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A STUDY OF COVARIANCE FUNCTIONS RELATED TO THE EARTH'S DISTURBING POTENTIAL

by

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ABSTRACT

The following quantities are considered: geoidal undulations N, gravity anomalies Δg, deflections of the vertical Δv, a fictitious surface density $\Delta\varphi$, the vertical gradient of gravity anomalies Δa. These quantities are interrelated by linear operators having the spherical harmonics as eigen-functions. If the covariance of one of these quantities is specified, that of the others can be computed. Thereby rigorous bounds for the ratios of the different variances can be established. These bounds demonstrate that Δg, Δv, $\Delta\varphi$ are quantities of equal smoothness. N is smoother and Δa is less smooth. These smoothness properties are important in various approaches to determine the earth's potential. Though the earth's disturbing potential can be represented by any of the above quantities, there are differences in the stability of the resulting solutions. Attention is focused on potentials obtained from a combination of satellite information and gravimetry. In that case the introduced quantities are considered as residuals with respect to a geoid resulting from the adjusted lower degree harmonic coefficients. It is shown that the covariance of any one of the residual quantities tends to have certain theoretical properties. These are a predetermined number of zeros as well as negative correlation at certain predetermined distances. A comparison has been performed between the gravity anomaly residuals with respect to a low order geoid and mean $5^{\circ} \times 5^{\circ}$ block anomalies having uncorrelated errors. Compared are the resulting errors in geoidal undulations and deflections of the vertical.

FOREWORD

This report has been prepared by Peter Meissl, Visiting Research Associate, Department of Geodetic Science, at the Ohio State University, under Air Force Contract No. F19628-C-69-0127, OSURF Project No. 2758, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen W. Williams and Mr. Bela Szabo, Project Scientists.

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CONTENTS

1. Introduction and Summary

It is a well established fact that satellite methods are very capable of yielding good information on the lower degree harmonics of the earth's potential. Details of the geoid or the gravity anomaly field can at the present time only be brought out by gravimetric methods. The reason why these details cannot be obtained from satellite perturbation is two-fold. First the variations of gravity at the earth's surface or close to it are attenuated at the altitude of the satellites. Second the perturbed motion of the satellite is a two-fold integral (solution of a second order differential equation) over the gravity disturbances along its path. This integration process generally favors the low frequent part of the anomalous gravity field. Low frequent gravity variations have a greater impact on the satellite motion than the short period variations. Exceptions are certain resonance effects. They allow to determine certain isolated coefficients of higher degree. However this does not contribute significantly to the knowledge of regional gravity variations. Those can only be obtained from gravity measurements. From gravimetry alone the low degree harmonic coefficients of the earth's potential are not so good determined at the present time because of the unsurveyed areas where no or only little gravity information is available.

Thus a combination of gravimetric and satellite information is the most promising approach toward a better knowledge of the earth's potential. Numerous combined solutions have been computed. The most recently published is that of Gaposchkin-Lambeck [1970]. Any of these solutions yields spherical harmonic coefficients complete and with good accuracy up to and including a certain degree N. Disregarding possible higher degree harmonics with lower accuracy as well as the aforementioned isolated coefficients one can compute a geoid based on the harmonics up to and including degree N. One can compute gravity anomalies with respect to this geoid. They are generally smaller than the original gravity anomalies with respect to a reference ellipsoid. The original gravity anomalies can be split into a low frequent portion associated with the geoid based on the coefficients up to N and a residual, mostly high frequent portion. The residual portion is unknown in areas with no gravity coverage. Substituting the low frequent part for the whole anomalies means that an error equal to the residual part is committed. Thus, it may be desirable to know about the statistical behavior of the residual part. This may not only be of interest for the unsurveyed areas. Replacing the true anomalies by the analytically and computationally simpler low frequent portion may be advantageous also for the surveyed areas provided that the error is tolerable for the purpose of using the anomalies.

The main objective of this report is to study the statistical behavior of the residual anomalies. It is a widely adopted rule of the thumb to assume that the residual anomalies after removal of the harmonics up to degree N have statistical properties comparable to those of uncorrelated $\frac{180^\circ}{N} \times \frac{180^\circ}{N}$ block averages having the same variance as the residuals. It will be shown that this is only partially justified.

First it will be shown that the covariance of the residual anomalies has certain theoretical properties which are quite general and hold for a wide class of stochastic processes on the sphere, if the contribution of the harmonics up to and including N is removed.

These properties are:

1. The covariance has at least $N + 1$ distinct zeros in the interval $0 \leq \psi \leq 180^\circ$;
2. The covariance is negative in the neighborhood of the smallest positive zero of $P_N(\cos\psi)$ (i.e. the zero for which $\cos\psi$ is largest and therefore closest to 1.)

   The covariance is positive around the second smallest zero, negative around the third smallest zero and so on.

Property (1) is a rigorous one. Property (2) is true only under certain additional conditions. Moreover, property (2) is most likely to hold for a few of the smallest zeros. For the larger zeros it may fail to hold.

These properties can be verified by looking at covariances found in the literature. Familiar are the two zeros of a covariance after removal of the zero and first order degree harmonics which bear no meaning for many questions within physical geodesy.

Rapp [1967] has encountered negative correlation between neighboring $15^\circ \times 15^\circ$ means of residuals. This is in agreement with property (2) if one considers that at the time of this investigation satellite methods yielded a satisfactory accuracy only up to $N = 8$. Under this assumption the covariance of the residuals should be definitely negative around $\psi = 16^\circ$.

In this report we base our numerical estimates mainly on the Gaposchkin-Lambeck [1970] solution. Though Gaposchkin and Lambeck give a complete solution up to $N = 16$ and in addition several isolated higher coefficients, we take $N = 12$ since we feel that the relative accuracy for the higher harmonics is inferior. As the basic gravity anomaly field we use that given by Kaula [1966] and which comprises averages over areas comparable to $5^\circ \times 5^\circ$ blocks at the equator. This field has also been used by Gaposchkin-Lambeck for their statistical analysis.

The $5^\circ \times 5^\circ$ mean anomalies with respect to the international ellipsoid have an overall standard deviation of $\sqrt{274} = 17$ mg. If we compute gravity anomalies, not with respect to the ellipsoid but with respect to a geoidal surface derived from the Gaposchkin-Lambeck harmonic coefficients up to degree 12, the residual anomalies have a standard deviation of about $\sqrt{150} = 12$ mg. From the theoretical properties (1), (2) above we expect a covariance of the residual anomalies which has value ≈ 150 at $\psi = 0$, which is negative around $\psi = 11^\circ$, positive around 25°. It is otherwise expected to ocsillate along the abscissa. The covariance estimated in section 6, Table 11, has indeed these properties. See also the figure on page 84.

The question is now the following. How do these anomaly residuals with respect to the Gaposchkin-Lambeck [1970] geoid, which residuals in unsurveyed areas have to be regarded as errors, compare with purely gravimetrically derived block means having errors of about 6 mg and no significant error correlation between blocks. An accuracy of 6 mg or better can be obtained by airborne and shipborne gravimetry methods.

If we take the errors in the resulting geoid as a basis of comparison then we have to say that the two sets of anomalies cause about the same error in the geoid. This error is about 3-4 m. The geoid depends mainly on the low frequent portion of the anomalies and these are determined very well by satellite observation. Though the anomaly residuals are of large standard deviations (12 mg) their impact upon geoid errors is much smaller due to their peculiar correlation which is significantly negative for certain distances. This peculiarity of the correlation is, as we have indicated above, nothing but a consequence of the fact that the influence of the lower degree harmonics has been (nearly) removed.

Uncorrelated block errors of 6 mg have still some small low frequent portions and those cause errors in the geoid which, relative to the error standard deviation, are larger.

The undulation of the geoid should however not be taken as the only basis for comparison. In inertial navigation the deflection of the vertical or in other words the inclination of the geoid is the most decisive quantity. A comparison of the two sets of anomalies performed on the basis of the deflection of the vertical gives a quite different picture.

First, we have to clarify that the deflections of the vertical which we compare are smoothed versions of the true deflections. They are smoothed in the same way as the underlying anomalies are, namely averaged over $5^{\circ} \times 5^{\circ}$ blocks. The same is true of the geoids discussed above. However, whereas the geoid varies little within $5^{\circ} \times 5^{\circ}$ areas, the deflection of the vertical shows much more variation.

The Gaposchkin-Lambeck geoid which has 12 mg correlated $5^{\circ} \times 5^{\circ}$ mean anomaly residuals has a residual error in the smoothed deflections of about 3".

Mean $5^{\circ} \times 5^{\circ}$ anomalies with uncorrelated errors of 6 mg yield an error in the smoothed deflections of 1"5. Thus we see that the slope of the geoid is by a factor of two better than in the case of Gaposchkin-Lambeck [1970] geoid with harmonics up to and including degree 12.

This is an outline of the main results of this report. The reader will find the derivation and a more detailed discussion in Part III. It remains to outline the purpose of Part I and II.

Part I contains a collection and unified presentation of the necessary mathematical tools. The geodesist is sometimes led to the presumption that as soon as a problem is specialized to the sphere, everything is a consequence of the special properties of spherical harmonics. In many treatizes one finds formal manipulations involving spherical harmonics which prove or demonstrate the same principle over and over again. This principle is that of isotropy or in other words the complete rotational symmetry of the sphere. Müller [1966] gave a treatize of spherical harmonics starting from this principle. Some of the ideas contained therein have been utilized for our presentation. The notion of what I call an isotropic operator (and what may be called by any other name elsewhere) having the spherical harmonics as eigen-function, greatly simplifies in my opinion most manipulations with spherical harmonics. The treatment also clearly shows the limitations of the approach. As soon as a problem looses its complete rotational symmetry the usefulness of spherical harmonics is diminished. The spectral decomposition of a stochastic process on the sphere involves spherical harmonics only if the covariance has the property of distance dependence. That means it depends only on the distance between two points. Distance dependence is in agreement with the rotational symmetry since two pairs of equidistant points can be brought to coincidence by a rotation. Such isotropic stochastic processes can be decomposed into uncorrelated harmonic components. Zero correlation between harmonic components is not always self-evident, especially dealing with satellite problems. On the contrary, correlation between different harmonics has often been verified. This observation shows that our model in Part III is a highly idealized one. Dropping the assumption of zero correlation between harmonics would, however, mean to abandon the isotropy property. The treatize would become much more involved.

Part II does not claim complete originality either. It is well known that the earth's disturbing potential can be represented by many different quantities like geoidal undulations, gravity anomalies, deflections of the vertical, a fictitious surface density, vertical gradient of gravity and many others. If these quantities are regarded with respect to the sphere and if the linear operators relating most of them are identified as isotropic operators then all further formal proofs are much simplified. The eigenvalues of the interrelating operators yield much more information than is usually utilized in the geodetic literature. First they allow to set up rigorous bounds for the mean square norm ratios of any two quantities. In case a stochastic process model is employed,the bounds hold also for the ratios of the standard deviations. Second, the qualitatively well-known fact that the geoidal undulations are much smoother than the gravity anomalies which in turn are smoother than their vertical gradients can be made precise in a quantitative sense. An absolute yardstick of smoothness can be introduced. It shows for example that gravity anomalies, surface density and deflections of the vertical are all quantities of equal smoothness. Their use in representing the earth's disturbing potential results in no differences in the encountered stability problems. A preference may only result from the viewpoint of computational convenience.

I have also tried to possibly clarify the inter-relations of the operators transferring the quantities into each other with what I call narrow-sense smoothing operators. By this I mean operators which remove or damp irregularities of a rapidly varying function. An example is the averaging operator over a certain area. If such an operator happens to be isotropic, like for example the averaging operator over a circular cap, then its interplay with the other isotropic operators is quite simple. The reason is that all isotropic operators commute. One consequence is that the above indicated bounds for the norm ratios (or standard deviation ratios) hold also true for the (narrow-sense) smoothed quantities.

Though I give full credit to previous workers in this field, from which I mention only Cook, Molodensky, Jeffreys, Moritz, De Witte, for establishing the spherical harmonic relationships from which everything else can also be deduced, I think that the presentation given here is helpful for a better understanding because it yields more insight into the underlying mathematical structure.

Part I

Mathematical Background

2. Hilbert spaces of functions on the unit sphere Γ.

It is not the purpose of this section to theorize. A few facts taken from the theory of Hilbert - (and Sobolev -) spaces of functions are specialized to the unit sphere Γ. The presentation is mathematically not rigorous. Proofs and arguments are only presented if they are formal and short. This and the following two sections shall support the understanding of the remainder and shall not introduce unnecessary sophistication.

Denote by ξ, η ... unit vectors, i.e. points on Γ. Consider the totality of real functions $f(\xi)$ which is quadratically integrable on Γ. These functions form the Hilbert space H_Γ . The square root $\|f\|_{H_\Gamma}$ of the following expression.

$$\|f\|^2_{H_\Gamma} = \int_\Gamma f(\xi)^2 d\Gamma(\xi) \qquad (2-1)$$

which is finite per assumption, is called the H_Γ - norm of $f(\xi)$. For any two functions $f(\xi)$, $g(\xi)$ one may form the inner product.

$$(f, g)_{H_\Gamma} = \int_\Gamma f(\xi) g(\xi) d\Gamma(\xi) . \qquad (2-2)$$

Then apparently

$$\|f\|_{H_\Gamma} = \sqrt{(f,f)} \qquad (2-3)$$

If $(f, g)_{H_\Gamma} = 0$, then f and g are called orthogonal. The geodesists usually deals with functions out of H_Γ in terms of spherical harmonics. Denote by $S_{nm}(\xi)$ $n = o, 1, \ldots$ $m = -n, \ldots, +n$ a system of orthogonal and normalized spherical harmonics. Then

$$\int_\Gamma S_{nm}(\xi) S_{pq}(\xi) d\Gamma(\xi) = \begin{cases} 4\pi & \text{for } n=p,\ m=q \\ 0 & \text{else} \end{cases}$$

The $S_{nm}(\xi)$ form then an orthogonal system of functions in H_Γ having equalnorm $\|S_{nm}\|_{H_\Gamma} = \sqrt{4\pi}$.

Remark: The reader should not be confused by the way the spherical harmonics are indexed. We have 2n + 1 harmonics for each n as required. We shall never need the explicit form of a specific harmonic. So we may index them in any way.

If we have an orthogonal system of vectors e_1, e_2, e_3 in the ordinary three-dimensional Euklidean space and if the e_i have equal length $\| e_1 \| = \| e_2 \| = \| e_3 \| = \ell$, then any other vector x of finite length $\| x \|$ can be represented in the form

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

with

$$x_i = \frac{1}{\ell^2} (x, e_i)$$

whereby

$$\| x \|^2 = \ell^2 (x_1^2 + x_2^2 + x_3^2)$$

The situation is formally the same in the Hilbert space H_Γ. Any function of "finite length", i.e. any quadratically integrable function (any function of H_Γ) can be expanded as

$$f(\xi) = \sum_{n=o}^{\infty} \sum_{m=-n}^{+n} f_{nm} S_{nm}(\xi) \qquad (2 - 4)$$

with

$$f_{nm} = \frac{1}{4\pi} (f, S_{nm})_{H_\Gamma} \qquad (2 - 5)$$

whereby

$$\| f \|^2_{H_\Gamma} = 4\pi \sum_{n=o}^{\infty} \sum_{m=-n}^{+n} f_{nm}^2 \qquad (2 - 6)$$

Returning to our three-dimensional anologon, we may say: any triplet of numbers x_1, x_2, x_3 gives rise to a vector x given by $x = x_1 e_1 + x_2 e_2 + x_3 e_3$. In H_Γ the situation is similar. Any sequence f_{nm}, having the property

$$\sum_{n,m} f_{nm}^2 < \infty$$

(note the abbreviation $\sum\limits_{n,m}$ for $\sum\limits_{n=o}^{\infty} \sum\limits_{m=-n}^{+n}$), gives rise to a function $f(\xi)$ given by (2 - 4) and having norm given by the square root of (2 - 6).

We say the $S_{nm}(\xi)$ form a complete orthogonal and normalized basis in H_Γ. The relationship (2 - 4), (2 - 5) shall be abbreviated in the future as

$$f \sim f_{nm} \tag{2 - 7}$$

If $f \sim f_{nm}$ and $g \sim g_{nm}$, then

$$(f, g)_{H_\Gamma} = 4\pi \sum_{n,m} f_{nm} g_{nm} . \tag{2 - 8}$$

This is sometimes called Parcevals' relation and has its well-known anologon in 3-space; namely, $(x, y) = \| x \| . \| y \| \cos(x,y) = \ell^2 (x_1 y_1 + x_2 y_2 + x_3 y_3)$.

