}{\partial x_i} = f_i^* - (w_{ik} w_{jk} + \dot{w}_{ij}) x_j + \ddot{b}_i \,. \tag{2-1}$$

Here all subscripts i, j, k assume values $1, 2, 3$; the Einstein summation convention is used. The vector f_i^* is the total measured force; $\ddot{b}_i$ is the inertial disturbance, the second time derivative of the position vector b_i of the origin of the local frame. In the present case, the vector b_i consists of the three coordinates of the aircraft (more precisely, of the center of mass of the measuring instrument) in a fixed coordinate system. The tensor w_{ij} describes the rotation of the local frame with respect to the fixed coordinate system; since we have assumed inertial stabilization, w_{ij} is zero.

Thus (2-1) reduces to

$$\frac{\partial V}{\partial x_i} = f_i^* + \ddot{b}_i \,. \tag{2-2}$$

Let us put for the gravitational gradients

$$\frac{\partial V}{\partial x_i} = V_i \,, \qquad \frac{\partial^2 V}{\partial x_i \partial x_j} = V_{ij} \tag{2-3}$$

and for the measured force,

$$f_i^* = F_i \quad . \tag{2-4}$$

Furthermore, let us introduce the velocity components

$$\dot{b}_i = u_i \; . \tag{2-5}$$

Then (2-2) becomes

$$V_i = F_i + \dot{u}_i \quad . \tag{2-6}$$

On the other hand, V_i may be obtained by integration of V_{ij}. In our new notation, (1-28) is written concisely as

$$V_i = (V_i)_o + \int_{P_o}^{P} V_{ij} \dot{b}_j \, dt \quad , \tag{2-7}$$

since the coordinates of the aircraft, b_i, have been denoted by x, y, z in section 1.

On introducing the velocity components (2-5) and denoting the times corresponding to positions P_o and P by t_o and t, eq. (2-7) becomes

$$V_i = (V_i)_o + \int_{t_o}^{t} V_{ij} u_j \, dt \quad . \tag{2-8}$$

Eliminating V_i between (2-6) and (2-8) we have

$$F_i + \dot{u}_i = (V_i)_o + \int_{t_o}^{t} V_{ij} u_j \, dt \; , \tag{2-9}$$

and differentiation with respect to t results in

$$\ddot{u}_i - V_{ij} u_j + \dot{F}_i = 0, \tag{2-10}$$

which is a second-order linear differential equation, or rather a system of linear differential equations, for the velocity u_i.

To get a clear picture of the basic equation (2-10), we shall write it explicitly:

$$\begin{aligned} \ddot{u}_x - (V_{xx}u_x + V_{xy}u_y + V_{xz}u_z) + \dot{F}_x &= 0, \\ \ddot{u}_y - (V_{yx}u_x + V_{yy}u_y + V_{yz}u_z) + \dot{F}_y &= 0, \\ \ddot{u}_z - (V_{zx}u_x + V_{zy}u_y + V_{zz}u_z) + \dot{F}_z &= 0, \end{aligned} \tag{2-11}$$

where u_x, u_y, u_z are the velocity components and F_x, F_y, F_z are the components of the measured force.

The quantities F_i and V_{ij} being given by measurement, eq. (2-10) may be solved by the usual numerical methods, for instance by a Runge-Kutta procedure, to get u_i.

This integration may be performed in real time.

In this way we have indeed effected a separation between gravitation and inertia:

The gravitational first-order gradient V_i, free from inertial noise, is obtained from (2-6) or, alternatively, from (2-8).

The inertial acceleration, free from gravitational disturbances, is obtained as

$$\ddot{b}_i = \dot{u}_i \ . \tag{2-12}$$

It may be twice integrated with respect to time to give the position vector b_i in a global Cartesian coordinate system. Alternatively we have simply

$$b_i = (b_i)_o + \int_{t_o}^{t} u_i \, dt \ . \tag{2-13}$$

Thus our combined accelerometer-gradiometer system acts at the same time as a purely gravitational gravimeter and a true inertial navigation system that is not affected by the gravitational field.

Additional Remarks.- In order to make the basic concepts transparent, we have introduced two simplifications which can, however, easily be taken into account.

First, with current gravimeter (or accelerometer) and gradiometer systems, the force F_i or the quantities V_{ij} are not the direct output when they are functions of time. In fact, the instruments may be considered as linear oscillating systems, for which the output is obtained as a linear operation on the input (F_i or V_{ij}):

$$\psi = L\varphi , \tag{2-14}$$

where ψ is the output and φ is the force F_i or gradient V_{ij} to be measured. If the linear operator can be inverted, then φ is obtained as

$$\varphi = L^{-1}\psi . \tag{2-15}$$

The form of the operators L and L^{-1} is characteristic of the measuring system under consideration.

Secondly, we have already mentioned in sec. 1 that the directions of the instrument axes, instead of being kept fixed, may be rotated in a prescribed way such that, for instance, one axis always coincides with the normal to the reference ellipsoid. Then the rotation tensor w_{ij}, instead of being zero, will be given, so that its effect can be fully taken into account using formulas such as (2-1). For a special case cf. (Hansen, 1971).

Finally we remark briefly on the relation of the present separation of gravitation and inertia to the principle of equivalence of these forces. As we have seen in (Moritz, 1967, p. 56), it is in fact impossible to separate gravitation and inertia as long as the force acting at a point only is considered. As soon as we have a region in space, even an arbitrarily small one, however, such a separation becomes feasible.

Thus we might use five independent components of the gradient tensor, e.g.

$$T_{xx},\ T_{xy},\ T_{yy},\ T_{xz},\ T_{yz}, \tag{2-21}$$

or the three components of the gradient vector plus two independent components of the gradient tensor, e.g.

$$T_x,\ T_y,\ T_z,\ T_{xx},\ T_{xy}, \tag{2-22}$$

or the potential plus four other independent quantities, e.g.

$$T,\ T_x,\ T_y,\ T_{xx},\ T_{xy}. \tag{2-23}$$

It would, however, be uneconomical to use only T_x, T_y, T_z or, a fortiori, only T.

This fact imposes a strong requirement on the geodetic computation method using these data: in order to take into account all available information, it should be able to use simultaneously five independent quantities, and it should use them in such a way that the result is the same regardless of which system of five independent quantities, e.g. (2-21) or (2-22) or (2-23) is taken as input.

In sec. 4 we shall present a method that satisfies these requirements.

In the present case, we are given the force along a line. This fact by itself is not yet sufficient for a separation, which becomes only possible through the additional measurement of the second-order gradients.

Here we have restricted ourselves to an approach through classical mechanics because analyses such as that in (Moritz, 1967) show that, to an extremely high accuracy, it gives the same result as the more rigorous but also more complicated approach through the General Theory of Relativity.

The Observational Data. - The measuring system under consideration gives as output the second-order gradient tensor

$$\begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix} , \tag{2-16}$$

the first-order gradient vector

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} , \tag{2-17}$$

and even the potential V: by integrating

$$dV = V_x dx + V_y dy + V_z dz$$

we get

$$V = V_0 + \int_{t_0}^{t} (V_x u_x + V_y u_y + V_z u_z) dz , \tag{2-18}$$

since the velocity components $dx/dt = u_x$ etc. are known.

This presupposes, of course, that initial values $(V_x)_o$, $(V_y)_o$, $(V_z)_o$ and V_o at some initial point P_o are given.

Likewise we obtain the position of the aircraft,

$$x = b_1 , \qquad y = b_2 , \qquad z = b_3 , \tag{2-19}$$

as a function of the time t, again presupposing suitable initial values that were required for the integration.

As we have seen in sec. 1, it is convenient to subtract from the quantities V_{ij} , V_i and V their normal values, corresponding to a normal gravity potential U, to obtain anomalous gradients and the anomalous potential:

$$\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} , \quad \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} , \quad T . \tag{2-20}$$

We remark that, since position is determined by inertial navigation, the normal values will refer to the same point P as the measured values, so that (T_x, T_y, T_z) represents the gravity disturbance vector; see sec. 1 and (Heiskanen and Moritz, 1967, pp. 227 and 245-6).

Which of the quantities (2-20) are used, will depend on the geodetic computation method chosen. For instance, we might use (T_x , T_y , T_z) to compute deflections of the vertical, to be used for astronomical leveling.

This would, however, be uneconomical because the available information is not fully used. In fact there are 5 independent quantities (2-20) since the second-order gradient tensor contains 5 independent quantities (see sec. 1) and the gradient vector and T are obtained by integration of this tensor.