Functions from H_Γ need not be finite everywhere. The relation (2 - 1) requires only the finiteness of the integral over the squared function. Functions from H_Γ need not be differentiable either. We are therefore interested in finding subclasses of functions in H_Γ which are smoother and better behaved in some sense. There are many ways in singling out such functions. We prefer the following approach. Consider the class of functions $f \sim f_{nm}$ in H_Γ for which not only

$$\| f \|^2_{H_\Gamma} = 4\pi \sum_{n,m} f^2_{nm} < \infty$$

but also

$$4\pi \sum_{n,m} n^2 f^2_{nm} < \infty$$

Since $n^2 \leq n(n+1) \leq 2n^2$, we may equivalently require

$$\| f \|^2_{H^1_\Gamma} = 4\pi \{ f^2_{oo} + \sum_{n,m} n(n+1) f^2_{nm} \} < \infty \tag{2 - 9}$$

As already indicated by the notation $\| f \|_{H^1_\Gamma}$ we comprise the totality of all these functions to another space H^1_Γ. It is also a Hilbert space with norm given by the square root of (2 - 9). The inner product for $f \sim f_{nm}$, $g \sim g_{nm}$ would then be

$$(f, g)_{H^1_\Gamma} = 4\pi \{ f_{oo} g_{oo} + \sum_{n,m} n(n+1) f_{nm} g_{nm} \} \tag{2 - 10}$$

The functions in H^1_Γ are certainly smoother than those functions in H_Γ which are not in H^1_Γ. The coefficients of the higher harmonics taper off more quickly. In fact, from (2 - 9) follows that at least $|f_{nm}| \leq \| f \|_{H^1_\Gamma} / (\sqrt{4\pi} \cdot n)$. Functions from H^1_Γ need not necessarily be continuous. However, they have the following remarkable property. They possess generalized derivatives of first order. Before discussing this we make a short vector - analytical digression.

Let $f(\xi)$ be a differentiable function on Γ. Then we denote by

$$\text{Grad } f(\xi)$$

its surface gradient. It is a vector, tangential to Γ and pointing into the direction of the largest increase of $f(\xi)$. The length of the vector Grad $f(\xi)$ is equal to the rate of increase of $f(\xi)$.

If one extends $f(\xi)$ to the whole 3-space (except the origin) by putting

$$f(x) = f(r\,\xi) = f(\xi)$$

then Grad $f(\xi)$ coincides with the usual gradient taken at $x = \xi$

$$\text{Grad } f(\xi) = \text{grad } f(x) \Big|_{x=\xi} \tag{2 - 11}$$

If f is twice differentiable then we can also apply the so-called surface Laplace operator toward $f(\xi)$. We have then

$$\text{Lap } f(\xi) = \text{Div Grad } f(\xi) \tag{2 - 12}$$

i.e. the Laplacean is the (surface -) divergence of the surface gradient. If we extend $f(\xi)$ in the same way as above then Lap $f(\xi)$ equals the ordinary three-dimensional Laplacean taken at $x = \xi$. Expressions for Grad, Div, and Lap in terms of differential geometric quantities are found in most textbooks on differential geometry. The geodesist may wish to consult Hotine (1969) or Strubecker (1969).

There are many interesting vector - analytical integral formulas involving these quantities. We present a few, specialized to the whole sphere Γ.

$$\int_\Gamma (\text{Grad } f, \text{ Grad } g)\, d\Gamma = -\int_\Gamma f \text{ Lap } g\, d\Gamma = -\int_\Gamma g \text{ Lap } f\, d\Gamma \quad \ldots \qquad (2-13)$$

$$f(\xi) = \frac{1}{4\pi}\int_\Gamma f(\eta)\, d\Gamma(\eta) + \frac{1}{4\pi}\int_\Gamma \left(\text{Grad}\left(2\ell n \frac{2}{|\xi-\eta|}\right), \text{ Grad } f(\eta)\right) d\Gamma(\eta) \quad \ldots \qquad (2-14)$$

$$f(\xi) = \frac{1}{4\pi}\int_\Gamma f(\eta) - \frac{1}{4\pi}\int_\Gamma 2\ell n\left(\frac{2}{|\xi-\eta|}\right) \cdot \text{ Lap } f(\eta)\, d\Gamma(\eta) \qquad (2-15)$$

These formulas are consequences of the so-called Green's formulas. Let B be a subarea of Γ and let ∂B denote its boundary. Let β be the arc length along the boundary and let the unit vector ν be tangential to Γ and normal (outward) to B. Green's first formula is then

$$\int_B (\text{Grad } f, \text{ Grad } g)\, d\Gamma = \int_{\partial B} f(\text{Grad } g, \nu)\, d\beta - \int_B f \text{ Lap } g\, d\Gamma \qquad (2-15a)$$

Specialized to the whole sphere $B = \Gamma$ this yields already (2 - 13). If one puts $g(\eta) = 2\ell n(2/|\xi-\eta|)$, where ξ is now a parameter, if one verifies $\text{Lap } g(\eta) \equiv 1$, if one lets B be the whole sphere except a small circular cap around ξ then one arrives in the usual manner at (2 - 15) by letting the cap contract toward ξ. (2 - 14) is in a similar way obtained from Green's second formula:

$$\int_\Gamma (f \text{ Lap } g - g \text{ Lap } f)\, d\Gamma = \int_{\partial B} [f(\text{Grad } g, \nu) - g(\text{Grad } f, \nu)]\, d\beta \qquad (2-15b)$$

We return now to our functions of H^1_Γ. If $f(\xi)$ out of H^1_Γ happens to be differentiable, then Grad $f(x)$ shall have the usual meaning. However, for any function $f(\xi) \sim f_{nm}$ in H^1_Γ there exists a (surface - tangential -) vector function Grad

$f(\xi)$ which acts in many ways like an ordinary gradient and is "squared integrable" fulfilling:

$$\int_\Gamma (\text{Grad } f, \text{Grad } f) d\Gamma = 4\pi \sum_{n,m} n(n+1) f_{nm}^2 < \infty \tag{2 - 16}$$

Thus from (2 - 9) and (2 - 5) specialized to f_{00} we get

$$\|f\|^2_{H^1_\Gamma} = \{[\int_\Gamma f d\Gamma]^2 + \int_\Gamma (\text{Grad } f, \text{Grad } f) d\Gamma\} \tag{2 - 17}$$

Moreover the inner product in H^1_Γ can be written as

$$(f, g)_{H^1_\Gamma} = \{\int_\Gamma f d\Gamma \int_\Gamma g d\Gamma + \int_\Gamma (\text{Grad } f, \text{Grad } g) d\Gamma\} \tag{2 - 18}$$

Let us motivate formula (2 - 16). Recalling from the theory of spherical harmonics that

$$\text{Lap } S_{nm}(\xi) = -n(n+1) S_{nm}(\xi) \tag{2 - 19}$$

we proceed in the following formal way. From

$$f(\xi) = \sum_{n,m} f_{nm} S_{nm}(\xi)$$

we derive by differentiation

$$\text{Grad } f(\xi) = \sum_{n,m} f_{nm} \text{ Grad } S_{nm}(\xi) \tag{2 - 20}$$

Now by (2-13) and (2 - 19)

$$\int_\Gamma (\text{Grad } S_{nm}, \text{Grad } S_{pq}) d\Gamma = -\int_\Gamma S_{nm} \text{ Lap } S_{pq} d\Gamma = n(n+1) \int_\Gamma S_{nm} S_{pq} d\Gamma$$

$$= \begin{cases} 4\pi \quad n(n+1) & \text{for } n = p, \ m = q \\ 0 & \text{else} \end{cases}$$

Thus

$$\int_\Gamma (\text{Grad } S_{nm}, \text{Grad } S_{pq}) d\Gamma = \begin{cases} 4\pi \ n(n+1) & \text{for } n = p, \ m = q \\ 0 & \text{else} \end{cases} \tag{2 - 21}$$

From (2 - 20) and (2 - 21) the formula (2 - 16) follows formally in an easy way, by inserting (2 - 20) into the left-hand side of (2-16). The justification of this formal approach as well as the extension to those functions of H^1_Γ for which (2 - 20) is problematic is a more difficult mathematical task which will not be undertaken here.

We mention that formula (2 - 14) holds for all f from H^1_Γ . Generally but loosely speaking, the generalized derivatives act like ordinary ones if they are involved in certain integrals.

Formulas (2 - 20), (2 - 21) show also the following. If we define the surface tangential system of vector functions U_{nm} by

$$U_{nm}(\xi) = \frac{1}{\sqrt{n(n+1)}} \text{ Grad } S_{nm}(\xi) \tag{2 - 22}$$

then the relation

$$\int_\Gamma (U_{nm}, U_{pq}) d\Gamma = \begin{cases} 4\pi & \text{for } n=m,\ p=q \\ 0 & \text{else} \end{cases} \tag{2 - 23}$$

holds. Thus the $U_{nm}(\xi)$ may be viewed as a orthogonal and normalized basis in a space of vector functions tangential to Γ. Call this space U_Γ and define it by all vector-functions representable in the form

$$u(\xi) = \sum_{n,m} u_{nm} U_{nm}(\xi) \tag{2 - 24}$$

with

$$\| u \|^2_{U_\Gamma} = 4\pi \quad \Sigma_{nm} u^2_{nm} < \infty \tag{2 - 25}$$

(Note that the u_{nm} are scalers and not vectors).

U_Γ is then a Hilbert space. It is the space of all generalized surface gradients of functions from H^1_Γ .

Given a sequence u_{nm} of constants for which (2 - 25) holds we get a vector function $u(\xi)$ by (2 - 24). Can we construct a function $f(\xi) \sim f_{nm}$ in H for which

$$\text{Grad } f(\xi) = u(\xi)$$

holds? Very easily. Put

$$f_{nm} = \frac{u_{nm}}{\sqrt{n(n+1)}} \tag{2 - 25a}$$

Then by (2 - 25)

$$4\pi \sum_{n,m} n(n+1) f_{nm}^2 = 4\pi \sum_{n,m} u_{nm}^2 < \infty$$

and

$$\text{Grad } f(\xi) = \text{Grad} \sum_{n,m} f_{nm} S_{nm}(\xi) = \sum_{n,m} f_{nm} \text{ Grad } S_{nm}(\xi)$$

$$= \sum_{n,m} f_{nm} \sqrt{n(n+1)}\, U_{nm}(\xi) = \sum_{n,m} u_{nm} U_{nm}(\xi) = u(\xi)$$

The norm $\| u \|_{U_\Gamma}$ in U_Γ is given by

$$\| u \|_{U_\Gamma}^2 = \int_\Gamma (u, u) d\Gamma \tag{2 - 26}$$

This follows from (2 - 23) and (2 - 24). The inner product in U_Γ is given by

$$(u, v)_{U_\Gamma} = \int_\Gamma (u, v) d\Gamma \tag{2 - 27}$$

From (2 - 17) we have

$$\| f \|_{H_\Gamma^1}^2 = \left(\int_\Gamma f \, d\Gamma\right)^2 + \| \text{Grad } f \|_{U_\Gamma}^2 \tag{2 - 28}$$

If we call $\tilde{H}_\Gamma^1$ the subspace of H_Γ^1 with the property that all its members $f \sim f_{nm}$ fulfill

$$f_{00} = \frac{1}{4\pi} \int_\Gamma f \, d\Gamma = 0$$

then we see that $\tilde{H}^1_\Gamma$ and U_Γ are isometric:

$$\|f\|_{H^1_\Gamma} = \|\mathrm{Grad}\ f\|_{U_\Gamma}, \quad (f, g)_{H^1_\Gamma} = (\mathrm{Grad}\ f, \mathrm{Grad}\ g)_{U_\Gamma} \qquad \ldots \qquad (2 - 29)$$

Remark: (a digression). The space U_Γ does not coincide with W_Γ, the space all surface vector functions $w(\xi)$ an Γ for which

$$\|w\|^2_W = \int_\Gamma (w(\xi), w(\xi))\, d\Gamma(\xi)$$

is finite. Only a vector function $u(\xi)$ which is the surface - gradient of some f in H^1_Γ is in U_Γ. If $u(\xi)$ is differentiable, then

$$\mathrm{Rot}\ u(\xi) = 0$$

is a sufficient condition for $u(\xi)$ to be in U_Γ. U_Γ is some sort of completion of such functions. $U_{nm}(\xi) = \mathrm{Grad}\ S_{nm}(\xi) / \sqrt{n(n+1)}$ is a basis in U_Γ.

Likewise one can form a space V_Γ which shall be a completion of differentiable vector functions $v(\xi)$ fulfilling

$$\mathrm{Div}\ v(\xi) = 0$$

One can show that the vector - functions $V_{nm}(\xi) = \xi \times U_{nm}(\xi)$ form an orthogonal and normalized basis in V_Γ and that W_Γ consists of all functions

$$w(\xi) = u(\xi) + v(\xi), \quad u \in U_\Gamma, \ v \in V_\Gamma$$

In other words, W_Γ is the direct sum of its (orthogonal) subspaces U_Γ, V_Γ.

We consider one more Hilbert space on Γ. It is the subspace of H_Γ and also of H^1_Γ for which

$$\|f\|^2_{H^2_\Gamma} = 4\pi \left\{ f^2_{oo} + \sum_{n,m} (n(n+1))^2 f^2_{nm} \right\} < \infty \qquad (2 - 30)$$

$\| f \|_{H^2_\Gamma}$ will then be the norm of this Hilbert space H^2_Γ. The inner product may be formed accordingly. H^2_Γ consists of all functions in H^1_Γ for which the Laplacean exists in a generalized sense and is squared integrable. The squared norm (2 - 30) may be written as

$$\| f \|^2_{H^2_\Gamma} = [\int_\Gamma f d\Gamma]^2 + \int_\Gamma (\text{Lap } f)^2 d\Gamma \tag{2 - 31}$$

This may be formally motivated in a way analogous to that described for H^1_Γ.

For functions of H^2_Γ all vector analytical formulas which have been listed hold true. Moreover functions of H^2_Γ are continuous and therefore bounded. Continuity can be derived from (2 - 15) or also from the spherical harmonics series. The argument for the latter approach runs as follows:

From (2 - 30) it follows that

$$f_{nm} = O(\frac{1}{n^2}) \tag{2 - 32}$$

Together with

$$S_{nm} (\xi) \leq \sqrt{2n + 1} \tag{2 - 33}$$

(see Müller (1966), Lemma 8 on p. 14) we see that the series of continuous functions

$$f(\xi) = \sum_{n,m} f_{nm} S_{nm} (\xi)$$

is absolutely and uniformly convergent. Hence $f(\xi)$ is continuous.

(2 - 32) shows again that the functions of H^2_Γ are smoother than those of H^1_Γ. We could now go on and define further Hilbert spaces H^k_Γ with the property that

$$\| f \|^2_{H^k_\Gamma} = 4\pi \left\{ f^2_\infty + \sum_{n,m} (n(n + 1))^k f^2_{nm} \right\} < \infty \tag{2 - 34}$$

These spaces certainly contain smoother and smoother functions. We shall not need them explicitly for $k > 2$.

What we shall be interested in is to identify some linear operators from H_Γ^k into $H_\Gamma^{k+\ell}$ or $H_\Gamma^{k-\ell}$ for some $\ell \geq 0$. We shall find some of these operators in the next section.

Thereby we shall sometimes use the following subspaces of H_Γ^k :

$\tilde{H}_\Gamma^k$ consisting of all functions $f \sim f_{nm}$ in H_Γ^k for which $f_{oo} = 0$.

$\tilde{H}_\Gamma^k$ consisting of all functions $f \sim f_{nm}$ in H_Γ^k for which $f_{oo} = f_{1,1} = f_{1,0} = f_{1,1} = 0$.

3. Isotropic linear operators on Γ.

We consider in this section linear operators transforming functions on the unit sphere into other such functions. We write this symbolically as

$$g = Lf \tag{3 - 1}$$

or sometimes as:

$$g(\xi) = L(\xi, \eta) f(\eta) \tag{3 - 1a}$$

Notation (3 - 1a) is suggested by the special case of an integral operator

$$g(\xi) = \int_{\Gamma} K(\xi, \eta) f(\eta) d\Gamma(\eta) \tag{3 - 1b}$$

We shall be particularly interested in operators with the following property. Denote by U an orthogonal 3×3 matrix. Define the linear operator R_U by

$$R_U(\xi, \eta) f(\eta) = f(U\xi) \tag{3 - 2}$$

We require now of L that it commutes with any R_U for arbitrary but orthogonal U:

$$R_U L = L R_U \tag{3 - 3}$$

In other symbols

$$L((U\xi), \eta) f(\eta) = L(\xi, \eta) f(U\eta) \tag{3 - 3a}$$

If we visualize the function $f(\xi)$ as a topographical surface over Γ, then the relationship (3 - 2) transforms this surface by rotating it around Γ. The shape of the surface is not altered. Application of L toward the rotated surface shall produce the same result as application of L toward the unshifted surface followed by the rotation. It is clear that L must have certain symmetry properties in order to fulfill this requirement.