PART B

GEODETIC USE OF MEASURED GRADIENTS

3. Review of Proposed Methods

First-order gradients are equivalent to gravity anomalies (or gravity disturbances) and deflections of the vertical, as we have remarked in sec. 1, so that their geodetic use may, in general, be reduced to problems familiar in physical geodesy.

Still, this is not the optimal procedure, especially if all first-order gradients are measured simultaneously. The main reason is that the familiar methods of physical geodesy use either the gravity anomaly or the deflection of the vertical, a combination of the two types of data not being directly possible. The simultaneous use of all three components raises new problems.

Such novel features are particularly prominent in the geodetic use of second-order gradients, which is also rather more difficult. We shall, therefore, limit ourselves to considering second-order gradients.

Various methods for their geodetic use have been proposed and discussed, e.g. in (Moritz, 1967). We shall now try to give a brief evaluation of proposed methods.

a) Line Integration. - We may integrate second-order gradients along the flight path to obtain first-order gradients. The basic formula is (1-28) or, written in a more gradient tensor are measured and if the velocity components u_j are known, either by external measurements of the flight path or by the method described in sec. 2. The first-order gradients so obtained are converted to gravity anomalies (or gravity disturbances) and deflections of the vertical, which are used in the conventional way (it is easy to derive an integral formula analogous to Stokes' integral but using gravity disturbances instead of gravity anomalies).

The advantage of this method is the reduction of the problem to problems familiar in physical geodesy. A disadvantage is that the available information is not completely used: The three components of the gradient vector are computed from the five independent components of the gradient tensor, so that two independent elements are not used.

Furthermore, as we have seen above, a combination of the data furnished by the gradient vector is not easily possible.

b) Torsion-Balance Type Computations. - The torsion balance invented by Eötvös, historically the first and still the only instrument in actual geodetic use, is measuring, not all components of the gradient tensor, but only the quantities

$$V_{xy} \quad \text{and} \quad V_{yy} - V_{xx} \tag{3-1}$$

(with, possibly, V_{xz} and V_{yz} in addition), the xy-plane being horizontal. The quantities (3-1) may be used to calculate deflections of the vertical by an integration method whose mathematical structure is clarified in (Moritz, 1967, sec. 1.2).

This method, classical and relatively widely applied, is appropriate to the torsion balance. For instruments that measure all components, the available information is only partially used. Furthermore, in this method the lines of integration do not coincide with the flight path, so that problems of interpolation and vertical reduction occur similar to those to be discussed for the next method.

c) Global Integration. - This was investigated in (Moritz, 1967, sec. 1.3). The relevant formula is equation (32) of that report:

$$T = \frac{R^2}{2\pi} \iint_{\sigma} T_{rr} \, S_1(\psi) \, d\sigma \quad . \tag{3-2}$$

This integral formula is completely analogous to the well-known Stokes formula: it expresses the anomalous potential T in terms of the second-order vertical (radial) gradient T_{rr} just as Stokes' formula expresses T in terms of the gravity anomaly Δg. The function $S_1(\psi)$ is a known function, R is a mean radius of the earth, $d\sigma$ is the element of solid angle, and the integration is to be extended over the full solid angle σ, that is, over the whole earth's surface.

This condition, that the integration be extended over the whole earth, is the more stringent as the effect of the remote zones on the integration decreases even less

Also the statistical meaning of the formal adjustment procedure is questionable. The α_{nm} and β_{nm} may be almost as irregular as the effect of the measuring errors, and almost as small. It would, therefore, be more satisfactory to have a method that takes this fact into account in a statistically well-founded manner.

Satellite gradiometry will probably be able to give harmonics of higher degrees than does orbital analysis; it seems, therefore, proper to combine these two techniques, possibly with other techniques such as satellite altimetry and satellite-to-satellite tracking.

This brief discussion of different methods shows some features that arise particularly in the geodetic use of second-order gradients:

1. A large amount of information is obtained simultaneously at the same point: the five independent components of the gradient tensor.

2. There are difficulties in the application of conventional horizontal interpolation and vertical reduction techniques owing to the irregular and fluctuating nature of higher gradients. To better overcome these difficulties, the additional information just mentioned should be used in an appropriate way.

3. By its very nature, gradiometer data are better suited to give fine details than to provide the large features. They are, therefore, best combined with other data. This directly calls for a method that is able to combine heterogeneous data in a natural way.

The classical methods of physical geodesy--astrogeodetic, gravimetric, dynamic satellite techniques--are always based on data of a single type. Attempts at combining them are more or less ad hoc. This is also true for methods using gradiometer data as discussed above, since they are modeled after those classical methods.

4. Statistical estimation and adjustment techniques have never penetrated very profoundly into classical physical geodesy. Adjustment techniques and methods of error theory have not been incorporated there in an entirely satisfactory and natural way. The same holds for the above-mentioned methods, which is particularly serious here because random errors may be comparable in magnitude to the quantity to be measured, and systematic effects have to be carefully eliminated.

(1-25) or, alternatively, the series for T_{xx} , T_{xy} , etc. by (1-25) and

$$\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} = A \begin{bmatrix} T_{\xi\xi} & T_{\xi\eta} & T_{\eta\eta} \\ T_{\eta\xi} & T_{\eta\eta} & T_{\eta\zeta} \\ T_{\zeta\xi} & T_{\zeta\eta} & T_{\zeta\zeta} \end{bmatrix} A^T , \tag{3-4}$$

which follows from (1-18), the matrix A being given by (1-22). In this way we may express all measured second-order gradients as series which contain the coefficients α_{nm} and β_{nm}, for instance,

$$\begin{aligned} T_{xx} &= f_1(r, \theta, \lambda; \alpha_{nm}, \beta_{nm}) , \\ T_{xy} &= f_2(r, \theta, \lambda; \alpha_{nm}, \beta_{nm}) , \\ &\ldots . \end{aligned} \tag{3-5}$$

Thus, every measurement gives one linear equation for the α_{nm} and β_{nm} in the form of an infinite series; note that r, θ, λ refer to the particular point at which the measurement is performed and are assumed to be known.

To determine the infinitely many α_{nm} and β_{nm}, a finite number of measurements and, therefore, of linear equations is certainly not sufficient. The conventional procedure in this case is to truncate the series at some $n = n_0$, such that the number of retained parameters α_{nm} and β_{nm} is smaller than the number of observations and these parameters can be determined by an adjustment.

Such a truncation is, however, a highly arbitrary procedure. In the present case this is even more problematic than in the usual determination of spherical harmonics from orbital analysis, since the magnitude of the terms in the series (3-5) decreases considerably less than, e.g., in the series (3-3), namely by a factor of order n^2. Truncation thus introduces "aliasing errors" and increases the mutual dependence of the resulting values.

than in Stokes' or Vening Meinesz' formulas.

Since the gradients are measured only at discrete points or along certain profiles, they must be interpolated in between. Unfortunately, second-order gradients fluctuate much more rapidly and are more irregular than gravity anomalies or deflections of the vertical, so that interpolation becomes more difficult and more problematic. The best that can be done to reduce interpolation errors is to use least-squares prediction which will give as accurate results as the data permit.

Another problem arises in this context. In (3-2), all the quantities T_{rr} should refer to the same level surface. Since it will hardly be feasible to perform all the measurements at the same level, they might be made at different levels and reduced to the same level. But because of the greater irregularity of second-order gradients, this reduction is even more problematic and less reliable than for gravity.

The main objection, however, is that most information remains unused: five quantities are measured and only one quantity, T_{rr}, is used.

d) Determination of Spherical Harmonics. - The anomalous potential T can be expressed as a series of spherical harmonics:

$$T = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} (\alpha_{nm} \cos m\lambda + \beta_{nm} \sin m\lambda) P_{nm}(\cos\theta) , \qquad (3\text{-}3)$$

where r(radius vector), θ(polar distance) and λ(longitude) are the spherical coordinates already used in sec. 1, a is the semi-major axis of the earth, $P_{nm}(\cos\theta)$ are Legendre's functions, and α_{nm} and β_{nm} are coefficients to be determined.

The use of spherical harmonics is most appropriate with satellite gradiometry because the convergence of the series (3-3) is satisfactory at satellite altitudes but is not so at flight elevations and a forteriori at the earth's surface, so that an excessive number of coefficients would be required to get a good approximation to the fine structure of the gravity field.