In order to give some name to these operators we call them isotropic on Γ. The class of these operators has the following remarkable characterization. The spherical harmonics form a complete system of eigen-functions. The eigen-value belonging to $S_{nm}(\xi)$ depends only on n and shall be called λ_n. In symbols

$$L(\xi, \eta)\, S_{nm}(\eta) = \lambda_n\, S_{nm}(\xi) \tag{3.- 4}$$

Denote by $\xi \cdot \eta$ the inner product of ξ and η: $\xi \cdot \eta = \cos\psi$, ψ...angle between ξ and η. A special spherical harmonic of degree n is then given by $P_n(\cos\psi) = P_n(\xi \cdot \eta)$, where $P_n(t)$ is the usual Legendre polynomial. $P_n(\xi \cdot \eta)$ is a spherical harmonic in ξ for fixed η and also vice versa. Since $S_{nm}(\xi)$ need not be normalized in (3 - 4) we may insert

$$S_{nm}(\xi) = P_n(\zeta \cdot \xi)$$

into (3 - 4). If we afterward put $\zeta = \xi$ and observe $P_n(\xi \cdot \xi) = P_n(1) = 1$, we get

$$\lambda_n = L(\xi, \eta)\, P_n(\xi \cdot \eta) \tag{3 - 5}$$

We shall verify that any operator having property (3 - 4) has property (3 - 3a). From (3 - 4) it follows for $f(\xi) \sim f_{nm}$

$$Lf = L \sum_{n,m} f_{nm} S_{nm} = \sum_{n,m} f_{nm}\, L S_{nm} = \sum_{n,m} \lambda_n\, f_{nm}\, S_{nm}$$

Thus

$$L(\xi, \eta)\, f(\eta) = \sum_{n,m} \lambda_n\, f_{nm} S_{nm}(\xi) \tag{3 - 5a}$$

Replace ξ by $U\xi$, then

$$L((U\xi), \eta)\, f(\eta) = \sum_{n,m} \lambda_n\, f_{nm} S_{nm}(U\xi) \tag{3 - 5b}$$

On the other hand

$$L(\xi, \eta)\, f(U\eta) = L(\xi, \eta) \sum_{n,m} f_{nm}\, S_{nm}(U\eta)$$

Now we utilize that (cf., Müller, (1966))

$$S_{nm}(U\eta) = \sum_{p=-n}^{+n} \alpha_{nm}^{p} S_{np}(\eta)$$

Then we deduce in much the same way as above that

$$L(\xi,\eta) f(U\eta) = \sum_{n,m} \lambda_n f_{nm} S_{nm}(U\xi)$$

which coincides with (3 - 5b) and shows that (3 - 4) entails (3 - 3a).

The reverse is also true. At least after exclusion of pathological operators, (3 - 3a) entails (3 - 4). The proof given in Müller (1966) for integral operators with kernels of the form $K(\xi.\eta)$ can be modified to cover the more general case of an operator fulfilling (3 - 3a). However this will not be done here.

We give now examples for isotropic linear operators.

We mentioned already that the surface Laplace operator has the S_{nm} as eigen-functions with eigen-values $\lambda_n = -n(n+1)$. The Laplacean is therefore an isotropic operator on Γ.

We consider integral operators (3 - 1b) where the kernel $K(\xi,\eta)$ depends only on the distance $|\xi-\eta|$ or, equivalently, only on $\xi.\eta$ (inner product: $\xi.\eta = \cos\psi$, ψ... angle between ξ,η):

$$g(\xi) = \int_{\Gamma} K(\xi.\eta)\, f(\eta) d\Gamma(\eta) \qquad (3 - 6)$$

The property (3 - 3a) is readily verified

$$g(U\xi) = \int_{\Gamma} K(U\xi.\eta) f(\eta) d\Gamma(\eta) = \int_{\Gamma} K(\xi.U^{-1}\eta) f(\eta) d\Gamma(\eta)$$

substituting

$$U^{-1}\eta = \eta'$$

and noting

$$d\Gamma(\eta') = d\Gamma(\eta)$$

we get

$$g(U\xi) = \int_{\Gamma} K(\xi.\eta') f(U\eta') d\Gamma(\eta'), \text{ q. e. d.}$$

The $S_{nm}(\xi)$ are then eigen-functions:

$$\int_\Gamma K(\xi \cdot \eta)\, S_{nm}(\eta)\, d\Gamma(\eta) = \lambda_n S_{nm}(\xi) \tag{3 - 7}$$

The λ_n are by (3 - 5) obtained as

$$\lambda = \int_\Gamma K(\xi \cdot \eta)\, P_n(\xi \cdot \eta)\, d\Gamma(\eta)$$

Assuming a ϑ, λ system with pole in ξ we derive with $t = \cos\psi$:

$$\lambda_n = 2\pi \int_{-1}^{+1} K(t)\, P_n(t)\, dt \tag{3 - 8}$$

This is called Funk-Hecke formula in Müller (1966).

We give now some specific examples for integral operators with distance dependent kernels. Put $\ell(\xi \cdot \eta) = |\xi - \eta|$

$$g(\xi) = \frac{1}{4\pi} \int_\Gamma \frac{1}{\ell(\xi \cdot \eta)} f(\eta)\, d\Gamma(\eta) \;\ldots\; \lambda_n = \frac{1}{2n+1} \tag{3 - 9}$$

$$g(\xi) = \frac{1}{4\pi} \int_\Gamma St(\xi \cdot \eta) f(\eta)\, d\Gamma(\eta) \;\ldots\; \lambda_n = \begin{cases} 0 & \text{for } n = 1, 2 \\ \frac{1}{n-1} & \text{else} \end{cases} \qquad \ldots (3 - 10)$$

$$St(\xi \cdot \eta) = St(\cos\psi) \;\ldots\; \text{Stokes' function}$$

$$g(\xi) = \frac{1}{4\pi} \int_\Gamma \frac{f(\xi) - f(\eta)}{\ell(\xi \cdot \eta)^3} d\Gamma(\eta) \;\ldots\; \lambda_n = \frac{n}{2} \tag{3 - 11}$$

$$g(\xi) = \frac{1}{4\pi} \int_\Gamma 2\ln\left(\frac{2}{\ell(\xi \cdot \eta)}\right) f(\eta)\, d\Gamma(\eta) \;\ldots\; \lambda_n = \frac{1}{n(n+1)}, \; n \geq 1 \qquad \ldots (3 - 12)$$

$$g(\xi) = \frac{1}{4\pi} \int_\Gamma \ell(\xi \cdot \eta)\, f(\eta)\, d\Gamma(\eta) \;\ldots\; \lambda_n = -\frac{2}{(2n+1)^2(n-\frac{3}{2})} \tag{3 - 13}$$

We see that all these listed operators are operators from some H_Γ^k into some

$H_\Gamma^{k+\ell}$. The Laplacean is an operator from H_Γ^k into H_Γ^{k-2}. If functions $f \sim f_{nm}$ with $f_{oo} = 0$ are excluded, then the Laplacean has an inverse which is given by the negative of (3 - 12). Confer (2 - 15). (3 - 12) is an operator from $\tilde{H}_\Gamma^k$ onto $\tilde{H}_\Gamma^{k+2}$ (by $\tilde{H}_\Gamma^k$ we denote the subspace of H_Γ^k comprising functions $f \sim f_{nm}$ with $f_{oo} = 0$). Stokes' operator is an operator from $\tilde{H}_\Gamma^k$ (functions $f \sim f_{nm}$ with $f_{oo} = 0$ and $f_{1m} = 0$) onto $\tilde{H}_\Gamma^{k+1}$, having an inverse.

Let us agree within this study to call an operator with the property $\lambda_n \to 0$ a smoothing operator in the wide sense. $\lambda_n \to 0$ means that the higher harmonics are subdued by the operator. Then (3 - 9), (3 - 10), (3 - 12), (3 - 13) are wide sense smoothing operators. The Laplacean and (3 - 11) have an opposite effect; they unsmooth because the higher harmonics are highly amplified.

We shall also deal with smoothing operators in a narrower sense. In addition, to $\lambda_n \to 0$ we require $\lambda_o = 1$. In that case, the constant function is reproduced. Thus if one wants to smooth a function in the sense of removing, damping or filtering out its irregularities, one has these narrow sense smoothing operators in mind.

An example is the averaging operator over a circular area of Γ with angular radius ψ_o. Its integral kernel is

$$B(\xi \cdot \eta) = \begin{cases} \dfrac{1}{2\pi} \cdot \dfrac{1}{1 - \cos\psi_o} & \text{for } \xi \cdot \eta \geq \cos\psi_o \\ 0 & \text{for } \xi \cdot \eta < \cos\psi_o \end{cases} \qquad (3 - 14)$$

Its eigen-values are by the Funk-Hecke formula:

$$\beta_n = \frac{1}{1 - \cos\psi_o} \int_{\cos\psi_o}^{1} P_n(t)\,dt \qquad (3 - 14a)$$

Using the recurrence relation (cf. Lense (1954), § 13):

$$\frac{d}{dt} P_{n+1}(t) - \frac{d}{dt} P_{n-1}(t) = (2n+1) P_n(t)$$

we get for $n \geq 1$ (recall $\lambda_o = 1$)

$$\beta_n = \frac{1}{1-\cos\psi_o} \cdot \frac{1}{2n+1}\left\{P_{n-1}(\cos\psi_o) - P_{n+1}(\cos\psi_o)\right\} \tag{3 - 14b}$$

Since for fixed $t \neq 1$ we have

$$P_n(t) = 0(\frac{1}{\sqrt{n}}) \tag{3 - 15}$$

(cf. Lense (1954), § 24) we see that

$$\beta_n = 0(\frac{1}{n\sqrt{n}}) \tag{3 - 16}$$

This shows that $\beta \to 0$ as it should be for a smoothing operator. It also shows that it is an operator from H_Γ^k into H_Γ^{k+1}.

Let us point out another remarkable property of these isotropic operators on Γ. Any of these operators commutes with any other, at least if applied to sufficient smooth functions. The reason is that all these operators have a common system of eigen-functions.

If $f \sim f_{nm}$ and L has eigen-values λ_n and M has eigen-values μ_n then we have $Lf \sim \lambda_n f_{nm}$, $Mf \sim \mu_n f_{nm}$, and

$$LMf = MLf \sim \lambda_n \mu_n f_{nm} \tag{3 - 17}$$

This shows that $LM = ML$ at least for functions for which

$$\| LMf \|^2 = 4\pi \sum_{n,m} (\lambda_n \mu_n f_{nm})^2 < \infty \tag{3 - 18}$$

From this we see that it is immaterial whether we apply a narrow-sense isotropic smoothing operator such as (3 - 14) toward a function, like gravity anomalies and compute from it a smoothed version of the geoidal undulations by Stokes' formula, or whether we make the transition from anomalies toward undulations first and apply the smoothing operator afterwards. The result is the same since the smoothing operator commutes with the Stokes' operator.

During this discussion the following question arises. We have introduced narrow-sense smoothing operators such as (3 - 14) acting on numerical valued functions $f(\xi)$.

How does the smoothing operator act on vector functions $u(\xi)$ out of U_Γ. For functions $f(\xi) \sim f_{nm}$ the operator B yields a function $\tilde{f}(\xi) \sim \tilde{f}_{nm}$ with $\tilde{f}_{nm} = \beta_n f_{nm}$. We want the same for $u(\xi) \sim u_{nm}$ (coefficients with respect to the system $U_{nm}(\xi) = (\sqrt{n(n+1)})^{-1} \text{Grad } S_{nm}(\xi).$). The smoothed function $\tilde{u}(\xi) \sim \tilde{u}_{nm}$ shall be given by $\tilde{u}_{nm} = \beta_n . u_{nm}$. If $f(\xi)$ happens to be the function for which

$$u(\xi) = \text{Grad } f(\xi) \tag{3 - 18a}$$

holds, then necessarily

$$\tilde{u}(\xi) = \text{Grad } \tilde{f}(\xi) \tag{3 - 18b}$$

In fact, (3 - 18a) is equivalent to $u_{nm} = \sqrt{n(n+1)}\, f_{nm}$ (cf. (2-25a)). Multiplication with β_n on both sides gives $\tilde{u}_{nm} = \sqrt{n(n+1)}\, \tilde{f}_{nm}$ and this in turn is nothing but (3 - 18b).

If the smoothing integral kernel $B(\xi . \eta)$ is specified as applying to functions $f(\eta)$, then the corresponding matrix kernel acting on vectors $u(\xi)$ can be analytically derived with the help of (2 - 14). The following result is obtained:

$$\tilde{u}(\xi) = \int_\Gamma M(\xi, \eta)\, u(\eta)\, d\Gamma(\eta) \tag{3 - 18c}$$

$M(\xi, \eta)$ is a 3 x 3 matrix given by

$$M(\xi, \eta) = \frac{1}{4\pi} \text{Grad}_\xi \ \text{Grad}_\eta^T \int_\Gamma B(\xi . \zeta)\, 2\ln \frac{2}{|\zeta - \eta|} \ d\Gamma(\zeta) \tag{3 - 18d}$$

The notation Grad_η^T shall indicate that the differentiation with respect to η shall produce rows whereas that with respect to ξ produces columns.

We shall not need later on this explicit form of the smoothing operator acting on $u(\xi)$.

Finally we shall be interested in the following. If two functions $f \sim f_{nm}$, $g \sim g_{nm}$ out of H_Γ are related by the isotropic operator L, having eigen-values λ_n

then the relationship

$$g = Lf$$

may, as we have seen in (3 - 5a), be written as

$$g_{nm} = \lambda_n f_{nm}$$

We shall be interested in the possible range of the ratio

$$\frac{\| g \|_{H_r}}{\| f \|_{H_r}}$$

From

$$\| f \|_{r}^{2} = 4\pi \sum_{n,m} f_{nm}^{2} \tag{3 - 19}$$

and

$$\| g \|_{H_r}^{2} = 4\pi \sum_{n,m} g_{nm}^{2} = 4\pi \sum_{n,m} (\lambda_n f_{nm})^2 \tag{3 - 20}$$

the following will be shown

$$GLB(|\lambda_n|) \leq \frac{\| g \|_{H_r}}{\| f \|_{H_r}} \leq LUB(|\lambda_n|) \tag{3 - 21}$$

GLB stands for greatest lower bound which is some sort of a generalized minimum, and LUB stands for least upper bound, some sort of a generalized maximum. Take for example $\lambda_n = \frac{1}{n}$, $n = 1, 2, \ldots$ Then the minimum of the λ_n does not exist. The greatest lower bound is however zero.

To see the validity of (3 - 21) as a consequence of (3 - 19), (3 - 20) consider the following simpler problem. Find by a proper choice of $\varphi_1 \ldots, \varphi_N$ the extrema of

$$\frac{\sum_{i=1}^{N} \lambda_i^2 \varphi_i^2}{\sum_{i=1}^{N} \varphi_i^2}$$

This is equivalent to finding the extrema

$$\sum_{i=1}^{N} \lambda_i^2 \varphi_i^2$$

subject to the constraint

$$\sum_{i=1}^{N} \varphi_i^2 = 1$$

Calculus yields then the extrema in the form

$$\underset{1 \le i \le N}{\mathrm{Min}} \ \lambda_i^2 \le \frac{\sum_{i=1}^{N} \lambda_i^2 \varphi_i^2}{\sum_{i=1}^{N} \varphi_i^2} \le \underset{1 \le i \le N}{\mathrm{Max}} \ \lambda_i^2$$

In the infinite case Max and Min have to be replaced by GLB and LUB.

It is remarkable that the inequalities (3 - 21) remain true, if we apply an isotropic operator B, with eigen values β_n to both functions g, f. Call

$$\tilde{f} = Bf$$

$$\tilde{g} = Bf$$

Then we have with $\tilde{f} \sim \tilde{f}_{nm}$, $\tilde{g} \sim \tilde{g}_{nm}$

$$\tilde{f}_{nm} = \beta_n f_{nm}, \ \tilde{g}_{nm} = \beta_n f_{nm}$$

i.e.

$$\tilde{g}_{nm} = \lambda_n \tilde{f}_{nm}$$

Hence

$$\tilde{g} = L\tilde{f}$$

and consequently

$$\mathrm{GLB}(|\lambda_n|) \le \frac{\|\tilde{g}\|_{H_\Gamma}}{\|\tilde{f}\|_{H_\Gamma}} \le \mathrm{LUB}(|\lambda_n|) \qquad (3 - 22)$$

If B is a narrow-sense smoothing operator then we see that the bounds of the norm ratios remain valid if we deal with smoothed versions of the functions f , g.

Remark: If some of the eigen-values β_n of B happen to be zero, then the corresponding λ_n may be excluded from consideration in (3 - 22). This may in some cases improve the bounds for the norm ratio of $\tilde{g}$ and $\tilde{f}$.