By differentiation we may find the corresponding series for the derivatives T_r, T_θ, T_λ; T_{rr}, $T_{r\theta}$, $T_{r\lambda}$, $T_{\theta\theta}$, $T_{\theta\lambda}$, $T_{\lambda\lambda}$ and then the series for $T_{\xi\xi}$, $T_{\xi\eta}$, etc. by

$$s_P = [\, C_{P1} \; C_{P2} \; \cdots \; C_{Pn} \,] \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \cdots & \overline{C}_{1n} \\ \overline{C}_{21} & \overline{C}_{22} & \cdots & \overline{C}_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \overline{C}_{n1} & \overline{C}_{n2} & \cdots & \overline{C}_{nn} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} . \tag{4-1}$$

Here $\overline{C}_{ij}$ is the covariance between the observations x_i and x_j, and C_{Pi} is the covariance between the signal s_P and the observation x_i $(i, j = 1, 2, \ldots, n)$. These covariances are basic; they carry, so to speak, the burden of the mathematical structure of the gravity field. Therefore, much will have to be said in the sequel, especially in the following section.

The above formula presupposes that the (suitably defined) average values of s_P and x_i are all zero:

$$M(s_P) = 0, \quad M(x_i) = 0, \tag{4-2}$$

which means that both s_P and x_i must be quantities of the anomalous gravity field (the systematic, "average" part of the gravity field being removed by subtracting the normal gravity field) and that, in addition, x_i must not be affected by systematic errors.

The measurements x_i can, however, contain the effect of random errors; eq. (4-1) is valid in this case as well as in the case of errorless observations.

The formula (4-1) is optimal in the sense that it determines s_P in such a way that the value so obtained is compatible with the given observations x_i and the mean square error of estimation is a minimum. This has the following meaning. The n given observations do not determine the gravitational field completely since this field depends on infinitely many parameters (e.g., the full infinite set of spherical harmonics). Therefore, there are infinitely many possible gravitational fields that are compatible with the given measurements. To each of these possible solutions there corresponds a mean square error of estimation, m_P, and eq. (4-1) singles out that solution for

4. Application of Least-Squares Collocation

The analysis of the preceding section has shown that none of the methods described there is fully satisfactory for the geodetic application of gradiometer measurements. We have also recognized some desiderata which a better method would have to satisfy:

1. It should be able to handle all occurring data -- first and second order gradients and any other data -- and combine them in a natural, objective and optimal way.

2. It should be able to handle discrete or profile data at different elevations directly, without interpolation or vertical reduction.

3. Methods described in the preceding section should be suitable limiting cases of it. For instance, if we assume that only T_{rr} has been measured, but that it is given without errors at very many points of a level surface, then the new method should give a result for T that tends, as a limit for infinitely dense coverage, to the result of eq. (3-2).

4. It should give the same results whether second-order gradients or quantities derived therefrom are used; cf. end of sec. 2.

5. It should, in a natural way, incorporate least-squares adjustment and give statistically meaningful accuracy estimates. It should be able to make optimal use even of "noisy" data.

Recently a new method of least-squares estimation of the gravitational field (least-squares collocation) has been developed which satisfies these requirements (Krarup, 1968, 1969; Moritz, 1970a, b). It may be described as follows.

Let n quantities of the anomalous gravity field be measured; the measurements will be denoted by $x_1, x_2, \ldots, x_n$. They might be anomalous gravity gradients but also, e.g., conventional gravity anomalies, astrogeodetic deflections of the vertical or geoidal heights derived from satellite altimetry. Denote by s_P (the "signal") the quantity of the anomalous gravity field that we wish to compute, for instance a geoid height or a component of the deflection of the vertical. Then s_P is given by the matrix equation which m_P is a minimum.

The formulas for this mean square error of estimation, m_P, and for the error covariances of any computed values s_P and s_Q, denoted by σ_{PQ}, are as follows:

$$m_P^2 = C_{PP} - [C_{P1}\ C_{P2}\ \dots\ C_{Pn}] \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \dots & \overline{C}_{1n} \\ \overline{C}_{21} & \overline{C}_{22} & \dots & \overline{C}_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \overline{C}_{n1} & \overline{C}_{n2} & \dots & \overline{C}_{nn} \end{bmatrix}^{-1} \begin{bmatrix} C_{P1} \\ C_{P2} \\ \cdot \\ \cdot \\ \cdot \\ C_{Pn} \end{bmatrix}, \quad (4\text{-}3)$$

$$\sigma_{PQ} = C_{PQ} - [C_{P1}\ C_{P2}\ \dots\ C_{Pn}] \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \dots & \overline{C}_{1n} \\ \overline{C}_{21} & \overline{C}_{22} & \dots & \overline{C}_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \overline{C}_{n1} & \overline{C}_{n2} & \dots & \overline{C}_{nn} \end{bmatrix}^{-1} \begin{bmatrix} C_{Q1} \\ C_{Q2} \\ \cdot \\ \cdot \\ \cdot \\ C_{Qn} \end{bmatrix}. \quad (4\text{-}4)$$

These quantities are analogous to the mean square error after adjustment and the (error) covariance of adjusted values in least-squares adjustment.

Note that in adjustment computations, "variance" and "covariance" always mean error variance and error covariance, whereas in the present method we have both field covariances (e.g., C_{Pi}) and error covariances (e.g., σ_{PQ}). More about this will be said in the next section.

For the derivation of all these formulas see (Moritz, 1970a, sec. 2). If the x_i are specialized to be errorless gravity anomalies, the well-known formulas for gravity prediction result; note the formal identity of the present equations (4-1), (4-3), and (4-4) with equations (7-63), (7-64), and (7-65) of (Heiskanen and Moritz, 1967, sec. 7-6).

Systematic Effects. - Especially in moving-base measurements, systematic trends such as instrumental drifts or systematic navigation errors, are likely to occur. They can also be easily incorporated in the present model by a method developed in (Moritz, 1969, sec. 10) for the case of aerial gravimetry.

If the measurements x_i are affected by systematic errors, they are split up into a purely random quantity ξ_i (comprising both signal and random error) and a systematic part, also called trend:

$$x_i = \xi_i + \sum_{\alpha=1}^{m} A_{i\alpha} X_\alpha , \tag{4-5}$$

where the X_α are m systematic parameters and $(A_{i\alpha})$ denotes a given matrix.

Thus the functional dependence on X_α is assumed to be linear; if it is originally non-linear, it is to be linearized in the usual way by means of Taylor's theorem.

The parameters X_α are determined by a least-squares adjustment with the result

$$\underline{X} = (\underline{A}^T \underline{\overline{C}}^{-1} \underline{A})^{-1} \underline{A}^T \underline{\overline{C}}^{-1} \underline{x} , \tag{4-6}$$

where

$$\begin{aligned} \underline{X} &= (X_\alpha) , \quad \underline{x} = (x_i) , \\ \underline{A} &= (A_{i\alpha}) , \quad \underline{\overline{C}} = (\overline{C}_{ij}) \end{aligned} \tag{4-7}$$

are vectors or matrices, respectively.

Then the trend is subtracted from the data x_i to get the "centered data"

$$\xi_i = x_i - \sum_\alpha A_{i\alpha} X_\alpha , \tag{4-8}$$

and these ξ_i may now be used in (4-1), in the place of x_i, to get again an optimal estimate.

A derivation of (4-6) by least-squares adjustment by parameters may be found in (Moritz, 1969, sec. 10). A more satisfactory simultaneous deduction of (4-1) and (4-6) from a unified minimum principle has been given in (Moritz, 1970b).

The basic equations (4-1), (4-3), and (4-4) need only be slightly modified when systematic errors are present. In (4-1) we must replace x_i by ξ_i as given by (4-8) as we have just seen. In (4-3) and (4-4), the matrix

$$\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \cdots & \bar{C}_{1n} \\ \bar{C}_{21} & \bar{C}_{22} & \cdots & \bar{C}_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \bar{C}_{n1} & \bar{C}_{n2} & \cdots & \bar{C}_{nn} \end{bmatrix}^{-1} = \bar{\underline{C}}^{-1}$$

is to be replaced by

$$\bar{\underline{C}}^{-1} [\underline{I} - \underline{A} (\underline{A}^T \bar{\underline{C}}^{-1} \underline{A})^{-1} \underline{A}^T \bar{\underline{C}}^{-1}] \quad , \tag{4-9}$$

where $\underline{I}$ is the n x n unit matrix, and $\underline{A}$ and $\bar{\underline{C}}$ are given by (4-7).

A derivation of (4-9) will be found in the Appendix.

Properties of the Solution. - As we have remarked above, the present solution is characterized by the fact that the mean square error of estimation is a minimum. This is reminiscent of an analogous property of least-squares adjustment. In fact, the present method is a generalization of least-squares adjustment for the case that there is not only a random "noise" (measuring errors) but also a random "signal" (elements of the anomalous gravity field). Cf. (Krarup, 1969) and (Moritz, 1970b). To distinguish it from ordinary adjustment, the least-squares estimation of the gravity field is called least-squares collocation.

As we have already mentioned, the quantities $x_1, x_2, \ldots, x_n$ entering in (4-1) can be any elements of the anomalous gravity field, affected or not by random errors. Thus, eq. (4-1) is able to handle and to combine any measurements of gravitational field elements, not only first and second order gradients. Applied to gravimetrically observed deflections of the vertical ξ, η it would, eg., give an optimally combined astrogeodetic and gravimetric geoid; cf. (Moritz, 1970a, sec. 9).

Eq. (4-1) could also be used with third-order gradients. The reason why third-order gradients are not dealt with explicitly in this report, is that they are probably of less geodetic usefulness. But the considerations of sec. 1 and the techniques of secs. 4, 6, and 7 could be readily applied to third and higher order gradients as well.

Also the signal s_P can be any desired element of the anomalous gravity field. The different quantities computed in this way are consistent with each other in the sense that they belong to one and the same gravity field.

In fact, the second and third factor in (4-1), depending only on the observations x_i and their covariances, are the same for all elements s_P to be computed. Thus the individual nature of the quantity s_P is expressed solely by the first factor, the vector (C_{Pi}), and the quantities s_P will be consistent if and only if the covariances C_{Pi} are compatible. The compatibility of these covariances is assured by computing them according to the law of propagation of covariances to be discussed below.

For instance, let all x_i be errorless measurements of the second vertical gradient T_{rr} at various points of a level surface, and use formula (4-1) to compute T_{rr} at every point of this level surface; this is, then, a pure case of least-squares interpolation in the usual sense. From the continuous global T_{rr}-field obtained in this way, compute T at some point of the same level surface by (3-2). Alternatively, compute T directly from the measured values x_i using again (4-1). The resulting value for T will be the same in both cases because the covariances entering in (4-1) are chosen so as to ensure this.

In this way we understand why conventional methods described in sec. 3 can, in fact, be considered as limiting cases of least-squares collocation for idealized data distributions.

As another example, consider the "problem of Bjerhammar": gravity anomalies are given at discrete points of the telluroid; for a definition of the telluroid cf. (Heiskanen and Moritz, 1967, p. 292). As a limiting case, for continuous coverage of the whole telluroid by gravity anomalies, this problem reduces to the "problem of Molodensky", the well-known boundary-value problem of physical geodesy (ibid, p. 291). The Bjerhammar problem may again be solved by (4-1) (Moritz, 1970a, sec. 5); if the gravity coverage becomes denser and denser, this solution tends to a solution of Molodensky's problem. As, under certain assumptions, the solution of Molodensky's problem is unique, this limiting solution will coincide with the usual solution of Molodensky's problem by integral formulas.

At first sight it may be difficult to believe that the simple matrix formula (4-1) is equivalent to complicated procedures such as the solution of Molodensky's problem. The reason is that all covariances C_{Pi} are based on the same covariance function $K(P,Q)$ (see next section), and that this covariance function may be selected to have a relatively simple analytical expression. Hence, all necessary operations may be performed analytically instead of numerically. Furthermore, starting from the covariance function of the potential, the covariances of all relevant quantities such as gravity anomalies, deflections of the vertical, or higher gradients are derived by differentiations. These are much simpler to perform than the integral operations necessary when going in the opposite direction as in the classical procedures of physical geodesy.

By taking for the covariance function a function that can be analytically continued down to sea level, all difficulties of analytical downward continuation are automatically avoided; such difficulties beset conventional reduction procedures.

These considerations help to understand why (4-1) is at the same time a generalization of classical procedures, so to speak with built-in interpolation and vertical reduction, and an essential simplification.

There remains to be discussed why the present method gives the same results with any of the data sets (2-21), (2-22) or (2-23) or with similar sets. The underlying fact is that least-squares collocation shares with least-squares adjustment the property of invariance with respect to linear transformations both of the signal s_P and of the data x_i. Invariance with respect to linear operations on field elements s_P is the reason why the method determines a consistent gravity field, as we have seen above; and invariance with respect to linear operations on the data x_i is the reason for obtaining the same results with the different data sets mentioned, since (1-28) and (2-18) are linear integral operations. Cf. also (Moritz, 1970a, pp. 12-13).

The equations of least-squares collocation are directly suited for high-speed computation. The biggest computational problem involved is the inversion of the matrix $\bar{C}$ for a great number of observations. For a given set of data x_i, however, such an inversion is to be performed only once for all quantities to be computed and for all accuracy evaluations, as formulas such as (4-1) and (4-3) show.

5. Covariances

As we have just seen, the covariances have to carry the whole burden of the mathematical structure of the problems under consideration. They need, therefore, be investigated more closely. This has been done in (Moritz, 1970a, sec. 4); we shall summarize the relevant results and apply them to the present problem of the use of gradients.

To ensure that all our computed quantities belong to one and the same gravity field, all covariances that enter into our computations must be derived from a single covariance function, for which we may take the covariance function of the anomalous potential T,

$$K(P,Q) = \mathrm{cov}(T_P, T_Q) = M(T_P T_Q) , \tag{5-1}$$

defined as the average product of the T-values at two points P and Q, the average being understood in a suitable way.

The covariance function (5-1) and the quantities derived therefrom are <u>field covariances</u>: they express the statistical behavior of the anomalous gravity field and should be carefully distinguished from <u>error covariances</u>, which express the statistical behavior of observational errors; only the latter are considered in adjustment computations. Cf. the remarks concerning the covariance function of the gravity anomalies in (Heiskanen and Moritz, 1967, pp. 267-8).

The results of the computations do not depend strongly on the choice of the basic covariance function(5-1) (as long as it is used consistently throughout!), in the same way as the results in adjustment computations do not depend strongly on the weights chosen. It is, therefore, possible to take for $K(P,Q)$ an analytically simple function.

K is a function of two points P and Q defined on and outside of some sphere of radius R (which we may take to represent sea level) that must be harmonic both as a function of P and as a function of Q:

$$\Delta_P K(P,Q) = 0 = \Delta_Q K(P,Q), \tag{5-2}$$

where Δ_P means the Laplace operator applied at the point P. This follows immediately from the definition (5-1). Furthermore, the function K is assumed to be rotationally symmetric: on the sea-level sphere of radius R, it depends only on the spherical distance ψ of P and Q. Thus

$$K(P,Q) = K(r_P, r_Q, \psi), \tag{5-3}$$

it is a function of the radius vectors, r_P and r_Q, of P and Q, and of the spherical distance ψ between P and Q.

Such a function has a spherical-harmonic expression of the form

$$K(P,Q) = \sum_{n=0}^{\infty} k_n \left(\frac{R^2}{r_P r_Q}\right)^{n+1} P_n(\cos\psi), \tag{5-4}$$

where the k_n are coefficients.

For example, we may take $k_o = k_1 = k_2 = 0$ and

$$k_n = \frac{A}{(n-1)(n-2)} \quad \text{for } n \geq 3. \tag{5-5}$$

With these coefficients, the series (5-4) may be summed so that a closed expression is obtained:

$$K(P,Q) = A\left(\frac{r_0^{\,2}}{r_P r_Q}\right)^3 \left[P_2(\cos\psi)\left(1 + \ell n \frac{2}{M}\right) + \frac{1}{4}\sin^2\psi \right] -$$

$$- A\left(\frac{r_0^{\,2}}{r_P r_Q}\right)^2 \cos\psi \, \ell n \frac{2}{M} +$$

$$+ A\left(\frac{r_0^{\,2}}{r_P r_Q}\right) \frac{N}{2} \left[3\left(\frac{r_0^{\,2}}{r_P r_Q}\right) \cos\psi - 1 \right] \tag{5-6}$$

where

$$L = \left[1 - 2\left(\frac{r_0^{\,2}}{r_P r_Q}\right) \cos\psi + \left(\frac{r_0^{\,2}}{r_P r_Q}\right)^2 \right]^{\frac{1}{2}},$$

$$M = 1 - L - \left(\frac{r_0^{\,2}}{r_P r_Q}\right) \cos\psi \quad , \tag{5-7}$$

$$N = 1 + L - \left(\frac{r_0^{\,2}}{r_P r_Q}\right) \cos\psi \quad ,$$

and A and r_0 are suitable constants. According to (Lauritzen, 1971), to whom this function is due, it fits excellently global gravity and satellite data, with

$$\begin{aligned} r_0 &= 0.9945\ R, \\ A &= 7.84888, \end{aligned} \tag{5-8}$$

R being again the mean radius of the earth.