4. Isotropic stochastic processes on the unit sphere.

Consider the series

$$x(\xi) = \sum_{n,m} x_{nm} S_{nm}(\xi) \tag{4 - 1}$$

and assume that the x_{nm} are uncorrelated random variables with variances

$$\sigma_{nm}^2 = \frac{1}{2n+1} \sigma_n^2 \tag{4 - 2}$$

Then $x(\xi)$ may be regarded as a random variable depending on ξ, i.e. as a stochastic process on Γ.

The covariance of $x(\xi)$ and $x(\eta)$ is computed as

$$C(\xi,\eta) = E\{x(\xi) \cdot x(\eta)\} = \sum_{n,m} \sigma_n^2 \frac{1}{2n+1} S_{nm}(\xi) S_{nm}(\eta) \tag{4 - 3}$$

By the addition theorem for spherical harmonics (cf. Müller, 1966, p. 9):

$$P_n(\xi \cdot \eta) = \frac{1}{2n+1} \sum_{m=-n}^{+n} S_{nm}(\xi) S_{nm}(\eta) \tag{4 - 4}$$

we obtain

$$C(\xi,\eta) = \sum_{n=o}^{\infty} \sigma_n^2 P_n(\xi \cdot \eta)$$

We see that $C(\xi,\eta)$ depends only on the inner product $\cos\psi = \xi \cdot \eta$ or equivalently only on the distance $|\xi - \eta|$. We write therefore $C(\xi \cdot \eta)$ instead of $C(\xi,\eta)$ and obtain

$$C(\xi \cdot \eta) = \sum_{n=o}^{\infty} \sigma_n^2 P_n(\xi \cdot \eta) \tag{4 - 5}$$

or, replacing $\xi \cdot \eta$ by $t = \cos\psi$

$$C(t) = \sum_{n=o}^{\infty} \sigma_n^2 P_n(t) \tag{4 - 6}$$

Because the correlation between $x(\xi)$ and $x(\eta)$ depends only on $\xi . \eta$ or on $|\xi - \eta|$, $x(\xi)$ is called an isotropic stochastic process on Γ. The σ_n^2 are called degree-variances. Isotropic stochastic processes are the counterpart of the more familiar stationary stochastic processes depending on the argument t(time).

Since the sequence of steps leading to equation (4 - 6) can be meaningfully reversed we see that any isotropic stochastic process on Γ can be represented in the form (4 - 1) which is called its spectral representation.

Let $L(\xi, \eta)$ denote an istropic linear operator on Γ. If $x(\xi)$ is a stationary stochastic process then we shall be interested in the linear transformation of this process.

$$y(\xi) = L(\xi, \eta)\, x(\eta) \tag{4 - 7}$$

Let λ_n be the eigen-values of $L(\xi . \eta)$. Then from (4 - 1)

$$y = Lx = L \sum_{n,m} x_{nm} S_{nm} = \sum_{n,m} x_{nm} L_{nm} S_{nm}$$

Thus by $LS_{nm} = \lambda_n S_{nm}$ (cf. (3 - 4))

$$y(\xi) = \sum_{n,m} \lambda_n x_{nm} S_{nm}(\xi) \tag{4 - 8}$$

It becomes apparent that $y(\xi)$ is also an isotropic stochastic process on Γ having degree variances

$$\sigma_n^2(y) = \lambda_n^2 \sigma_n^2(x) \tag{4 - 9}$$

The covariance of $y(\xi)$ is obtained as

$$C_y(\xi . \eta) = L(\xi, \xi')\, L(\eta, \eta')\, C_x(\xi' . \eta') \tag{4 - 10}$$

This follows either from (4 - 7) and the propagation law for covariances, or by forming $E(y(\xi) . y(\eta))$ using (4 - 8) and comparing with $L(\xi, \xi') L(\eta, \eta')$ $C_x(\xi' . \eta')$ based on (4 - 5).

Thus we see: an isotropic operator on Γ transforms an isotropic process on Γ into another such process. The degree variances of the new process are obtained from those of the old one by multiplication with the squares of the eigen-values of the operator.

If the operator has smoothing properties then we see that the higher degree variances are subdued. Operators with $\lambda_n \to \infty$ have the opposite effect. Let us have a closer look at this phenomena by comparing the two formulas

$$\sigma^2(x) = \sum_{n=0}^{\infty} \sigma_n^2 \qquad (4 - 11)$$

$$\sigma^2(y) = \sum_{n=0}^{\infty} \lambda_n^2 \sigma_n^2 \qquad (4 - 12)$$

which follow for

$$y(\xi) = L(\xi, \eta)\ x(\xi)$$

from (4 - 5) and (4 - 9) by putting ξ equal to η. λ_n are of course the eigen-values of L.

We can pose the following question. What is the range of values for the ratio $\sigma(y)/\sigma(x)$. The problem is analogous to that encountered in section 3 where we were dealing with the ratio $\|g\|_{H_\Gamma} / \|f\|_{H_\Gamma}$. Using the same approach as there we find

$$\mathrm{GLB}\ (|\lambda_n|) \leq \frac{\sigma(y)}{\sigma(x)} \leq \mathrm{LUB}\ (|\lambda_n|) \qquad (4 - 14)$$

We recall that GLB stands for greatest lower bound which is a generalization of the concept of a minimum. LUB stands for least upper bound which is a generalization of maximum. For example the GLB of the numbers $\frac{1}{n}$, $n = 1, 2, \ldots$ is zero, a minimum does not exist. Note that inequalities (4 - 14) are formally the same as the inequalities (3 - 21) for the ratio of the norms of two functions related by the operator L. For a smoothing operator we have $\lambda_n \to 0$. Hence GLB ($|\lambda_n|$) = 0. The LUB ($|\lambda_n|$) must necessarily be finite (and constitute a true maximum). Thus we have for smoothing operators

$$0 \leq \frac{\sigma(y)}{\sigma(x)} \leq \operatorname{Max}(|\lambda_n|) \tag{4 - 15}$$

For operators with the opposite property: $\lambda_n \to \infty$ (as for example for the Laplacean), we have

$$\operatorname{Min}(|\lambda_n|) \leq \frac{\sigma(y)}{\sigma(x)} \leq \infty \tag{4 - 16}$$

Such operators may badly blow up the variance, even to the limiting case of infinity. This can never happen with smoothing operators which in turn may yield image-processes $y = Lx$ which are nearly deterministic $(\sigma^2(y) = 0)$. All kinds of intermediate cases are also possible and we shall encounter some in the sequel. We shall see that the transition from gravity anomalies to undulations of the geoid has property (4 - 15). The transition from anomalies to the vertical gradient of gravity has property (4 - 16). The transition from anomalies to deflections of the vertical, however, involves non-zero and finite upper and lower bounds in the form

$$0 < \operatorname{GLB}(|\lambda_n|) \leq \frac{\sigma(y)}{\sigma(x)} \leq \operatorname{LUB}(|\lambda_n|) < \infty \tag{4 - 17}$$

It may be interesting to throw some more light onto the inequalities of type (4 - 15), (4 - 16), (4 - 17) by asking for processes $x(\xi)$ and operators L for with the σ - ratio comes close to the upper or lower bounds.

Assume first that we have a (wide sense) smoothing operator with $\lambda_n \geq \lambda_{n+1}$ and $\lambda_n = 0(\frac{1}{n})$. The operators (3 - 9), (3 - 13), (3 - 12) fulfill this assumption. Stokes' operator (3-10) is another example if λ_o, λ_1 are excluded from consideration.

It is clear that $\operatorname{LUB}(|\lambda_n|) = \lambda_o$ (or = λ_2 in the case of Stokes' operator). If we take the process

$$x(\xi) = x_{oo} S_{oo}(\xi)$$

with

$$\sigma(x) = \sigma_o(x) \quad (\sigma_o^2(x) \ldots \text{ zero degree variance of } x(\xi))$$

then

$$\sigma(y) = |\lambda_o| \, \sigma(x_{oo} S_{oo}) = |\lambda_o| \, \sigma_o(x)$$

and

$$\frac{\sigma(y)}{\sigma(x)} = |\lambda_o|$$

Hence, we have an example of a process for which the upper boundary of the σ - ratio is reached. In the case of Stokes' operator we have to take

$$x(\xi) = \sum_{m=-2}^{+2} x_{2m} S_{2m}(\xi)$$

with

$$\sigma(x) = \sigma_2(x)$$

We find then

$$\sigma(y) = |\lambda_2| \, \sigma_2(x)$$

and

$$\frac{\sigma(y)}{\sigma(x)} = |\lambda_2|$$

Processes of this type are not very realistic. However, the following is clear. If the contribution toward $\sigma(x)$ comes mainly from a few low degree harmonics then the ratio $\sigma(y) / \sigma(x)$ will be close to the upper bound.

Introducing on the other hand the sequence of operators

$$x^{[N]}(\xi) = \sum_{m=-N}^{+N} x_{Nm} S_{Nm}(\xi)$$

it is seen that the ratio $\sigma(y^{[N]})/\sigma(x^{[N]})$ tends toward zero for $N \to \infty$. Processes of this type are even more unrealistic. However we see that a concentration of variance in the higher harmonics tends to push the σ-ratio near to its lower limit zero.

Consider as a special case the process

$$x^{(N)}(\xi) = \alpha^2(N) \sum_{n=o}^{+N} \sum_{m=-n}^{+N} x_{nm} S_{nm}(\xi) \tag{4 - 18}$$

where the variances of all x_{nm} are equal to unity

$$\sigma(x_{nm}) = 1 \tag{4 - 19}$$

Such a process has degree variances $(2n + 1)\alpha^2(N)$ and if we choose

$$\alpha^2(N) = \frac{1}{(N+1)^2} \tag{4 - 20}$$

then the variance of $x^{(N)}(\xi)$ is unity:

$$\sigma(x^{(N)}) = 1 \tag{4 - 21}$$

The covariance of this process is

$$C^{(N)}(\xi \cdot \eta) = \alpha^2(N) \sum_{n=o}^{N} (2n+1) P_n(\xi \cdot \eta) \tag{4 - 22}$$

The significance of these covariance is that for $N \to \infty$ it tends to the degenerate covariance of a process having variance equal to 1 and no correlation between different points. We may talk of this limit case as of "white noise" on the unit sphere.

In order to demonstrate this consider the integral kernel

$$P^{(N)}(\xi \cdot \eta) = \frac{1}{4\pi} \sum_{n=o}^{\infty} (2n+1) P_n(\xi \cdot \eta) \tag{4 - 23}$$

$P^{(N)}$ differs only by a factor from $C^{(N)}$. The effect of applying this integral kernel toward a function $f(\xi) \sim f_{nm}$ is simply the following

$$\int_\Gamma P^{(N)}(\xi.\eta) \sum_{n=o}^{\infty} f_{nm} S_{nm}(\eta) d\Gamma(\eta) = \sum_{n=o}^{N} f_{nm} S_{nm}(\xi) \qquad (4-24)$$

Thus the kernel $P^{(N)}$ transforms a function into its truncated spherical harmonics expansion.

As it is known and as it is rigorously proved in Hobson (1931) we have for sufficiently smooth functions

$$\lim_{N\to\infty} \int_\Gamma P^{(N)}(\xi.\eta) f(\eta) d\Gamma) = f(\xi) \qquad (4-25)$$

This means that the kernels concentrate more and more around $\eta = \xi$ whereas the values for $\eta \neq \xi$ taper off. In the limit they act similar to Dirac's delta-function on the line.

Since for $\sigma^2(x^{(N)})$ we have the expansion

$$\sigma^2(x^{(N)}) = \alpha^2(N) \sum_{n=o} (2n+1)$$

we have for $\sigma^2(y)$ the expansion

$$\sigma^2(y^{(N)} = \alpha^2(N) \sum_{n=o}^{N} (2n+1) \, . \, \lambda_n^2$$

since

$$\lambda_n^2 = 0(\frac{1}{n^2})$$

and

$$\sum_{n=o}^{N} \frac{2n+1}{n^2} = 2\ln N + 0(1)$$

(see Knopp (1947), chapter 4) we get with $\alpha^2(N) = 1/(N+1)^2$

$$\sigma^2(y^{(N)}) = 0(\frac{\ln N^2}{(N+1)^2})$$

This tends to zero for $N \to \infty$. Because of $\sigma^2(x^{(N)}) = 1$, the ratio $\sigma(y^{(N)})/\sigma(x^{(N)})$ tends to zero in the same way.

Dropping the assumption $\lambda_n \to 0$ now, one can more generally show the following (cf. Kantorowitsch-Akilov (1964) chapter VII, § 2).

Suppose that the operator L has eigen values λ_n such that

$$\lim_{n \to \infty} \lambda_n = \lambda \tag{4 - 26}$$

exists and is either finite or infinite. Then we have for $x^{(N)}(\xi)$ as in (4 - 18), (4 - 20) and $y^{(N)}(\xi) = L(\xi, \eta)\, x^{(N)}(\eta)$ the relation

$$\lim \frac{\sigma(y^{(N)})}{\sigma(x^{(N)})} = \lambda \tag{4 - 27}$$

For operators like the Laplacean or (3 - 12) where the limit in (4 - 26) is infinite, we see that the image of a near-white noise process has a very large variance becoming infinite in the limit.

We shall also use processes obtained by averaging $x^{(N)}(\xi)$ over circular areas of spherical radius ψ_0. This process $Bx^{(N)}$ (cf. (3 - 14), (3 - 14a)) has variance

$$\sigma^2(Bx^{(N)}) = \alpha^2(N) \sum_{n=0}^{N} (2n+1)\beta_n^2 \tag{4 - 28}$$

which necessarily tends to zero for $N \to \infty$ because $\beta_n \to 0$. Since we need a process with a prescribed variance equal to, lets say, unity we apply an appropriate factor obtaining a process

$$z^{(N)} = k(N) Bx^{(N)} \tag{4 - 29}$$

where $k(N)$ is chosen in a way that

$$\sigma^2(z^{(N)}) = k^2(N)\alpha^2(N) \sum_{n=0}^{N} (2n+1)\beta_n^2 = 1 \tag{4 - 30}$$

For N increasing the process $z^{(N)}$ tends to have less and less correlation at points ξ, η for which

$$\xi \cdot \eta > \cos 2\psi_0$$

i.e. for points which cannot be covered by one circular area.

We shall mainly need modified versions of these processes which are obtained by removing the harmonics of degree zero and one. The factors $\alpha(N)$ and $k(N)$ will then be adjusted in a way that the resulting processes have certain prescribed variances. The processes $z^{(N)}$ after removal of the zero and first degree harmonics will have <u>some</u> correlation even between non-overlapping caps. However, this correlation comes in only through the removal procedure.

Part II

Various quantities related to the earth's disturbing potential.

5) Representation of the earth's disturbing potential.

The disturbing potential of the earth may be represented by its values $T(\xi)$ at the surface of the geoid (or some equipotential surface). $T(\xi)$ is then the deviation of the true potential from the normal potential associated with the reference surface. The undulation $N(\xi)$ of the geoid with respect to the reference surface is then given by Bruns' formula.

$$N(\xi) = \frac{T(\xi)}{G} \tag{5 - 1}$$

where G is a mean gravity value. Certain manipulations involving the quantities $N(\xi)$ and $T(\xi)$ allow it to regard them as functions over a sphere which may be the unit sphere. The errors committed in doing so are small since the deviations of the reference surface from a sphere are small. By (5 - 1) the disturbing potential may be equivalently represented by the geoidal undulations. We may expand them in spherical harmoics

$$N(\xi) = \sum_{n=2}^{\infty} \sum_{m=-n}^{+n} N_{nm} S_{nm}(\xi) \tag{5 - 2}$$

We have assumed that the zero and first order terms are eliminated which may always be accomplished by a change in scale and position of the reference surface.

The undulations of the geoid are connected with the gravity anomalies by Stokes' formula.

$$N(\xi) = \frac{R}{4\pi G} \int_{\Gamma} St(\xi.\eta)\, \Delta g(\eta)\, d\Gamma(\eta) \tag{5 - 3}$$

which (cf. (3 - 10) can be written as

$$N_{nm} = \frac{R}{G} \cdot \frac{1}{n-1} \Delta g_{nm}, \quad n \geq 2 \tag{5 - 4}$$

where Δg_{nm} are of course the spherical harmonic coefficients of $\Delta g(\xi)$:

$$\Delta g(\xi) = \sum_{n=2}^{\infty} \sum_{m=-n}^{+n} \Delta g_{nm} S_{nm}(\xi) \tag{5 - 5}$$

Stokes' formula shows that the disturbing potential may also be represented by gravity anoamlies. If zero and first order harmonics are excluded then there is a one to one correspondence between the anomalies and the undulation which is seen by (5 - 4).