A simpler function which might also be useful in appropriate cases is

$$K(P,Q) = B\left[\left(\frac{r_P r_Q}{r_0^{\,2}}\right)^2 - \frac{r_P r_Q}{r_0^{\,2}} \cos\psi + 1 \right]^{-\frac{1}{2}}, \tag{5-9}$$

given by Krarup (1969), with suitable constants B and r_0.

The equivalent, for the plane, of the spherical expression (5-9), is the function

$$K(P,Q) = C\,[(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_P + z_Q + b)^2\,]^{-\frac{1}{2}}, \tag{5-10}$$

with constants C and b. It is readily verified that this function is harmonic with respect to P and Q.

Propagation of Covariances.- The law of propagation of covariances states how the covariances between any two elements of the anomalous gravity field are derived from the basic covariance function (5-1). Perhaps the easiest way to express it is verbally as follows:

Let u and v be two quantities derived from T by linear operations. Then the covariance between u and v,

$$\mathrm{cov}\,(u, v)\ , \tag{5-11}$$

is obtained as follows. Apply to the covariance function K(P, Q), considered as a function of Q, the operation that determines the quantity v from T. To the result, considered as a function of P, apply the operation that determines the quantity u from T. The result is cov(u, v).

An example will clarify this rule. Let

$$u = T_{xy}\ , \qquad v = T_z\ . \tag{5-12}$$

Then u is determined by successive partial differentiation with respect to x and y, and v is derived from T by partial differentiation with respect to z.

Then, by the verbal rule just given, $\mathrm{cov}(T_{xy}, T_z)$ is found as follows. Apply to the covariance function K(P, Q), considered as a function of Q, the operation that determines v from T, that is, partial differentiation with respect to z, obtaining

$$\frac{\partial^2 K(P,Q)}{\partial z_Q} .$$

To this result, considered as a function of P, apply successive partial differentiation with respect to x and y, obtaining

$$\frac{\partial^2}{\partial x_P \, \partial y_P} \left(\frac{\partial K}{\partial z_Q} \right) = \frac{\partial^3 K(P,Q)}{\partial x_P \, \partial y_P \, \partial z_Q} .$$

Thus the desired covariance is given by

$$\operatorname{cov}(T_{xy}, T_z) = \frac{\partial^3 K(P,Q)}{\partial x_P \, \partial y_P \, \partial z_Q} . \tag{5-13}$$

Putting $x = x_1$, $y = x_2$, $z = x_3$ and letting i, j, k, l take the values 1, 2, 3, we obviously have

$$\begin{aligned}
\operatorname{cov}\left(T, \frac{\partial T}{\partial x_i}\right) &= \frac{\partial K(P,Q)}{\partial x_{i,Q}} , \\
\operatorname{cov}\left(\frac{\partial T}{\partial x_i}, T\right) &= \frac{\partial K(P,Q)}{\partial x_{i,P}} , \\
\operatorname{cov}\left(T, \frac{\partial^2 T}{\partial x_i \, \partial x_j}\right) &= \frac{\partial^2 K(P,Q)}{\partial x_{i,Q} \, \partial x_{j,Q}} , \\
\operatorname{cov}\left(\frac{\partial^2 T}{\partial x_i \, \partial x_j}, T\right) &= \frac{\partial^2 K(P,Q)}{\partial x_{i,P} \, \partial x_{j,P}} , \\
\operatorname{cov}\left(\frac{\partial T}{\partial x_i}, \frac{\partial T}{\partial x_j}\right) &= \frac{\partial^2 K(P,Q)}{\partial x_{i,P} \, \partial x_{j,Q}} , \qquad (5\text{-}14) \\
\operatorname{cov}\left(\frac{\partial^2 T}{\partial x_i \, \partial x_j}, \frac{\partial T}{\partial x_k}\right) &= \frac{\partial^3 K(P,Q)}{\partial x_{i,P} \, \partial x_{j,P} \, \partial x_{k,Q}} , \\
\operatorname{cov}\left(\frac{\partial T}{\partial x_i}, \frac{\partial^2 T}{\partial x_j \, \partial x_k}\right) &= \frac{\partial^3 K(P,Q)}{\partial x_{i,P} \, \partial x_{j,Q} \, \partial x_{k,Q}} , \\
\operatorname{cov}\left(\frac{\partial^2 T}{\partial x_i \, \partial x_j}, \frac{\partial^2 T}{\partial x_k \, \partial x_l}\right) &= \frac{\partial^4 K(P,Q)}{\partial x_{i,P} \, \partial x_{j,P} \, \partial x_{k,Q} \, \partial x_{l,Q}} .
\end{aligned}$$

Here we have used x_i to denote coordinates x, y, z, as we did in sec. 2. Otherwise throughout Part B of the present report x_i($i = 1, 2, \ldots, n$) always denotes measurements, so that no confusion should arise.

These formulas give the covariances between T and its first and second partial derivatives. Extensions to higher derivatives are obvious.

Sometimes linear combinations occur. For instance, the gravity anomaly is a linear combination of T and $\partial T/\partial r$:

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r} T \tag{5-15}$$

(Heiskanen and Moritz, 1967, p. 89). Another example is represented by (1-25).

Thus let us, for instance, find

$$\mathrm{cov}(\Delta g, T_{\xi\xi}),$$

$T_{\xi\xi}$ being given by (1-25):

$$T_{\xi\xi} = \frac{1}{r^2} T_{\theta\theta} + \frac{1}{r} T_r \quad . \tag{5-16}$$

We shall use the rule for the propagation of covariances as given above. Apply to the covariance function $K(P,Q)$, considered as a function of Q, the operation that determines $T_{\xi\xi}$ from T by (5-16), obtaining

$$\frac{1}{r_Q^2} \frac{\partial^2 K}{\partial \theta_Q^2} + \frac{1}{r_Q} \frac{\partial K}{\partial r_Q} \quad .$$

To this result, considered as a function of P, apply the operation that determines Δg from T by (5-15). Thus we obtain

$$- \frac{1}{r_Q^2} \frac{\partial^3 K}{\partial r_P \, \partial \theta_Q^2} - \frac{2}{r_P \, r_Q^2} \frac{\partial^2 K}{\partial \theta_Q^2} -$$

$$- \frac{1}{r_Q} \frac{\partial^2 K}{\partial r_P \, \partial r_Q} - \frac{2}{r_P \, r_Q} \frac{\partial K}{\partial r_Q} =$$

$$= \operatorname{cov}(\Delta g, \, T_{\xi\xi}) \; . \tag{5-17}$$

In this way we are in a position to express all covariances that occur in the geodetic use of gradients, in terms of partial derivatives of the basic covariance function $K(P,Q)$.

Finally we consider briefly how these partial derivatives are evaluated. If K is given as a function of rectangular coordinates x, y, z, then the evaluation is straightforward. A fully worked out example will be found in sec. 7; for another example see (Moritz, 1970a, sec. 7).

If K is given as a function of three variables r_P, r_Q, ψ as in (5-3), then the differentiations must be performed as

$$\frac{\partial K}{\partial x_P} = \frac{\partial K}{\partial r_P} \frac{\partial r_P}{\partial x_P} + \frac{\partial K}{\partial r_Q} \frac{\partial r_Q}{\partial x_P} + \frac{\partial K}{\partial \psi} \frac{\partial \psi}{\partial x_P} \; . \tag{5-18}$$

Now

$$\begin{aligned} r_P^2 &= x_P^2 + y_P^2 + z_P^2 \; , \\ r_Q^2 &= x_Q^2 + y_Q^2 + z_Q^2 \; , \\ \cos \psi &= \frac{x_P \, x_Q + y_P \, y_Q + z_P \, z_Q}{\sqrt{x_P^2 + y_P^2 + z_P^2} \, \sqrt{x_Q^2 + y_Q^2 + z_Q^2}} \; , \end{aligned} \tag{5-19}$$

so that, by straightforward differentiation,

$$\begin{aligned} &\frac{\partial r_P}{\partial x_P} = \frac{x_P}{r_P} \; , \qquad \frac{\partial r_Q}{\partial x_P} = 0 \; , \\ &\frac{\partial \psi}{\partial x_P} = \frac{1}{\sin \psi} \left(\frac{x_P}{r_P^2} \cos \psi - \frac{x_Q}{r_P \, r_Q} \right) \; . \end{aligned} \tag{5-20}$$

In this way, all occurring differentiations may be performed without any mathematical difficulties, although the analytical work may be laborious.

Observational Errors. - If the observations x_i are errorless, then all covariances C_{Pi} and $\overline{C}_{ij}$ entering into the basic collocation formulas (4-1), (4-3) and (4-4) should be field covariances as we have just considered.