What is the difference between representing the disturbing potential by undulations $N(\xi)$ and anomalies $\Delta g(\xi)$? From our discussion in section 3 we know that the Stokes' operator has a smoothing property. If we start with a field of anomalies which has a spherical harmonic expansion, then necessarily

$$\|\Delta g\|^2_{H_\Gamma} = 4\pi \sum_{n,m} \Delta g_{nm}^2 < \infty$$

Substituting for Δg_{nm} from (5 - 4) shows that the coefficients of N_{nm} fulfill

$$4\pi \sum_{n,m} (n-1)^2 N_{nm}^2 = \frac{R^2}{G^2} \|\Delta g\|^2_{H_\Gamma} < \infty \tag{5 - 6}$$

This means that $N(\xi)$ is a function out of H^1_Γ having first order generalized derivatives. The H^1_Γ norm of N is given by the square root of

$$\|N\|^2_{H^1_\Gamma} = 4\pi \sum_{n,m} n(n+1) N^2_{nm} < \infty \tag{5 - 7}$$

(The convergence of (5 - 7) is indeed a consequence from that of (5 - 6) since for $n \geq 2$ we have $n(n+1) \leq 6(n-1)^2$.

On the other hand, if we start with some function $N(\xi) \sim N_{nm}$, requiring only

$$\|N\|^2_{H_\Gamma} = 4\pi \sum_{n,m} N^2_{nm} < \infty$$

then we end up with gravity anomalies which are not necessarily square integrable because

$$\|\Delta g\|^2_{H_\Gamma} = 4\pi \sum_{n,m} \Delta g^2_{nm} = 4\pi \left(\frac{G}{R}\right)^2 \sum_{n,m} (n-1)^2 N^2_{nm}$$

need not be convergent. N has to be at least a function out H_Γ^1 in order that $\|\Delta g\|_{H_\Gamma}$ is finite.

Using the inequalities

$$1 \le (n-1)^2 \le n(n+1) \le 6(n-1)^2, \; n \ge 2$$

it is easy to verify the following estimates

$$\|N\|_{H_\Gamma} \le \frac{R}{G} \|\Delta g\|_{H_\Gamma} \le \|N\|_{H_\Gamma^1} \le \sqrt{6} \; \frac{R}{G} \|\Delta g\|_{H_\Gamma} \qquad (5 - 8)$$

Confer also (3 - 21).

The discussion may appear pure academic. It has, however, the following consequence. If we try to determine the earth's potential by measuring gravity (anomalies) then we end up with a smooth geoid. The geoid depends mainly on the low degree harmonics of the anomalies. The higher harmonics are smoothed out by a factor proportional to $1/(n-1)$.

On the other hand if we use satellite perturbations in order to obtain information on the potential then we have to be aware that the disturbance of the satellite path is a two-fold integral over gravity anomalies (at satellite altitude). This two-fold integration has certainly a smoothing effect. The consequence is that only the lower degree harmonics of the potential (or of $N(\xi)$) can be accurately determined. (Resonance effects yield some higher coefficients too. But since the latter are isolated they do not significantly contribute to a better knowledge of $N(\xi)$ in regional areas.) From our previous discussion it is clear that gravity anomalies derived from such a satellite derived geoid are bad. The higher harmonics, which are not or only badly determined are multiplied by a factor proportional to $(n - 1)$.

Only in so-called combination solutions a representation of the potential in terms of gravity anomalies is feasible. In that case both quantities namely gravity anomalies in surveyed areas and satellite derived coefficients should be improved. The resulting anomalies in unsurveyed areas are still of dubious value.

We shall discuss these questions in more detail later on. Here we wanted only to point out the close connection between the stability of various approaches to determine quantities depending on the potential and the smoothing or unsmoothing properties of operators relating these quantities.

Let us now turn to the derivatives of $N(\xi)$ i.e. to the quantity $\operatorname{Grad} N(\xi)$. This quantity, measuring the slope of the geoid is closely related to the deflections of the vertical. Introducing the surface tangential vector

$$\Delta v(\xi) = -\frac{1}{R} \operatorname{Grad} N(\xi) \tag{5 - 9}$$

it is readily verified that the quantity $\Delta v(\xi)$ is nothing but the deflection of the vertical on coordinate free representation. (Recall that the operator Grad is performed with respect to the unit sphere. Therefore the multiplication with 1/R, R ... mean earth radius).

Recall from section 2 that $\Delta v(\xi) = -\frac{1}{R} \cdot \operatorname{Grad} N(\xi)$ may be viewed as a member of the Hilbert space U_Γ. The norm in this space is the square root of

$$\|\Delta v\|^2_{U_\Gamma} = \int_\Gamma (\Delta v(\xi), \Delta v(\xi))\, d\Gamma(\xi) \tag{5 - 10}$$

Since $N(\xi)$ has no zero order component (i.e. $N_{oo} = 0$) we conclude from equation (2 - 29) that

$$\|\Delta v\|_{U_\Gamma} = \frac{1}{R} \|N\|_{H^1_\Gamma} \tag{5 - 11}$$

We can therefore substitute for $\|N\|_{H^1_\Gamma}$ in (5 - 8) obtaining

$$\frac{1}{G} \|\Delta g\|_{H_\Gamma} \leq \|\Delta v\|_{U_\Gamma} \leq \frac{\sqrt{6}}{G} \|\Delta g\|_{H_\Gamma} \tag{5 - 12}$$

which may also be written as

$$\frac{1}{G} \leq \frac{\|\Delta v\|_{U_\Gamma}}{\|\Delta g\|_{H_\Gamma}} \leq \frac{\sqrt{6}}{G} \tag{5 - 13}$$

This shows that the ratio of the norms of Δg and Δv is within non-zero fixed and relatively narrow bounds.

Deflections of the vertical may also be used to represent the disturbing potential. Equation (2 - 14) gives at once the following representation of $N(\xi)$ in terms of $\Delta v(\xi) = -R^{-1}$ Grad $N(\xi)$

$$N(\xi) = -\frac{R}{4\pi}\int_{\Gamma}(\text{Grad}\,(2\ln\frac{2}{|\xi-\eta|}),\,\Delta v(\eta))\,d\Gamma(\eta) \tag{5 - 14}$$

The relationship can also be written in terms of spherical harmonics: Expand

$$N(\xi) = \sum_{n,m} N_{nm}\,S_{nm}(\xi)$$

$$\Delta v(\xi) = \sum_{n,m} \Delta v_{nm}\,U_{nm}(\xi) \tag{5 - 15}$$

where $U_{nm}(\xi) = (\sqrt{n(n+1)})^{-1}$ Grad $S_{nm}(\xi)$ is the orthonormal system introduced in (2 - 22). Note that the Δv_{nm} are scalar coefficients and not vectors. From

$$\text{Grad}\,N(\xi) = \sum_{n,m} N_{nm}\,\text{Grad}\,S_{nm}(\xi)$$

we get by $\Delta v = -R^{-1}$Grad N

$$\Delta v_{nm} = -\frac{1}{R}\sqrt{n(n+1)}\,N_{nm}\,. \tag{5 - 16}$$

From (5 - 4) we also see that

$$\Delta v_{nm} = -\frac{1}{G}\,\frac{\sqrt{n(n+1)}}{n-1}\,\Delta g_{nm} \tag{5 - 17}$$

This shows that $\Delta v(\xi)$ is equivalent to $\Delta g(\xi)$ in representing the disturbing potential. There is no smoothing or unsmoothing during the transition from Δg to Δv. The factor $-\frac{1}{G}\,\frac{\sqrt{n(n+1)}}{n-1}$ does not increase to infinity nor does it tend toward zero. It is absolutely bounded by $\frac{1}{G}$ and $\frac{\sqrt{6}}{G}$ which is in perfect agreement with (5 - 13).

Another quantity which serves the purpose to represent the potential but which has no physical significance is a surface density function. Representing $T(\xi)$ by a surface density function means to write it in the form

$$T(\xi) = \frac{R}{4\pi} \int_{\Gamma} \frac{\Delta\varphi(\eta)}{\ell(\xi.\eta)} d\Gamma(\eta) \tag{5 - 18}$$

Equivalently:

$$N(\xi) = \frac{R}{4\pi G} \int_{\Gamma} \frac{\Delta\varphi(\eta)}{\ell(\xi.\eta)} d\Gamma(\eta) \tag{5 - 18a}$$

In spherical harmonics (cf. (3 - 9))

$$N_{nm} = \frac{R}{G} \frac{1}{2n+1} \Delta\varphi_{nm}, \quad n \geq 2 \tag{5 - 18b}$$

It turns out that $\Delta\varphi$ is just in the same way as Δg and Δv suited to represent the potential. We have from (5 - 4) and (5 - 18b)

$$\Delta g_{nm} = \frac{n-1}{2n+1} \Delta\varphi_{nm} \tag{5 - 19}$$

Hence (confer (3 - 21)

$$\frac{1}{5} \leq \frac{\|\Delta g\|_{H_\Gamma}}{\|\Delta\varphi\|_{H_\Gamma}} \leq \frac{1}{2} \tag{5 - 20}$$

We turn to a last quantity which compared to $N(\xi)$ is even rougher than Δg, Δv, $\Delta\varphi$. It is the vertical gradient of the anomalies. Call this quantity Δa and take its definition from Heiskanen-Moritz, (1967) p. 116:

$$\Delta a(\xi) = \frac{\partial \Delta g}{\partial r} = \frac{G}{R^2} \left\{ 2N(\xi) + \text{Lap}\, N(\xi) \right\} \tag{5 - 21}$$

Lap. is the surface Laplacean operator with respect to the unit sphere Γ. From section 3 we know that the eigen-values of the Laplacean are given by $-n(n+1)$.

The spherical harmonics equivalent of (5 - 21) is therefore

$$\Delta a(\xi) = \sum_{n,m} \Delta a_{nm} S_{nm}(\xi) \tag{5 - 22}$$

with

$$\Delta a_{nm} = \frac{G}{R^2}\left\{2N_{nm} - n(n+1)N_{nm}\right\} = -\frac{G}{R^2}[(n-1)(n+2)]N_{nm} \tag{5 - 23}$$

This shows that for $\Delta a \in H_\Gamma$, i.e. for $\Delta a \sim \Delta a_{nm}$ with

$$\|\Delta a\|^2_{H_\Gamma} = 4\pi \sum_{n,m} \Delta a^2_{nm} < \infty$$

we get $N(\xi) \in H^2_\Gamma$. For we have

$$\|\Delta a\|^2_{H_\Gamma} = 4\pi \frac{G^2}{R^4} \sum_{n,m} [(n-1)(n+2)]^2 N^2_{nm} < \infty$$

one can verify the following inequalities:

$$\frac{2}{3}\frac{G}{R^2} \le \frac{\|\Delta a\|_{H_\Gamma}}{\|N\|_{H^2_\Gamma}} \le \frac{G}{R^2} \tag{5 - 24}$$

The transition from Δa to N involves heavy smoothing. The transition from N to Δa therefore presupposes that N is a very smooth function, i.e. at least out of H^2_Γ .

Table 1 on the following page summarizes and complements the results of this section.

The entries in this table give the relationship between the spherical harmonic coefficients (for Δv the coefficients with respect to the $U_{nm}(\xi) = \text{Grad } S_{nm}/\sqrt{n(n+1)}$) and the bounds for the norm ratios. Norms are with respect to H_Γ (for Δv with respect to U_Γ). Thus for example from the row Δv and the column $\Delta\varphi$

$$\Delta v_{nm} = -\frac{1}{G}\frac{\sqrt{n(n+1)}}{2n+1}\Delta\varphi_{nm}$$

$$\frac{1}{G}\frac{\sqrt{6}}{5} \le \frac{\|\Delta v\|_{U_\Gamma}}{\|\Delta\varphi\|_{H_\Gamma}} \le \frac{1}{G}\frac{1}{2}$$

	N	Δg	Δv	Δφ	Δa
N		$\frac{R}{G}\frac{1}{n-1}$ $0, \frac{R}{G}$	$-\frac{R}{\sqrt{n(n+1)}}$ $0, \frac{R}{\sqrt{6}}$	$\frac{R}{G}\frac{1}{2n+1}$ $0, \frac{R}{G}\frac{1}{5}$	$-\frac{R^2}{G}\frac{1}{(n-1)(n+2)}$ $0, \frac{R^2}{G}\frac{1}{4}$
Δg	$\frac{G}{R}(n-1)$ $\frac{G}{R}, \infty$		$-G\frac{n-1}{\sqrt{n(n+1)}}$ $\frac{G}{\sqrt{6}}, G$	$\frac{n-1}{2n+1}$ $\frac{1}{5}, \frac{1}{2}$	$-R \cdot \frac{1}{n+2}$ $0, \frac{R}{4}$
Δv	$-\frac{1}{R}\sqrt{n(n+1)}$ $\frac{\sqrt{6}}{R}, \infty$	$-\frac{1}{G}\frac{\sqrt{n(n+1)}}{n-1}$ $\frac{1}{G}, \frac{\sqrt{6}}{G}$		$-\frac{1}{G}\frac{\sqrt{n(n+1)}}{2n+1}$ $\frac{1}{G}\frac{\sqrt{6}}{5}, \frac{1}{G}\frac{1}{2}$	$\frac{R}{G}\frac{\sqrt{n(n+1)}}{(n-1)(n+2)}$ $0, \frac{R}{G}\frac{\sqrt{6}}{4}$
Δφ	$\frac{G}{R}(2n+1)$ $5\frac{G}{R}, \infty$	$\frac{2n+1}{n-1}$ $2, 5$	$-G\frac{2n+1}{\sqrt{n(n+1)}}$ $2G, \frac{5}{\sqrt{6}}G$		$-R\frac{2n+1}{(n-1)(n+2)}$ $0, R\frac{5}{4}$
Δa	$-\frac{G}{R^2}(n-1)(n+2)$ $4\frac{G}{R^2}, \infty$	$-\frac{1}{R}(n+2)$ $\frac{4}{R}, \infty$	$\frac{G}{R}\frac{(n-1)(n+2)}{\sqrt{n(n+1)}}$ $\frac{G}{R}\frac{4}{\sqrt{6}}, \infty$	$-\frac{1}{R}\frac{(n-1)(n+2)}{2n+1}$ $\frac{1}{R}\frac{4}{5}, \infty$	

Table 1

Note the narrow bounds for the norm ratios in this particular case.

As we have demonstrated at the end of section 3, all the operators involved in Table 1 commute. They commute in turn with any isotropic operator and therefore with any isotropic smoothing operator in the narrow sense. Hence the relationship between spherical harmonics coefficients is the same whether we regard the quantities like $N(\xi)$, $\Delta g(\xi)$ ... themselves or smoothed versions of them.

Let B denote a narrow-sense smoothing operator having eigen-values β_n then we have for example

$$\Delta\tilde{g} = B\Delta g \sim \beta_n \Delta g_{nm} = \Delta\tilde{g}_{nm} \text{, say}$$

$$\tilde{N} = BN \sim \beta_n N_{nm} = \tilde{N}_{nm} ,$$

and the relationship between the smoothed versions $\Delta\tilde{g}$, $\tilde{N}$ of Δg and N is still the same,

$$\tilde{N}_{nm} = \frac{R}{G}\frac{1}{n-1}\Delta\tilde{g}_{nm} ,$$

as it was for the unsmoothed quantities (confer (5 - 4))

It turns out that the bounds for the norm ratios remain valid. This follows from the considerations in section 3 leading to equation (3 - 22). Therefore the bounds for the norm ratios exhibited in the above table hold true also in the case of smoothed versions subject to the same smoothing operator. Recall in this context what smoothing means for the vector function $\Delta v(\xi)$. The narrow-sense smoothed version $\Delta\tilde{v}(\xi)$ is the surface gradient of the smoothed geoid $\tilde{N}(\xi)$. $\tilde{N}(\xi)$ is obtained by smoothing $N(\xi)$. $N(\xi)$ and $\Delta v(\xi)$ are connected by (5 - 14).

Remark: Most of the spherical harmonics relationships contained in Table 1 are very well known. Several are found in the textbook by Heiskanen-Moritz (1967). The interrelation between Δg and Δv has been discussed in papers as old as Cook (1950). There, as in may other treatises, the deflection vector is decomposed into two components. Schwarz (1970) gives a brief disucssion of the relationship between Δg and $\Delta\varphi$. He also arrives at the conclusion that the variation in Δg and $\Delta\varphi$ in regional areas (i.e. after exclusion of the lower degree components) is nearly proportional. Shaw et al. (1969) use a local model for their discussion of the Δv process. In their case, which is a limiting one, the norm ratio between Δg and Δv is a strict constant. (Actually they consider a variance ratio in the sense of our next section. However, these variance ratios will have the same bounds as the norm ratios.).

Our aim in this presentation is three-fold. First, to outline a systematic approach based on isotropic operators which have the spherical harmonics as eigen-functions. Second, to emphasize the (wide-sense) smoothing or unsmoothing properties of these operators, especially to derive the bounds for the norm ratios. Third, to discuss and possibly clarify the interaction of these operators with the narrow-sense smoothing operators such as the averaging operator over a circular cap.

The bounds for the norm ratios in Table 1 are based on the assumption that harmonics of degree $n = o, 1$ are disregarded. Numerical values of these bounds are exhibited in Table 1a, (top line in each box). The bounds hold for the indicated measuring units, i.e. meters for N, arc seconds for Δv, milligal for Δg, $\Delta\varphi$ and Eotvos units for Δa. Table 1a contains also the bounds for the case that harmonics up to and including degree 12 are disregarded (bottom line in each box). One sees that these bounds are much narrower in many cases.

	Δg mg	Δv "	$\Delta\varphi$ mg	Δa E.U.
N meters	0, 6.5 0, 0.7	0, 12.6 0, 2.5	0, 1.3 0, 0.3	0, 1000 0, 27
Δg milligal		2.0, 4.8 4.2, 4.8	0.2, 0.5 0.4, 0.5	0, 160 0, 46
Δv arc seconds			.10, .11 .11, .11	0, 82 0, 11
$\Delta\varphi$ milligal				0, 800 0, 104

Table 1a

Thus, for example, if the norm of Δv is 1", the norm of Δg is between 2 and 4.8 mg for harmonics of degree $n = 0,1$ excluded and between 4.2 and 4.8 mg if harmonics of degree $n \leq 12$ are excluded.

We conclude this section with a sidelight on truncation errors. These result from the fact that in the various integral formulas like that of Stokes, Vening Meinesz and others, the integration is not carried out over the whole sphere. The reason for omitting certain areas is either incomplete knowledge of the integrand due to unsurveyed areas or a reduction in computing time. In either case the resulting error must be tolerable. In previous investigations of the truncation error it was mostly assumed that the omitted area is (1) a circular cap and (2) is centered at the antipode of the point in which the integral transform is to be evaluated. See Molodensky et al, (1962), chapter VII, Cook (1950), (1951), De Witte (1965), (1966), Heiskanen-Moritz (1967), chapter 7.

Though attention in this report is not focused on truncation errors, an alternative approach should perhaps be indicated. It is based on Green's second formula. Confer (2 - 15b).

Take for example Stokes' formula (5 - 3). If an arbitrarily shaped area B with boundary ∂B is omitted, then the resulting error is

$$\delta N(\xi) = \frac{R}{4\pi G} \int_B St(\xi . \eta)\, \Delta g(\eta)\, d\Gamma(\eta) \qquad (5 - 25)$$

Assume that the point ξ is not contained in B. Find a function $G(\eta)$ having the following properties

$$\mathrm{Lap}\ G(\eta) = \Delta g(\eta) \quad \text{in B}$$

$$G(\eta) = 0 \quad \text{on} \quad \partial B$$

From our previous discussions we can expect that $G(\eta)$ is rather smooth and small compared to $\Delta g(\eta)$. Apply now Green's second formula to (5 - 25) and obtain

$$\delta N(\xi) = \frac{R}{4\pi G} \int_B \mathrm{Lap}_\eta \left\{ St(\xi . \eta) \right\} G(\eta) d\Gamma(\eta)$$

$$- \frac{R}{4\pi G} \int_{\partial B} St(\xi . \eta)\, (\mathrm{Grad}\ G(\eta), \nu(\eta))\, d\Gamma(\eta) \qquad (5 - 26)$$

If ξ is at a sufficient distance from ∂B then $\mathrm{Lap}\,\{St(\xi . \eta)\}$ is found to be rather small. Thus the first term on the right can be expected to be small. The second term may be larger in some cases. If $St(\xi . \eta)$ happens to vanish on

∂B , then the second term disappears. This is the reason why De Witte found small truncation errors for a B being a circular cap and one of the zeros of $St(\xi . \eta)$ located at ∂B. I feel that much more can be learned by thoroughly discussing both right-hand terms in (5 - 25). However, this will be done elsewhere.

6. Gravimetric quantities and their errors viewed as isotropic stochastic process on the unit sphere.

It is a widely adopted practice to view gravity anomalies and related quantities such as undulations, deflections, as isotropic stochastic processes, on Γ.

I see no easy way to give a rigorous justification for this practice. An isotropic stochastic process on Γ can, as we have seen in section 4, be generated by a series of uncorrelated random coefficients x_{nm} having variance $\sigma_n^2/(2n+1)$. The random process $x(\xi)$ is then formed by

$$x(\xi) = \sum_{n,m} x_{nm} S_{nm}(\xi) \tag{6 - 1}$$

What meaning shall we give to the random variables $x_{nm} = N_{nm} = \Delta g_{nm}$?

I give four motivations for regarding the gravimetric quantities as stationary stochastic process without attempting to fully solve this philosophical question.

a. Behind the terms random or stochastic is the idea of some repetitive pattern, some experiment which can be repeated yielding different outcomes. One can visualize such a repetitive pattern by viewing e.g. anomalies, in a population of limited areas. Thus we have a repetitive pattern in space since the earth's surface may be regarded as consisting of many such areas, for example $5^\circ \times 5^\circ$ blocks. This is a reasonable justification for viewing Δg as stochastic process in a limited area. However, the global random function $\Delta g(\xi)$ has still no meaning. Nor have the coefficients Δg_{nm} as random varianbles.

b. One can regard $\Delta g(\xi)$ as a fixed function and introduce a randomization by selecting points at random on Γ. Thus for example selecting random points at fixed distances and computing the sample correlation should give a value in reasonable agreement with the overall covariance function.

But why shall we choose points at random on the earth's surface?

c. One can adopt the following point of view. If the gravity anomalies are completely known all over the earth, then there is no need to regard them as a stochastic process. However, as long as this is not the case, I may think of a population of functions $\Delta g(\xi)$ to which I assign certain subjective probabilities. If I do not know the outcome of a football game, even though it may already have taken place, then I may assign subjective probabilities to the different outcomes, probabilities which reflect my judgment and background knowledge. I may use these probabilities for bets with friends who, hopefully, do not know the exact result either. If I learn more about the gravity anomalies by new measurements or evaluations of satellite - observations, I may change my subjective probabilities in the same way as I change my probabilities for the football game - result when I obtain additional information for example, the score after the first quarter or an injury of a certain key player. As soon as the full result of the game or anomalies is known, to me, I abandon my probabilities or, as some would like to say, I assign probability 1 to the true outcome.

d. I may use the concept of stochastic process just for the sake of a formal anology, as a heuristic guideline for working out procedures such as interpolation, prediction, adjustment. In doing so I hope that the procedures turn out to average and weigh my observations in a reasonable way. Another probability assumption would just be a transition to different weights. Quantities like means or variances bear then no stochastic meaning. They are numbers computed from observations and weights, characterizing them in a certain way. The stochastic process - terminology is used merely to make the language more picturesque (or the reader more confused).

There could be an endless discussion of these and possible other ways to view gravimetric quantities as stochastic processes. I confine myself to a few remarks.

* If gravity anomalies all over the earth are viewed as a single realization of an isotropic process (with whatsoever underlying probability structure) then the ergodic law does not hold for the covariance function. This has been pointed out for example by Krarup (1969). This means that the sample covariance i.e. the covariance computed from gravity anomalies need not be the same as that implied by the underlying probability structure. Even in the case that we have a world wide gravity coverage.

* Only if one focuses attention on very small areas in the sense of a.), one may assume that the ergodic law nearly holds. In other words, the sample covariance has much more meaning for small distances than for larger. This presupposes on the other hand that gravity anomalies are fairly homogenous all over the world. Certain

regions have certainly to be excluded from this assumption.

Let us now view the gravimetric quantities as stochastic processes. Since the quantities N, Δg, Δv, $\Delta \varphi$, Δa are interrelated by operators, we may start with probability assumptions for one of them and derive the stochastic properties of the others. Usually Δg is taken as starting points since most observations are in terms of gravity.

Assume that $\sigma_n^2(\Delta g)$ are the degree variances of $\Delta g(\xi)$. The variance of Δg is then given by (confer (4 - 11))

$$\sigma^2(\Delta g) = \sum_{n=2}^{\infty} \sigma_n^2(\Delta g)$$

The degree variances of the other quantities are then obtained by multiplying with the λ_n^2 of the appropriate operators. The λ_n are found in column Δg of Table 1 in section 5.

Thus

$$\sigma^2(N) = \left(\frac{R}{G}\right)^2 \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} \sigma_n^2(\Delta g) \tag{6 - 3}$$

$$\sigma^2(\Delta v) = \frac{1}{G^2} \sum_{n=2}^{\infty} \frac{n(n+1)}{(n-1)^2} \sigma_n^2(\Delta g) \tag{6 - 4}$$

$$\sigma^2(\Delta \varphi) = \sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^2 \sigma_n^2(\Delta g) \tag{6 - 5}$$

$$\sigma^2(\Delta a) = \frac{1}{R^2} \sum_{n=2}^{\infty} (n+2)^2 \sigma_n^2(\Delta g) \tag{6 - 6}$$

The formal relationship between the degree-standard deviations, e.g.:

$$\sigma_n(N) = \frac{R}{G} \frac{1}{n-1} \sigma_n(\Delta g)$$

is the same as that for the harmonic coefficients, e. g.:

$$N_{nm} = \frac{R}{G} \frac{1}{n-1} \Delta g_{nm}$$

Therefore the upper and lower bounds for the ratios of the standard deviations are the same as those for the H_Γ norms. They are listed in Table 1, section 5.

Take for example row Δg and column Δv in Table 1 and obtain

$$\frac{G}{\sqrt{6}} \leq \frac{\sigma(\Delta g)}{\sigma(\Delta v)} \leq G$$

The degree variances of N taper off more quickly than those of Δg. This is in agreement with the smoothing properties of the Stokes' operator. The degree variances of Δa blow up compared to those of Δg. The sum

$$\sum_n n^2 \sigma_n^2(\Delta g)$$

has to be finite, in order that the process Δa may be formed from the process Δg. Since we may equivalently require

$$4\pi \sum_n n(n+1)\sigma_n^2(\Delta g) < \infty \tag{6 - 7}$$

we can also say, that Δg has to be a function out of H_Γ^1 with probability 1. If (6 - 7) fails to hold Δa does not have a finite variance.

If we study smoothed versions of the gravimetric quantities with respect to an isotropic (narrow-sense) smoothing operator B having eigen-values β_n then the degree variances are multiplied with the squares of the eigen-values. Denoting the smoothed quantities by tildas we have for example:

$$\sigma^2(\Delta\tilde{g}) = \sum_n \beta_n^2 \sigma_n^2(\Delta g) = \sum_n \sigma_n^2(\Delta\tilde{g})$$

The relationships (6 - 3) to (6 - 6) remain true also for the smoothed quantities. For example

$$\sigma^2(\tilde{N}) = \left(\frac{R}{G}\right)^2 \sum_{n=o}^{\infty} \frac{1}{(n-1)} \sigma_n^2(\Delta\tilde{g}) .$$

Again, the upper and lower bounds for the variance ratios exhibited in Table 1 remain valid also for the smoothed quantities.

Sometimes it is not so much desirable to regard the gravimetric quantities as stochastic processes rather than errors in these quantities. Sometimes even a superposition of two stochastic processes, e.g. gravity anomalies plus error in gravity anomalies is assumed.

If $x(\xi)$ denotes any gravimetric quantity we shall denote by $\delta x(\xi)$ the error in $x(\xi)$. The covariance of $\delta x(\xi)$ will be called the error covariance of $x(\xi)$. Likewise we talk of error variance, error-standard deviation or error degree-variances. Error covariances depend not so much on the quantity itself (as a matter of fact, one frequently assumes zero correlation between $x(\xi)$ and $\delta x(\xi)$) as in measurement methods. Since measurements may be repeated and since they are affected by random errors there is sometimes a stronger motivation to regard the errors as stochastic processes than there is for the quantities x themselves.

If we represent an observed quantity in the form

$$x(\xi) + \delta x(\xi)$$

i.e. as a sum of true value plus error, and if we make a transition to another quantity by a stationary linear operator, then we have because of the linearity

$$y + \delta y = L(x + \delta x) = Lx + L\delta x$$

This means that not only

$$y = Lx$$

but also

$$\delta y = L\delta x$$

Hence everything which has been said about the stochastic processes N, Δg, Δv and so on can be transferred to the error processes δN, $\delta\Delta g$, $\delta\Delta v$, ...

We need not repeat the discussion how the degree variances transform under linear operators, how the σ-ratios are bounded and how things interact with the narrow-sense smoothing operators.

Needless to say, Table 1 holds also for the σ-ratios of the error processes and their smoothed versions.

7. A qualitative discussion of various approaches to determine the earth's disturbing potential.

We consider two sources of information: gravity anomalies and orbit perturbations of satellites.

As it is known and as we have already pointed out earlier, the orbit perturbations are much smoother than the variations in gravity anomalies since the former are so to speak a two-fold time integral over them. Orbit perturbations are generally more sensitive to the lower degree harmonics of the earth's potential if we disregard certain isolated harmonics causing resonance effects. Isolated higher harmonics do not significantly contribute to a detailed regional knowledge of the field. The consequence is that the satellite observations contribute considerably to a better knowledge of the lower degree harmonics of the field. The higher harmonics cannot be determined except for a very few ones.

In a pure satellite approach toward the determination of the earth's potential, the higher harmonics are usually forced to be zero. In such a solution we can hope to get a good approximation of the coefficients N_{nm} of $N(\xi)$ up to degree $n=N$ say. Since $N(\xi)$ is a rather smooth surface, the error in $N(\xi)$ mainly due to truncation may not be large. If N is about 12, the standard error may be about 3-5m. Now $N(\xi)$ is the smoothest of gravimetric quantities considered here. If we try to compute other quantities like $\Delta g(\xi)$, $\Delta v(\xi)$, we face serious trouble. The transition from $N(\xi)$ toward $\Delta g(\xi)$ amplifies the higher harmonics. Since the higher harmonics are not determined, the resulting error is much larger. Even if we confine ourselves to (narrow-sense) smoothed versions of the quantities $\Delta g(\xi)$, $\Delta v(\xi)$ we end up with considerable errors. We shall devote several later sections toward this question.

If we still consider a pure satellite approach, representing, however, the potential in a different way by gravity anomalies (or equivalently, as should be clear by now, by a surface density or by deflections of the vertical) then we face a problem with poor stability. Large variations in local anomalies cause little variations in $N(\xi)$, especially in its lower harmonics, and even lesser variation in the satellite motion. Thus the anomalies are poorly determined. They are, however, poorly determined in a certain way. Namely, in a way that the resulting geoid $N(\xi)$ is rather good. If we manage to overcome the computational difficulties in getting numerical values for the anomalies then whese values have a rather large error. The error correlation shows, however, a peculiar pattern as we shall make more clear later on. This correlation is due to the fact that the errors in the lower degree harmonic components are small. The consequence is then that $N(\xi)$ which depends mainly on the lower degree harmonics is of much better accuracy than we could hope from the standard deviation of the anomaly errors.

If we use a pure gravimetric approach trying to determine the potential from gravity anomalies alone, then we face at the present time the problem of large areas with poor coverage. These uncovered areas prevent a development into spherical harmonics except if we extrapolate gravity anomalies all over the earth. This is possible by prediction procedures based on, or in formal analogy to, stochastic process concepts. If the whole earth is covered by good gravity measurements, then a good geoid with good deflections of the vertical results. At least if one accepts Stokes' formula as a suitable tool to compute the geoid from anomalies.

If the coverage is good only in certain areas, then the geoid suffers in those areas only from the influence of the distant zones. Because this influence is nearly the same in limited areas, the relative accuracy of the geoid there will be good.

In combining the gravimetric and the satellite approach one gets certainly a distinct improvement over each of the two separate approaches. The satellite information will insure good lower degree harmonics and a good overall $N(\xi)$. The gravimetric data will insure good detail, especially good deflections in well-covered areas. The influence of distance zones which was encountered there previously will be diminished. In areas with poor coverage the detail of the geoid especially its slope (deflections of vertical) will hardly be better than in a pure satellite solution.

The question arises how to represent the potential during the combination procedure. If one takes gravity anomalies Δg (or a comparable quantity like $\Delta\varphi$, Δv) then things will be stable in well-covered areas and unstable in the rest.

If one uses the spherical harmonic coefficients of $N(\xi)$, i.e. the N_{nm} up to a very high order then one would have to impose some conditions on the N_{nm} in order that they give a good fit to $\Delta g(\xi)$ in the well-covered area. The numerical solution of this problem would probably be a little more stable. If, however, one goes back to gravity anomalies in unsurveyed areas, then large fluctuations in the computed anomalies will be observed. These are similar to those for pure satellite solutions.

The best approach would probably be a hybrid one, representing the potential by the coefficients N_{nm} up to $n=N$ and by residual anomalies with respect to the (still unknown) surface given by the N_{nm} up to $n=N$. Residual anomalies would only be assumed in areas with good coverage. Care has to be taken that the residuals are free of spherical harmonic components up to degree N.

In section 9 we will use a very simplified procedure of this nature. This will only be done to provide the necessary background for some error estimates. Solution procedures designed for actual data may be much more sophisticated.