If the observations x_i are affected by random errors, then the covariances C_{Pi} remain field covariances, whereas the covariances $\overline{C}_{ij}$ are now given by

$$\overline{C}_{ij} = C_{ij} + D_{ij} , \tag{5-21}$$

where C_{ij} are the field covariances corresponding to the observed elements, and D_{ij} are the error covariances of the observational errors. In the terminology of adjustment computations, the matrix (D_{ij}) is the variance-covariance matrix of the observations.

The simple relation (5-21) presupposes that the errors are uncorrelated to the anomalous gravity field. This will be true if the observations have not yet been subjected to a preliminary collocation, for instance, a least-squares filtering. In the latter case, the covariance matrix $(\overline{C}_{ij})$ is to be taken from this preliminary collocation; cf. (Moritz, 1969, sec. 9). This is in complete analogy to least-squares adjustment.

6. Determination of Spherical Harmonics

The collocation method described in sec. 4 may also be used to determine spherical harmonics from gradiometer measurements.

Let the spherical harmonic expansion of the anomalous potential T again be given in the form (3-3), which we shall write in terms of fully normalized spherical harmonics (Heiskanen and Moritz, 1967, sec. 1-14):

$$T(r, \theta, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} \left[\overline{\alpha}_{nm} \overline{R}_{nm}(\theta, \lambda) + \overline{\beta}_{nm} \overline{S}_{nm}(\theta, \lambda)\right]. \tag{6-1}$$

Then the "signal" s_P in (4-1) is any coefficient $\overline{\alpha}_{nm}$ or $\overline{\beta}_{nm}$; let us assume

$$s_P = \bar{\alpha}_{nm} \; . \tag{6-2}$$

Then

$$C_{Pi} = \operatorname{cov}(\bar{\alpha}_{nm}, x_i), \tag{6-3}$$

$$\bar{C}_{ij} = \operatorname{cov}(x_i, x_j), \tag{6-4}$$

x_i being again any measured second-order gradient (or any other measured field element).

The computation of the covariances $\bar{C}_{ij}$ has already been considered in the preceding section; it remains to study the covariances (6-3).

The spatial covariance function of T may again be expressed in the form (5-4):

$$K(P,Q) = \sum_{n=2}^{\infty} k_n \left(\frac{a^2}{r_P r_Q} \right)^{n+1} P_n(\cos\psi) \; . \tag{6-5}$$

Then we have by (Moritz, 1970a, p. 45)

$$\operatorname{cov}(\bar{\alpha}_{nm}, \bar{\alpha}_{nm}) = \operatorname{cov}(\bar{\beta}_{nm}, \bar{\beta}_{nm}) = \frac{k_n}{2n+1} \, ,$$

$$\operatorname{cov}(\bar{\alpha}_{nm}, \bar{\alpha}_{pq}) = \operatorname{cov}(\bar{\beta}_{nm}, \bar{\beta}_{pq}) = 0$$

$$\text{if } p \neq n \text{ or } q \neq m \text{ or both}, \tag{6-6}$$

$$\operatorname{cov}(\bar{\alpha}_{nm}, \bar{\beta}_{pq}) = 0 \quad \text{always.}$$

In the report just quoted, these formulas have been derived for the covariance function of the gravity anomaly. It is, however, obvious that they are valid for the covariance function of the potential as well.

Any gradient is obtained by single or multiple differentiation of T (or by a linear combination of such derivatives), to be symbolized by DT. Thus from (6-1) we get

$$DT = \sum_{n=2}^{\infty} \sum_{m=o}^{n} a^{n+1} \left[\overline{\alpha}_{nm} D\left(\frac{\overline{R}_{nm}}{r^{n+1}}\right) + \overline{\beta}_{nm} D\left(\frac{\overline{S}_{nm}}{r^{n+1}}\right)\right] . \qquad (6\text{-}7)$$

Let DT denote the gradient the measurement of which is x_i. Then

$$C_{Pi} = \text{cov}(\overline{\alpha}_{nm}, DT)$$

$$= \text{cov}\left\{ \overline{\alpha}_{nm}, \sum_{p=2}^{\infty} \sum_{q=o}^{p} a^{p+1} \left[\overline{\alpha}_{pq} D\left(\frac{\overline{R}_{pq}}{r^{p+1}}\right) + \overline{\beta}_{pq} D\left(\frac{\overline{S}_{pq}}{r^{p+1}}\right)\right]\right\}$$

$$= \sum_{p} \sum_{q} a^{p+1} \left[\text{cov}(\overline{\alpha}_{nm}, \overline{\alpha}_{pq}) D\left(\frac{\overline{R}_{pq}}{r^{p+1}}\right) + \right.,$$

$$\left. + \text{cov}(\overline{\alpha}_{nm}, \beta_{pq}) D\left(\frac{\overline{S}_{pq}}{r^{p+1}}\right) \right]$$

$$= a^{n+1} \text{cov}(\overline{\alpha}_{nm}, \overline{\alpha}_{nm}) D\left(\frac{\overline{R}_{nm}}{r^{n+1}}\right) ,$$

since all covariances between coefficients are zero except one, by (6-6). Thus we have

$$C_{Pi} = \frac{a^{n+1}}{2n+1} k_n D\left(\frac{\overline{R}_{nm}(\theta,\lambda)}{r^{n+1}}\right) . \qquad (6\text{-}8)$$

Since

$$\frac{\overline{R}_{nm}(\theta,\lambda)}{r^{n+1}} = r^{-(n+1)} \overline{P}_{nm}(\theta) \cos m\lambda$$

is a simple function of r, θ, λ, any differentiations with respect to r, θ, λ are easily carried out, e.g.,

$$\frac{\partial^2}{\partial r^2}\left(\frac{\overline{R}_{nm}(\theta,\lambda)}{r^{n+1}}\right) = (n+1)(n+2) r^{-(n+3)} \overline{P}_{nm} \cos m\lambda ,$$

$$\frac{\partial^2}{\partial\theta\partial\lambda}\left(\frac{\overline{R}_{nm}(\theta,\lambda)}{r^{n+1}}\right) = - m r^{-(n+1)} \frac{d\overline{P}_{nm}}{d\theta} \sin m\lambda ,$$

and the components in rectangular coordinates follow from equations such as (1-25).

For the determination of the coefficient $\bar{\beta}_{nm}$ we need cov($\bar{\beta}_{nm}$, DT), which is again given by (6-8), with $\bar{R}(\theta, \lambda)$ replaced by $\bar{S}(\theta, \lambda)$.

After these preparations we are ready for the application of the collocation formulas such as (4-1), (4-3) and (4-4) for the derivation of spherical harmonics from gradiometer measurements.

Random measuring errors are automatically taken into account if the covariance matrix (6-4) is properly computed, in the way outlined at the end of the preceding section.

Systematic effects can also be incorporated into our computations as discussed in sec. 4.

The advantages of the collocation method over the conventional procedure described in sec. 3 (Item C) are as follows.

1. Every harmonic is determined independently, without aliasing errors, since the infinite series (6-1) is not directly used and, consequently, no truncation occurs. Convergence problems do not affect the present solution.

2. The statistical meaning of the new procedure is transparent: it is an optimal procedure in the sense that it gives the most accurate results obtainable on the basis of the given data. The statistical behavior of the anomalous gravity field is properly taken into account.

It is said to be a disadvantage of spherical harmonics in satellite geodesy that their orthogonality properties cannot be used as efficiently as it would be desirable. The collocation method takes full advantage of these orthogonality properties, in the form (6-6), to separate the individual coefficients.

Combination with any other observations--from classical techniques such as direction, range and range-rate observations or from new techniques such as satellite altimetry or satellite-to-satellite ranging--are straightforward because (4-1) can be used with heterogenous observations as well, systematic parts being eliminated as discussed in sec. 4.

7. Use of Profile Measurements

In (Moritz, 1969, sec. 6) we have discussed at length the use of aerial gravity measurements along parallel profiles. Since the least-squares estimation formulas hold for any type of measurements, also of different kinds, the formulas given there are also valid for measurements of first and second order gradients along parallel profiles.

To keep our problem simple we assume, as we did in the case of aerial gravimetry, that the profiles are parallel straight lines; they need not be equally spaced, and they may be at different elevations. Let t be the distance counted along the direction of the profiles, such that the lines $t = \text{const.}$ are straight lines perpendicular to this direction; cf. Figure 2 in (Moritz, 1969, p. 26). Denote by $x_i(t)$ the measurement of some field element (in our case, of some first or second order gradient), recorded along a profile as a function of t. In (Moritz, 1969, sec. 6), subscripts such as i or j $(i, j = 1, 2, \ldots, n)$ have labeled the profiles; now they label the different quantities measured. All n measurements $x_i(t)$ might, in principle, be performed along the same profile; or they might be performed along different parallel profiles: the formulas are the same.