Part III

The Covariance of Anomaly-Residuals with Respect to a Satellite Derived Geoid

8. The covariance of an isotropic stochastic process on Γ having degree variances zero for $n \le N$.

Satellite observations establish very well the low degree harmonics of the earth's potential. The disturbing part of the potential may be represented by the geoidal undulations

$$N(\xi) = \sum_{n,m} N_{nm} S_{nm}(\xi)$$

If one tries to derive more and more coefficients from satellites alone than their relative error, i.e. the ratio, error divided by coefficients, certainly increases.

The reasons for this are mainly the general decrease in the degree variances. with increasing n and the attenuation effect which reduces the influence of the higher coefficients even more at satellite altitude.

Let us disregard this gradual decrease in relative accuracy and let us study a limiting case. Assume that the coefficients N_{nm} are exactly known for $n \le N$ and that they are completely unknown for $n > N$.

From the N_{nm} we can compute the Δg_{nm}, i.e. the coefficients in the expansion of $\Delta g(\xi)$ by (cf. (5 - 4)).

$$\Delta g_{nm} = \frac{G}{R} (n-1) N_{nm}$$

We can then assume that the Δg_{nm} are known for $n \le N$. We now have the following situation

$$\Delta g(\xi) = \Delta g^{t}(\xi) + \Delta g^{r}(\xi) = \sum_{\substack{n \le N \\ m}} \Delta g_{nm} S_{nm}(\xi) + \sum_{\substack{n > N \\ m}} \Delta g_{nm} S_{nm}(\xi) \qquad \ldots (8 - 1)$$

$\Delta g^{t}(\xi)$ represents the truncated part and $\Delta g^{r}(\xi)$ the residual part. The residual part may also be regarded as the gravity anomalies with respect to the surface implied by the truncated part, i.e.

$$N^{t}(\xi) = \sum_{\substack{n \le N \\ m}} N_{nm} S_{nm}(\xi) \qquad (8 - 2)$$

We shall study now $\Delta g^r(\xi)$. Since the considerations hold for any isotropic process, we change the notation from Δg to x .

Let us split an isotropic stochastic process $x(\xi)$ on Γ in the following way

$$x(\xi) = x^t(\xi) + x^r(\xi) = \sum_{\substack{n\leq N\\ m}} x_{nm} S_{nm}(\xi) + \sum_{\substack{n>N\\ m}} x_{nm} S_{nm}(\xi) \qquad (8-3)$$

$x^t(\xi)$ represents then the truncated part and $x^r(\xi)$ the residual part. Let $C(t) = C(\xi \cdot \eta)$ denote the covariance of $x(\xi)$. It splits in the same way as

$$C(t) = C^t(t) + C^r(t) = \sum_{n\leq N} \sigma_n^2 P_n(t) + \sum_{n>N} \sigma_n^2 P_n(t) \qquad (8-4)$$

$C^r(t)$ is then the covariance of the residual part.

We shall now show that $C^r(t)$ has two general properties which hold for a wide class of stationary processes on Γ having zero degree variances up to and including degree N.

The first property holds for any function $f(t)$ of the form

$$f(t) = \sum_{n>N} \varphi_n P_n(t) \qquad (8-5)$$

with not necessarily non-negative coefficients. We shall show that

(1) A continuous function of the form

$$f(t) = \sum_{n>N} \varphi_n P_n(t), \quad -1 \leq t \leq +1$$

has at least $N+1$ distinct zeros in $-1 < t < +1$.

Remark: The harmonics of degree 0, 1 are frequently removed from the Δg. In any case they are small. The covariance should then have at least two zeros in the interval $-1 < t < +1$ or equivalently with $t = \cos\psi$ in $0 < \psi < 180°$. We find this verified in Heiskanen-Moritz, p. 254 or in Kaula (1966), p. 5304. In the latter case the degree variance σ_2^2 is rather small. We are therefore not surprised to find even more than two zeros.

Thus we shall expect covariances $C^r(t)$ oscillating around the abscissa if N is greater. This holds true regardless of the special shape of the original covariance $C(t)$.

Proof of (1). Put

$$F_o(t) = f(t)$$

$$F_k(t) = \int_{-1}^{t} F_{k-1}(\tau)d\tau \ , \quad k \geq 1$$

We have then trivially

$$F_k(-1) = 0, \quad k \geq 1.$$

We shall show that also

$$F_k(+1) = 0, \quad k = 1, \dots, N+1$$

Assume for the moment that this has been shown. From $F_{N+1}(-1) = F_{N+1}(+1)$ it follows that

$$F_N(t) = \frac{d}{dt} F_{N+1}(t)$$

has a zero somewhere in $0 < t < 1$. Denote it by z

From

$$F_N(-1) = F_N(z) = F_N(+1) = 0$$

it follows in a similar way that $F_{N-1}(t)$ has a zero in $-1 < t < z$ and in $z < t < +1$.

Proceeding in that fashion we arrive at $N+1$ zeros for $f(t) = F_o(t)$. We show now $F_k(+1) = 0$. From

$$\int_{-1}^{+1} f(t)P_o(t)dt = 0$$

we deduce, since $P_o(t)$ is a constant that $F_1(1) = 0$. By partial integration we find

$$\int_{-1}^{+1} f(t)P_1(t)dt = F_1(t)P_1(t)\Big|_{-1}^{+1} - \int_{-1}^{+1} F_1(t)P_1'(t)dt = 0$$

Because of $F_1(+1) = 0$, we have

$$\int_{-1}^{+1} P_1'(t)F_1(t)dt = 0$$

Since P_1' is a constant we have $F_2(1) = 0$. Proceeding in that fashion we verify the desired relation.

Having established the oscillatory behavior of $C^r(t)$ we would like to know a little more about the location of these zeros. We are particularly interested in possible zeros in the neighborhood of $t = 1$. These correspond by $t = \cos\psi$ to zeros for small ψ i.e. to points at a close distance. We shall not prove an exact result. However utilizing the fact that the coefficients in the expansion of $C^r(t)$ are non-negative, we shall demonstrate the following.

(2) There is a strong tendency that $C^r(t)$ is of alternating sign at the larger zeros of $P_N(t)$. At the largest zero of $P_N(t)$ it would then have negative sign.

Remark: The large zeros of $P_N(t)$ are close to 1 and correspond to small ψ. For N=12, the largest zero is at $t = 0.981561$ or $\psi = 11.02^\circ$. Thus, at distances around 11° we should expect negative correlation.

Since (2) is not a precise theorem, we cannot prove it. So we give a motivation. It is known that the zeros of $P_{n-1}(t)$ separate those of $P_n(t)$. Denote by z the largest zero of $P_N(t)$. Since $P_n(1) = 1$ for all n, we see that $P_{N-1}(z) > 0$. We use now the recursion formula. (See Abramowitz (1964), equ. 8.5.3.)

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t)$$

to compute $P_{N+1}(z)$, $P_{N+2}(z)$ and perhaps a few more. We obtain

$$P_{N+1}(z) = -\frac{N}{N+1} P_{N-1}(z) < 0$$

This shows that $P_{N+1}(z)$ is negative and of about the same modulus as $P_{N-1}(z)$.

Now:

$$P_{N+2}(z) = - \frac{(2N+3)N}{(N+2)(N+1)} \cdot z P_{N-1}(z) < 0$$

For large N and therefore z close to 1 we see that $P_{N+2}(z)$ is about twice as large (absolutely) as $P_{N+1}(z)$. Now it goes on

$$P_{N+3}(z) = \left[- \frac{(2N+5)(2N+3)N}{(N+3)(N+2)(N+1)} z^2 + \frac{(N+2)N}{(N+3)(N+1)} \right] P_{N-1}(z)$$

this is for larger N still < 0 and about three times as large as $P_{N-1}(z)$. $P_{N+4}(z)$ would then be roughly four times as large as $P_{N+1}(z)$. This certainly cannot go on that way. Eventually the increase in modulus has to taper off since always $|P_n(t)| \leq 1$. However, we see that there are a certain number of $P_n(z)$, $n = N+1, \ldots, N + K_N$ which are definitely negative. Return now to $C^r(t)$ for $t = z$

$$C^r(z) = \sum_{n>N} \sigma_n^2 P_n(z)$$

The σ_n^2 are ≥ 0. If $\sigma_{N+1}^2 > 0$, $\sigma_{N+2}^2 > 0, \ldots$, and moreover if for larger n the σ_n^2 taper off to zero quickly enough then we have

$$C^r(z) < 0$$

which was to be motivated in the case of being largest zero of $P_N(t)$.

The motivation for the following zeros is similar. Since z is then smaller the motivation is somewhat weaker.

Let us look at $P_{12}(t)$ for example. It has a zero at $z = 0.9816$ which corresponds to $\psi = 11^{\circ}02$. We have $P_{13}(z) = -0.0957$, $P_{14}(z) = -0.18124$, $P_{15}(z) = 0.2546$, $P_{16}(z) = -0.3142$ .... One sees clearly the tendency to follow the pattern $1 \times p$, $2 \times p$, $3 \times p$, ... which has been motivated above.

We shall further exemplify these effects using two covariance functions published by Kaula in 1959 and 1966.

Example 1. Kaula (1959) estimated the following correlation function for the gravity anomalies (See Table 2.)

Arc distance, degrees	Covariance, $mgal^2$	Arc distance, degrees	Covariance, $mgal^2$	Arc distance, degrees	Covariance, $mgal^2$	Arc distance, degrees	Covariance, $mgal^2$
0.0	+1201	21	+35	59	−23	97	+10
0.5	+751	23	+10	61	−38	99	+13
1.0	+468	25	+20	63	−17	101	+15
1.5	+356	27	+18	65	−34	103	+16
2.0	+332	29	+6	67	−17	105	+8
2.5	+306	31	+8	69	−19	107	+13
3.0	+296	33	+5	71	−20	109	−2
4	+272	35	−8	73	−7	111	+19
5	+246	37	−10	75	−6	113	+1
6	+214	39	−13	77	0	115	+10
7	+174	41	−11	79	+3	117	+31
8	+124	43	−7	81	−6	119	+5
9	+104	45	−18	83	+6	122	+26
10	+82	47	−18	85	−6	126	+14
11	+76	49	−18	87	+4	130	+4
13	+54	51	−23	89	−7	134	+2
15	+47	53	−12	91	0	138	−4
17	+45	55	−32	93	−2	145	−23
19°	+34	57°	−23	95°	+4	155	−20
						167°	+5

Table 2

He also computed the first 32-degree variances.

n	σ_n^2	n	σ_n^2	n	σ_n^2	n	σ_n^2
2	7.3	9	22	16	6	23	9
3	43.6	10	15	17	12	24	11
4	29.8	11	18	18	19	25	9
5	10.5	12	7	19	10	26	11
6	24.2	13	15	20	7	27	4
7	2.8	14	23	21	14	28	8
8	22.7	15	22	22	10	29	5

Table 3

We have listed them only up to $n = 29$. Kaula's σ_{30}^2 is negative and therefore not acceptable.

Using these degree variances, we have removed the harmonics up to order N from C(t). The results are shown for $N = 2, 3, \ldots, 12$ in Table 4. Arguments are $\leq 51^\circ$ in Table 4. We know that for greater arguments we have to expect oscillations around the ψ axis. These show up in the results very well but are not exhibited here.

According to our "theorem" (2) in section 2 we should expect negative correlation at the smallest zeros of $P_N(\cos\psi)$. The first zeros in increasing order are listed in Table 5 for various N.

N	ψ_1	ψ_2	ψ_3
2	$54^\circ 7$	-	-
3	$39^\circ 2$	-	-
4	$30^\circ 5$	$70^\circ 2$	-
5	$25^\circ 2$	$57^\circ 4$	-
6	$21^\circ 8$	$47^\circ 0$	$76^\circ 2$
7	$18^\circ 4$	$42^\circ 2$	$66^\circ 0$
8	$16^\circ 2$	$37^\circ 2$	$58^\circ 3$
9	$14^\circ 5$	$33^\circ 3$	$52^\circ 2$
10	$13^\circ 1$	$30^\circ 1$	$47^\circ 2$
12	$11^\circ 0$	$25^\circ 3$	$39^\circ 7$

Table 5

We can verify alternating signs at these zeros. For example, take $N = 12$. $C(\cos\psi)$ is negative around $\psi_1 = 11^\circ$, positive around $\psi_2 = 25^\circ$ and again negative around $\psi_3 = 40^\circ$.

Example 2. In 1966 Kaula estimated the following covariance based on mean $5^\circ \times 5^\circ$ blocks. (See Table 6 on page 72.)

ψ	N=0	2	3	4	5	6	7	8	9	10	11	12
0.0	1201.	1194.	1150.	1120.	1110.	1086.	1083.	1060.	1038.	1023.	1005.	998.
0.5	751.	744.	700.	670.	660.	636.	633.	610.	588.	573.	555.	548.
1.0	468.	461.	417.	387.	377.	353.	350.	327.	306.	291.	273.	266.
1.5	356.	349.	305.	275.	265.	241.	238.	216.	194.	179.	162.	155.
2.0	332.	325.	281.	252.	241.	217.	215.	192.	171.	156.	139.	133.
2.5	306.	299.	255.	226.	216.	192.	189.	167.	146.	132.	115.	108.
3.0	296.	289.	245.	216.	206.	182.	180.	158.	137.	123.	107.	101.
4.0	272.	265.	222.	193.	183.	160.	157.	136.	117.	104.	88.	83.
5.0	246.	239.	196.	167.	158.	135.	133.	113.	95.	83.	69.	64.
6.0	214.	207.	165.	136.	127.	105.	103.	84.	68.	57.	45.	40.
7.0	174.	167.	125.	98.	88.	68.	65.	48.	33.	24.	14.	10.
8.0	124.	117.	76.	49.	40.	20.	18.	3.	-10.	-18.	-27.	-29.
9.0	104.	97.	57.	30.	22.	3.	1.	-12.	-24.	-30.	-36.	-38.
10.0	82.	75.	35.	10.	2.	-15.	-17.	-29.	-38.	-43.	-47.	-48.
11.0	76.	69.	30.	6.	-2.	-18.	-19.	-29.	-36.	-40.	-42.	-42.
13.0	54.	47.	10.	-12.	-19.	-32.	-33.	-39.	-42.	-42.	-40.	-39.
15.0	47.	40.	5.	-15.	-21.	-30.	-31.	-33.	-32.	-30.	-25.	-23.
17.0	45.	39.	6.	-12.	-17.	-23.	-23.	-22.	-18.	-13.	-7.	-4.
19.0	34.	28.	-2.	-18.	-21.	-25.	-25.	-20.	-13.	-8.	-0.	2.
21.0	35.	29.	1.	-11.	-14.	-14.	-13.	-7.	2.	8.	15.	17.
23.0	10.	4.	-20.	-31.	-32.	-29.	-28.	-20.	-11.	-5.	0.	1.
25.0	20.	15.	-7.	-15.	-15.	-10.	-9.	1.	9.	14.	17.	17.
27.0	18.	13.	-6.	-11.	-9.	-3.	-1.	8.	15.	18.	19.	18.
29.0	6.	1.	-14.	-16.	-15.	-6.	-5.	3.	9.	10.	8.	6.
31.0	8.	4.	-9.	-8.	-6.	4.	5.	12.	15.	14.	10.	8.
33.0	5.	1.	-9.	-6.	-2.	8.	9.	14.	14.	12.	6.	4.
35.0	-8.	-12.	-18.	-13.	-9.	1.	2.	4.	2.	-2.	-7.	-9.
37.0	-10.	-13.	-17.	-10.	-5.	4.	5.	5.	1.	-4.	-9.	-10.
39.0	-13.	-16.	-16.	-8.	-3.	5.	6.	4.	-2.	-7.	-11.	-11.
41.0	-11.	-14.	-11.	-1.	3.	11.	11.	7.	-0.	-4.	-6.	-6.
43.0	-7.	-9.	-4.	7.	12.	17.	17.	11.	4.	1.	1.	2.
45.0	-18.	-20.	-12.	-0.	4.	8.	7.	0.	-6.	-8.	-6.	-4.
47.0	-18.	-19.	-9.	3.	7.	8.	8.	1.	-5.	-5.	-1.	1.
49.0	-18.	-19.	-7.	6.	9.	8.	8.	1.	-3.	-1.	3.	5.
51.0	-23.	-24.	-10.	3.	5.	3.	2.	-4.	-5.	-3.	2.	3.

Table 4

Rapp (1967) has observed positive correlation between neighboring $5^\circ \times 5^\circ$ blocks of residual anomalies obtained after removal of satellite observed harmonics. Since at that time the degree up to which the satellite derived harmonics had a reasonable accuracy (i.e. a standard error considerably less than their value) was certainly not above N=8 we see from Table 5 that this positive correlation is in agreement with our results (positive correlation for $\psi = 5^\circ$.). Rapp observed negative correlation between $15^\circ \times 15^\circ$ blocks. This is again in agreement with Table 5 if we assume that N is greater or equal to 6.