The computational formulas derived in the previous report just mentioned may be summarized as follows.

Denote by

$$\overline{C}_{ij}(t) = \operatorname{cov}(x_i, x_j) \tag{7-1}$$

the autocovariance function of the measurements. More precisely, $\overline{C}_{ij}(t)$ is the covariance between the value of x_i for the argument $u + t$ and the value of x_j for the argument u, u being any real number. Similarly,

$$C_{Pj}(t) = \operatorname{cov}(s_P, x_j) \tag{7-2}$$

denotes the cross-covariance function between signal and measurement; more precisely, $C_{Pj}(t)$ is the covariance between $s_P(u + t)$ and $x_j(u)$.

consists of the field covariance $C_{ij}(t)$ and the error covariance $D_{ij}(t)$.

The error covariances for second-order gradients and for first-order gradients obtained by the method of sec. 2 should be very much smaller than in the case of aerial gravimetry, because there $D_{ij}(t)$ also includes the inertial noise which is now absent.

An Example. - This method will be illustrated by a simple example. We assume two parallel profiles 1 and 2, the first at elevation z_1, the second at elevation z_2. Along profile 1, the second-order gradient T_{xy} is measured, along profile 2, the first-order gradient T_z is measured; these measurements are errorless.

This example differs from Example 2 in (Moritz, 1969, pp. 35-36) only by different observational data; the geometrical configuration (Fig. 1) and the

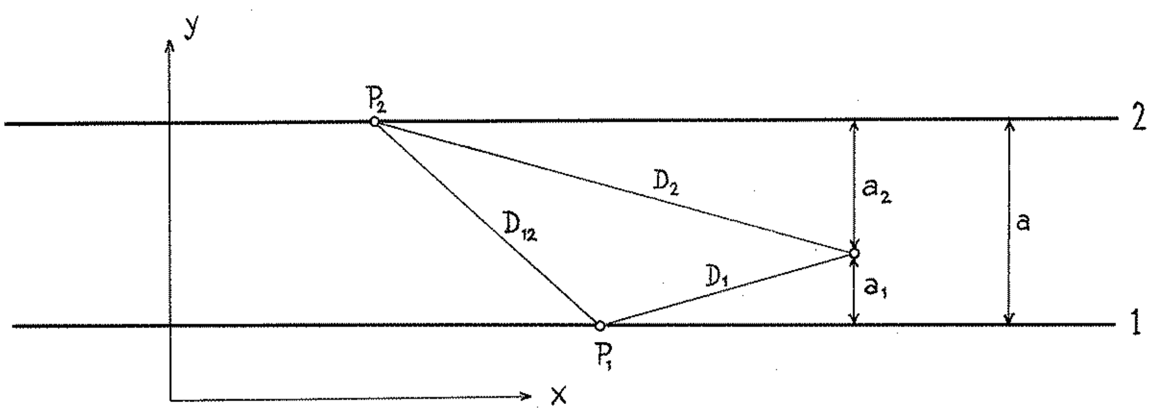

Figure 1

mathematical structure are the same. The solution is represented by equations (7-12) and (7-13) on p. 36 of that report. Only the covariance functions are different because of the different observational data; they will be computed now.

As the basic covariance function, let us take the function (5-10), with $C = 1$:

$$K(A,B) = \frac{1}{D} \tag{7-8}$$

with

$$D^2 = (x_B - x_A)^2 + (y_B - y_A)^2 + (z_A + z_B + b)^2. \tag{7-9}$$

The assumption that these covariance functions do not depend on u but only on the argument difference $(u + t) - u = t$ means that our measurements $x_i(t)$ are considered as "stationary stochastic processes"; cf. (Meissl, 1970).

Now we form the Fourier transforms of the covariances, the spectra

$$\bar{S}_{ij}(\omega) = \int_{-\infty}^{\infty} \bar{C}_{ij}(t) e^{-i\omega t} dt ,$$
$$S_{Pj}(\omega) = \int_{-\infty}^{\infty} C_{Pj}(t) e^{-i\omega t} dt . \tag{7-3}$$

Next we compute the "system functions"

$$H_{Pj}(\omega) = \sum_{i=1}^{n} S_{Pi}(\omega) \bar{S}_{ij}^{(-1)}(\omega) , \tag{7-4}$$

where $\bar{S}_{ij}^{(-1)}(\omega)$ are the elements of the matrix inverse to the $n \times n$ matrix with elements $\bar{S}_{ij}(\omega)$. Applying the inverse Fourier transformation we obtain the "weighting functions"

$$h_{Pj}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{Pj}(\omega) e^{i\omega t} d\omega , \tag{7-5}$$

and the optimum estimate of the signal $s_P(t)$ is finally given by

$$s_P(t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} h_{Pj}(t - \alpha) x_j(\alpha) d\alpha . \tag{7-6}$$

For the validity of this method it is essential that the covariances be appropriately computed. If the measurements can be considered as errorless, then all covariances are directly given by the law of propagation of covariances as discussed in sec. 5. If the measurements are affected by random errors, $C_{Pj}(t)$ is again a pure field covariance, whereas now

$$\bar{C}_{ij}(t) = C_{ij}(t) + D_{ij}(t) \tag{7-7}$$

Both the example and the covariance function have been chosen for simplicity; they are obviously not very realistic.

By differentiation we find:

$$\mathrm{cov}(T,\ T_{xy}) = \frac{\partial^2 K}{\partial x_B \partial y_B} = \frac{3}{D^5}(x_B - x_A)(y_B - y_A)\ , \tag{7-10a}$$

$$\mathrm{cov}(T,\ T_z) = \frac{\partial K}{\partial z_B} = -\frac{1}{D^3}(z_A + z_B + b)\ , \tag{7-10b}$$

$$\begin{aligned}\mathrm{cov}(T_{xy}, T_{xy}) &= \frac{\partial^4 K}{\partial x_A \partial y_A \partial x_B \partial y_B} = \\ &= \frac{3}{D^5} - \frac{15}{D^7}(x_B - x_A)^2 - \frac{15}{D^7}(y_B - y_A)^2 + \\ &+ \frac{105}{D^9}(x_B - x_A)^2 (y_B - y_A)^2\ ,\end{aligned} \tag{7-10c}$$

$$\mathrm{cov}(T_{xy}, T_z) = \frac{\partial^3 K}{\partial x_A \partial y_A \partial z_B} = \frac{15}{D^7}(x_B - x_A)(y_B - y_A)(z_A + z_B + b)\ , \tag{7-10d}$$

$$\mathrm{cov}(T_z, T_z) = \frac{\partial^2 K}{\partial z_A \partial z_B} = -\frac{1}{D^3} + \frac{3}{D^5}(z_A + z_B + b)^2\ . \tag{7-10e}$$

Substituting

$$A = P_1\ , \qquad B = P\ ;$$

$$x_B - x_A = t\ , \quad y_B - y_A = a_1\ ;$$

$$D_1^2 = t^2 + a_1^2 + (z_P + z_1 + b)^2$$

we obtain from (7-10a)

$$C_{P1}(t) = \frac{3 a_1 t}{D_1^5}\ ; \tag{7-11a}$$

substituting

$$A = P_2 \ , \quad B = P \ ;$$

$$x_B - x_A = t \ , \ y_B - y_A = -a_2 \ ;$$

$$D_2^2 = t^2 + a_2^2 + (z_P + z_2 + b)^2$$

we find from (7-10b)

$$C_{P2}(t) = -\frac{1}{D_2^3} (z_P + z_2 + b) \quad ; \tag{7-11b}$$

substituting

$$x_B - x_A = t, \quad y_B - y_A = 0 \ ;$$

$$D_{11}^2 = t^2 + (2z_1 + b)^2$$

we have from (7-10c)

$$C_{11}(t) = \frac{3}{D_{11}^5} - \frac{15\,t^2}{D_{11}^7} \quad ; \tag{7-11c}$$

substituting

$$A = P_2 \ , \quad B = P_1 \ ;$$

$$x_B - x_A = t \ , \ y_B - y_A = -a \ ;$$

$$D_{12}^2 = t^2 + a^2 + (z_1 + z_2 + b)^2$$

we find from (7-10d)

$$C_{12}(t) = -\frac{15\,a\,t}{D_{12}^7} (z_1 + z_2 + b) \ ; \tag{7-11d}$$

and substituting

$$x_B - x_A = t\,, \quad y_B - y_A = 0\,;$$

$$D_{22}^{2} = t^2 + (2z_2 + b)^2$$

we finally obtain from (7-10e)

$$C_{22}(t) = -\frac{1}{D_{22}^{3}} + \frac{3}{D_{22}^{5}}(2z_2 + b)^2\,. \tag{7-11e}$$

For the notations cf. Figure 1; note that a, a_1, a_2, being measured in the xy-plane, are shown in true size, whereas the spatial distances D_1, D_2, D_{12} are shown as projected onto the xy-plane.

Now we can form the Fourier transforms of these covariance functions to get the spectra $S_{P1}(\omega)$, $S_{P2}(\omega)$, $S_{11}(\omega)$, $S_{12}(\omega)$, and $S_{22}(\omega)$. Then we find $H_{P1(\omega)}$ and $H_{P2}(\omega)$ by eq. (7-13) of (Moritz, 1969, p. 36), which are nothing else than our present eq. (7-4) specialized for the example under consideration, and $h_{P1}(t)$ and $h_{P2}(t)$ by (7-5). Finally, (7-6) gives T as the signal to be computed.

APPENDIX

Error Variances and Covariances in the Presence of Systematic Effects

We shall derive the modification of the formulas (4-3) and (4-4) for error variances and covariances of the result when systematic effects are present, arriving at (4-9).

Equation (4-1) may be written

$$y_P = \underline{h}_P \underline{x} \tag{A-1}$$

with

$$\underline{h}_P = \underline{C}_P \underline{\bar{C}}^{-1} ; \tag{A-2}$$

we are using a matrix notation similar to the notation in (Moritz, 1969, p. 11), writing y_P for the estimated value of s_P to distinguish it from the true value s_P.

If systematic effects are present, then in (A-1) the observation $\underline{x}$ is to be replaced by the centered observation

$$\underline{\xi} = \underline{x} - \underline{A}\,\underline{X} , \tag{A-3}$$

so that

$$y_P = \underline{h}_P(\underline{x} - \underline{A}\,\underline{X}) , \tag{A-4}$$

with $\underline{h}_P$ again given by (A-2).

The error of estimation is then the difference true minus estimated value:

$$\epsilon_P = s_P - y_P ,$$

and by (A-4),

$$\epsilon_P = s_P - \underline{h}_P(\underline{x} - \underline{A}\,\underline{X}) . \tag{A-5}$$

Let us now introduce the true values of the parameters, X', and the corresponding true values of the centered observations, ξ', for which we have

$$\underline{\xi}' = \underline{x} - \underline{A}\,\underline{X}' , \tag{A-6}$$

in analogy to (A-3). Substituting

$$\underline{x} = \underline{\xi}' + \underline{A}\,\underline{X}' \tag{A-7}$$

in (A-5) we find

$$\epsilon_P = s_P - \underline{h}_P \underline{\xi}' - \underline{h}_P \underline{A}(\underline{X}' - \underline{X}) \,. \tag{A-8}$$

The estimated values of the parameters are given by (4-6), which may be abbreviated as

$$\underline{X} = \underline{H}\,\underline{x}\,, \tag{A-9}$$

with

$$\underline{H} = (\underline{A}^T \overline{\underline{C}}^{-1} \underline{A})^{-1} \underline{A}^T \overline{\underline{C}}^{-1} \,. \tag{A-10}$$

Thus by (A-7),

$$\underline{X} = \underline{H}\,\underline{x} = \underline{H}\,\underline{\xi}' + \underline{H}\,\underline{A}\,\underline{X}',$$

and by (A-10),

$$\underline{H}\,\underline{A} = \underline{I} \tag{A-11}$$

($\underline{I}$ denotes again the unit matrix), so that

$$\underline{X}' - \underline{X} = -\underline{H}\,\underline{\xi}'\,, \tag{A-12}$$

which is substituted into (A-8) to give

$$\epsilon_P = s_P - \underline{h}_P (\underline{I} - \underline{A}\,\underline{H})\,\underline{\xi}' \quad . \tag{A-13}$$

With the abbreviation

$$\overline{\underline{h}}_P = \underline{h}_P (\underline{I} - \underline{A}\,\underline{H}) \tag{A-14}$$

we may write this as

$$\epsilon_P = s_P - \underline{\bar{h}}_P \underline{\xi}' . \tag{A-15}$$

Thus

$$\epsilon_P \epsilon_Q = s_P s_Q - s_P \underline{\bar{h}}_Q \underline{\xi}' - \underline{\bar{h}}_P \underline{\xi}' s_Q + \underline{\bar{h}}_P \underline{\xi}' \underline{\xi}'^T \underline{h}_Q^T ,$$

and on forming the mean value:

$$\sigma_{PQ} = C_{PQ} - \underline{C}_P \underline{\bar{h}}_Q^T - \underline{\bar{h}}_P \underline{C}_Q^T + \underline{\bar{h}}_P \underline{\bar{C}} \underline{\bar{h}}_Q^T . \tag{A-16}$$

Now

$$\underline{\bar{h}}_P = \underline{h}_P (\underline{I} - \underline{A}\underline{H}) = \underline{C}_P \underline{\bar{C}}^{-1} \left[\underline{I} - \underline{A} (\underline{A}^T \underline{\bar{C}}^{-1} \underline{A})^{-1} \underline{A}^T \underline{\bar{C}}^{-1} \right]$$

is substituted into (A-16) to give, after some straightforward manipulations,

$$\sigma_{PQ} = C_{PQ} - \underline{C}_P \underline{\bar{C}}^{-1} \left[\underline{I} - \underline{A} (\underline{A}^T \underline{\bar{C}}^{-1} \underline{A})^{-1} \underline{A}^T \underline{\bar{C}}^{-1} \right] \underline{C}_Q^T , \tag{A-17}$$

which is (4-4) with $\bar{C}^{-1}$ replaced by (4-9); and setting $Q = P$ gives the corresponding result for the error variance m_P^2.

REFERENCES

Anthony, D. (1971): Gradiometer applications and status of sensor development. AFCRL-71-0314, Environmental Research Papers, No. 355, Air Force Cambridge Research Lab.

Bradley, C.L. (1970): Identification of vertical deflections utilizing an inertial navigator and optimum data processing. Advances in Dynamic Gravimetry, W.T. Kattner, ed. Instrument Society of America, Pittsburgh, pp. 95-104.

Forward, R.L. (1971): Geodesy with orbiting gravity gradiometers. Presented at Symposium on the Use of Artificial Satellites in Geodesy, Washington, D. C., April, 1971.

Hansen, S.H. (1971): Use of a horizontal gravity gradiometer to improve inertial navigation. Lecture notes, Hughes Research Laboratories.

Heiskanen, W.A., and H. Moritz (1967): Physical Geodesy, W.H. Freeman, San Francisco.

Krarup, T. (1968): A framework for least-squares determination of the potential of the earth. Report to the members of SSG 5:31 of the IAG.

Krarup, T. (1969): A contribution to the mathematical foundation of physical geodesy. Publ. No. 44, Danish Geodetic Institute, Copenhagen.

Lauritzen, S.L. (1971): The probabilistic background of some statistical methods in physical geodesy. Danish Geodetic Institute, Copenhagen.

Meissl, P. (1970), Probabilistic error analysis of airborne gravimetry. Report No. 138, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Moritz, H. (1967): Kinematical geodesy. Report No. 92, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Moritz, H. (1969): A general theory of gravity processing. Report No. 122, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Moritz, H. (1970a): Least-squares estimation in physical geodesy. Report No. 130, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Moritz, H. (1970b): A generalized least-squares model. Studia geophysica et geodaetica, 14, pp. 353-362.

Moritz, H. (1970c): Combination of satellite harmonics and gravimetry. Report No. 146, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Mueller, I.I. (1964): The horizontal gradients of gravity in geodesy. Department of Geodetic Science, The Ohio State University, Columbus, Ohio.

Savet, P.H. (1970): New developments in gravity gradiometry. Advances in Dynamic Gravimetry, W.T. Kattner, ed. Instrument Society of America, Pittsburgh, pp. 48-58.

Schultz, O.T. and J.A. Winokur (1969): Shipboard or aircraft gravity vector determination by means of a three-channel inertial navigator. Journal of Geophysical Research, 74(20), pp. 4882-4896.

Szabo, B. and D. Anthony (1971): Results of AFCRL's experimental aerial gravity measurements. Bulletin géodésique, No. 100, pp. 179-202.

Trageser, M.B. (1970): A gradiometer system for gravity anomaly surveying. Advances in Dynamic Gravimetry, W.T. Kattner, ed. Instrument Society of America, Pittsburgh, pp. 1-35.