Table 8

ψ	original	0	2	3	4	5	6	7	8	9
0.0	274.	271.	265.	233.	215.	206.	184.	173.	164.	154.
5.0	116.	113.	107.	76.	58.	50.	30.	20.	12.	3.
9.0	89.	86.	80.	51.	34.	27.	11.	3.	-2.	-8.
13.0	51.	48.	42.	15.	1.	-4.	-16.	-20.	-23.	-24.
18.0	34.	31.	26.	3.	-8.	-11.	-15.	-16.	-14.	-12.
23.0	20.	17.	12.	-6.	-12.	-13.	-11.	-8.	-4.	-0.
29.0	5.	2.	-2.	-13.	-14.	-13.	-5.	-1.	3.	5.
34.0	-4.	-7.	-10.	-16.	-13.	-10.	-1.	3.	4.	4.
39.0	-10.	-13.	-15.	-16.	-10.	-7.	1.	3.	2.	-1.
44.0	-9.	-12.	-13.	-9.	-1.	2.	6.	5.	3.	-0.
49.0	-9.	-12.	-13.	-4.	4.	7.	6.	3.	0.	-1.
54.0	-10.	-13.	-13.	-1.	6.	8.	3.	-0.	-2.	-1.
59.0	-13.	-16.	-15.	-1.	4.	4.	-3.	-6.	-5.	-3.
64.0	-11.	-14.	-12.	2.	5.	3.	-4.	-5.	-3.	-0.
69.0	-6.	-9.	-7.	7.	7.	5.	-1.	1.	3.	4.
74.0	-5.	-8.	-5.	6.	4.	1.	-1.	2.	4.	3.
80.0	-5.	-8.	-5.	3.	-2.	-4.	-1.	2.	2.	-1.
85.0	-3.	-6.	-3.	1.	-5.	-6.	-0.	2.	-0.	-2.
90.0	-1.	-4.	-1.	-1.	-8.	-8.	-1.	-1.	-3.	-3.
95.0	5.	2.	5.	1.	-5.	-4.	2.	0.	-2.	0.
100.0	9.	6.	9.	1.	-4.	-1.	2.	-1.	-2.	1.
105.0	16.	13.	16.	5.	2.	5.	4.	1.	3.	4.
111.0	17.	14.	16.	3.	3.	6.	1.	-0.	2.	1.
116.0	16.	13.	15.	0.	4.	6.	-2.	-1.	1.	-1.
121.0	14.	11.	12.	-2.	4.	5.	-2.	0.	1.	-2.
126.0	11.	8.	8.	-4.	4.	3.	-2.	2.	0.	-1.
131.0	7.	4.	3.	-5.	3.	0.	-0.	3.	-0.	1.
136.0	1.	-2.	-3.	-8.	-1.	-4.	0.	1.	-2.	1.
141.0	-1.	-4.	-6.	-6.	-1.	-4.	4.	2.	1.	4.
146.0	-8.	-11.	-14.	-8.	-6.	-9.	1.	-3.	-2.	-1.
151.0	-4.	-7.	-11.	1.	-1.	-2.	6.	1.	5.	2.
156.0	-9.	-12.	-16.	1.	-5.	-4.	-1.	-4.	-1.	-5.
162.0	-2.	-5.	-10.	13.	2.	6.	1.	2.	3.	0.
167.0	2.	-1.	-7.	21.	6.	12.	0.	5.	2.	3.
172.0	1.	-2.	-8.	22.	5.	13.	-5.	3.	-3.	3.
175.0	1.	-2.	-8.	23.	5.	13.	-7.	3.	-5.	3.

9. The covariance of the gravity anomalies after imperfect removal of the lower degree harmonics.

The lower degree harmonics are not perfectly determined by satellite methods, but have relative errors generally increasing with the order n. Therefore the effects outlined in section 8 cannot be expected to show up unperturbed. Whereas those attributed to the very low harmonics should still be recognizable, the effects due to the rather unprecise determined harmonics of order 10 and more are certainly somewhat blurred. Nevertheless the characteristic feature of negative correlation between relatively close points is expected to be still present.

In the following we discuss a simplified procedure for combining satellite and gravity information. Then we make some experiments with the examples of section 8.

Let $\Delta g(\xi)$ be the observed gravity anomalies (complemented in some way for the unsurveyed areas). $\Delta g(\xi)$ is superimposed by observation and prediction errors. We can calculate the spherical harmonics coefficients Δg^{g}_{nm} defined through

$$\Delta g(\xi) = \sum_{n,m} \Delta g^{g}_{nm} S_{nm}(\xi)$$

We can only calculate them up to a certain degree without committing too large an error.

On the other hand, satellite methods provide us with harmonics Δg^{s}_{nm} up to a certain degree N.

We can view Δg^{g}_{nm} and Δg^{s}_{nm} as observations of the true harmonic coefficients Δg_{nm}. The observations are superimposed by errors. Thus

$$\Delta g^{g}_{nm} = \Delta g_{nm} + \delta \Delta g^{g}_{nm} \tag{9 - 1}$$

$$\Delta g^{s}_{nm} = \Delta g_{nm} + \delta \Delta g^{s}_{nm} \tag{9 - 2}$$

We are now going to estimate or predict the true coefficients combining the two observations. Thereby we assume

$$\Delta g_{nm} \quad \text{has variance} \qquad \sigma_{nm}^2 = \frac{\sigma_n^2}{2n+1} \tag{9 - 3}$$

$$\delta\Delta g_{nm}^G \quad \text{has variance} \qquad \tau_{nm}^2 = \frac{\tau_n^2}{2n+1} \tag{9 - 4}$$

$$\delta\Delta g_{nm}^s \quad \text{has variance} \qquad \omega_{nm}^2 = \frac{\omega_n^2}{2n+1} \tag{9 - 5}$$

We assume no correlation between all these quantities.

This assumption is certainly a simplification. We know that there is some correlation between satellite derived coefficients, i.e. between different $\delta\Delta g_{nm}^s$. We ignore this here. The reader too ready to condemn such simplification shall however notice that this assumption is the only one to insure the estimated anomaly process (after the combination) to be an isotropic process on Γ. As soon as there are correlations between different $\delta\Delta g_{nm}^s$ the estimated combined process would no longer be an isotropic process. Its covariance would no longer be dependent only on the distance between two points but also on latitude and possibly longitude of the points.

Thus the problem would have to be attacked by means of general, i.e. non-isotropic, processes on Γ.

Since in this study we want to retain the isotropic property of the process, our assumptions are nearly the most general one to insure this.

If we employ now the methods for linear prediction, (Papoulis, (1965), p. 390), we find by the orthogonality principle that the problem decomposes into an estimation problem for each single coefficient. Moreover we see that we have a non-trivial estimation problem only for those coefficients for which we have two observations, i.e. for which we have a value from anomalies and from satellites. Coefficients obtained from anomalies only remain unchanged. They need not even be evaluated during the combination procedure.

The prediction problem for one coefficient looks as follows:

Find an estimate $\Delta\tilde{g}_{nm}$ for Δg_{nm} as a linear combination

$$\Delta\tilde{g}_{nm} = \alpha_{nm}^G \Delta g_{nm}^G + \alpha_{nm}^s \Delta g_{nm}^s \qquad n \leq N \tag{9 - 6}$$

The criterion for the choice of the α's is

$$E\{(\Delta g_{nm} - \Delta \tilde{g}_{nm})^2\} = \text{Minimum}$$

The problem is solved by the orthogonality principle leading to

$$E\{(\Delta g_{nm} - \Delta \tilde{g}_{nm})\,\Delta g^{g}_{nm}\} = 0, \quad E\{(\Delta g_{nm} - \Delta \tilde{g}_{nm})\Delta g^{s}_{nm}\} = 0 \qquad \ldots (9-7)$$

Inserting

$$\Delta \tilde{g}_{nm} = \alpha^{g}_{nm}(\Delta g_{nm} + \delta \Delta g^{g}_{nm}) + \alpha^{s}_{nm}(\Delta g_{nm} + \delta \Delta g^{s}_{nm})$$

we obtain for fixed n, m the two equations:

$$(\sigma^2_{nm} + \tau^2_{nm})\,\alpha^{g}_{nm} + \sigma^2_{nm}\alpha^{s}_{nm} = \sigma^2_{nm}$$

$$\sigma^2_{nm}\alpha^{g}_{nm} + (\sigma^2_{nm} + \omega^2_{nm})\,\alpha^{s}_{nm} = \sigma^2_{nm} \qquad (9-8)$$

Their solution yields α^{g}_{nm}, α^{s}_{nm}:

$$\alpha^{g}_{nm} = \frac{\sigma^2_n}{\tau^2_n}\;\frac{1}{1 + \frac{\sigma^2_n}{\tau^2_n} + \frac{\sigma^2_n}{\omega^2_n}} \qquad (9-9)$$

$$\alpha^{s}_{nm} = \frac{\sigma^2_n}{\omega^2_n}\;\frac{1}{1 + \frac{\sigma^2_n}{\tau^2_n} + \frac{\sigma^2_n}{\tau^2_n}} \qquad (9-10)$$

$$0 \le n \le N$$

$$-n \le m \le +n$$

The best estimate for $n > N$, i.e. for the coefficients with no satellite information is merely

$$\Delta \tilde{g}_{nm} = \Delta g^{g}_{nm} \qquad (9-11)$$

However, the Δg^{g}_{nm}, $n > N$ need not be evaluated. For we have for the best estimate $\Delta\tilde{g}(\xi)$ for $\Delta g(\xi)$:

$$\Delta\tilde{g} = \sum_{\substack{n,m\\ m}} \Delta\tilde{g}_{nm} S_{nm} = \sum_{\substack{n \le N\\ m}} \Delta\tilde{g}_{nm} S_{nm} + \sum_{\substack{n > N\\ m}} \Delta g^{g}_{nm} S_{nm}$$

$$= \sum_{n,m} \Delta g^{s}_{nm} S_{nm} - \sum_{\substack{n \le N \\ m}} (\Delta g^{s}_{nm} - \Delta \tilde{g}_{nm}) S_{nm}$$

Thus

$$\Delta \tilde{g}(\xi) = \Delta \tilde{g}(\xi) - \sum_{\substack{n \le N \\ m}} (\Delta g^{s}_{nm} - \Delta \tilde{g}_{nm}) S_{nm}(\xi) \tag{9 - 12}$$

This serves the evaluation of the estimated gravity anomalies after combination with satellites.

The (unknown) true anomaly process is

$$\Delta g^{\text{true}} = \sum_{n,m} \Delta g_{nm} S_{nm}(\xi)$$

The residual process after subtraction of the estimated low degree component is

$$\Delta g^{r} = \sum_{\substack{n \le N \\ m}} (\Delta g_{nm} - \Delta \tilde{g}_{nm}) S_{nm} + \sum_{\substack{n > N \\ m}} \Delta g_{nm} S_{nm} \tag{9 - 13}$$

The quantity $\Delta g_{nm} - \Delta \tilde{g}_{nm}$ is the error in Δg_{nm}, its variance ϵ^2_{nm} is according to prediction theory obtained as $\epsilon^2_{nm} = E\{(c_{nm} - \tilde{c}_{nm})^2\} = E\{(c_{nm} - \tilde{c}_{nm}) c_{nm}\}$, or

$$\epsilon^2_{nm} = \sigma^2_{nm}(1 - \alpha^{g}_{nm} - \alpha^{s}_{nm})$$

This is according to (9 - 9), (9 - 10) equal to

$$\epsilon^2_{nm} = \frac{\epsilon^2_n}{2n+1}, \quad \epsilon^2_n = \sigma^2_n \frac{1}{1 + \frac{\sigma^2_n}{\tau^2_n} + \frac{\sigma^2_n}{\omega^2_n}} \tag{9 - 14}$$

The covariance of the residual process is therefore

$$C^{r}(\xi.\eta) = \sum_{n \le N} \epsilon^2_n P_n(\xi.\eta) + \sum_{n > N} \sigma^2_n P_n(\xi.\eta) \tag{9 - 15}$$

The second term on theright has been investigated in section 8. The first term blurrs the picture obtained there somewhat. As long as the ϵ^2_n are small, the effects discussed earlier should still be visible. However they need not necessarily be present up to degree N.

Remark: The combination procedure outlined in Rapp (1968) is a limiting case of the one presented here. It is obtained for $\sigma_n \to \infty$ (Cf. ibid. equ. (31)).

Examples continued.

Kaula (1966) has compared various satellite derived potentials with gravity anomalies. The statistics which he used during this comparison are clearly based on the concept of isotropic processes. This means that any discussion based on these statistics neglects correlation between harmonics.

Gaposchkin and Lambeck (1970) used the same set of statistics to compare their 1969 Smithsonian standard earth with surface gravity anomalies. The gravity anomalies were the same as those used in Kaula (1966). Their estimated correlation function is listed in section 8, Table 6.

We shall use these statistics to obtain an idea of the covariance of the residual gravity anomalies which are obtained after removal of the harmonics derived by Gaposchkin and Lambeck. Equivalently we may say that we are interested in the covariances of gravity anomalies with respect to the 1969 Smithsonian standard earth.

Though Gaposchkin-Lambeck derived a complete set of harmonic coefficients up to degree 16 and in addition several higher coefficients, we shall base our discussion only on coefficients up to degree 12. The harmonics above $n = 12$ certainly serve to better interpolate and predict the motion of the satellites which had been used to determine the field. They also may give a better fit to surface gravity data in well surveyed areas. It is however, questionable whether they contribute favorably to a better fit in areas with poor gravity coverage.

The problem is to find estimates for the quantities ϵ_n^2 in (9-15). From (9-14) we have

$$\epsilon_n^2 = \sigma_n^2 \frac{1}{1 + \frac{\sigma_n^2}{\tau_n^2} + \frac{\sigma_n^2}{\omega_n^2}}$$

Since the ratios $\frac{\sigma_n^2}{\tau_n^2}, \frac{\sigma_n^2}{\omega_n^2}$ increase with increasing n we expect ratios ϵ_n / σ_n in the form of

$$\frac{\epsilon_n}{\sigma_n} = \frac{1}{1 + f(n)}$$

where f(n) is an increasing function.

Figure 5 in Gaposchkin-Lambeck yields the following values of these ratios: (the coefficients in Figure 5 refer to the potential rather than the gravity anomalies. However this is immaterial for the ratios).

n	ϵ_n / σ_n	
2		
3		practically zero
4		
5		
6	0.1	
7	0.15	
8	0.25	
9	0.28	
10	0.34	
11	0.48	
12	0.67	

Table 9

We may check them using the statistic $E(\varepsilon_s^2)$ given and explained in Gaposchkin-Lambeck. We should have

$$E(\varepsilon_s^2) = \sum_{n=2}^{12} \epsilon_n^2$$

For $E(\varepsilon_s^2)$ we have the value 9 taken from Table 20 in Gaposchkin-Lambeck (1970). The right-hand side can be evaluated if the degree variances are given. Gaposchkin-Lambeck (1970) lists the following degree variances: (See Table 10 on following page).

The first column should coincide with Table 7 since the same gravity data have been used. Apparently a re-evaluation has taken place. The differences with Kaula's results in Table 7 are small.

n	gravimetry	from satellites alone	after combination gravimetry-satellites
0	2.9		
1	0		
2	5.9	7.4	7.4
3	31.0	33.3	33.0
4	18.2	19.7	20.0
5	7.3	17.5	17.8
6	20.7	14.4	15.7
7	9.2	16.4	15.5
8	7.0	8.5	6.7
9	8.7	15.1	12.7
10	9.4	17.7	12.9
11	5.7	13.7	12.2
12	3.5	8.4	5.1

Table 10

We use the last column together with the ratios ϵ_n/σ_n in Table 9 to evaluate the ϵ_n^2:

n	ϵ_n^2
2	-
3	-
4	-
5	-
6	0.2
7	0.3
8	0.4
9	1.0
10	1.5
11	2.8
12	2.3

Table 11

The sum $\sum_{n=2}^{12} \epsilon_n^2$ is evaluated to 8.5 and compares well with $E(\epsilon_s^2) = 9$.

We use Table 11 to evaluate the covariance of the residual anomalies by perturbing the results of section 8 according to formula (9 - 15). For N = 12 we would have to add to the last column in Table 4 the expression

$$0.2\ P_6(\cos\psi) + 0.3\ P_7(\cos\psi) + \ldots + 2.3\ P_{12}(\cos\psi)$$

This is essentially done in Table 12 except that Table 4 has been modified according to the differences between Table 7 and Table 10 concerning the gravimetric degree variances. Results are exhibited for $N \leq 12$ in Table 12 on page 83.

The results in Table 12 refer to the case where satellite derived harmonics have been combined with gravimetric data. One may also be interested in the covariance of the anomaly residuals after removal of harmonic coefficients which are purely satellite derived.

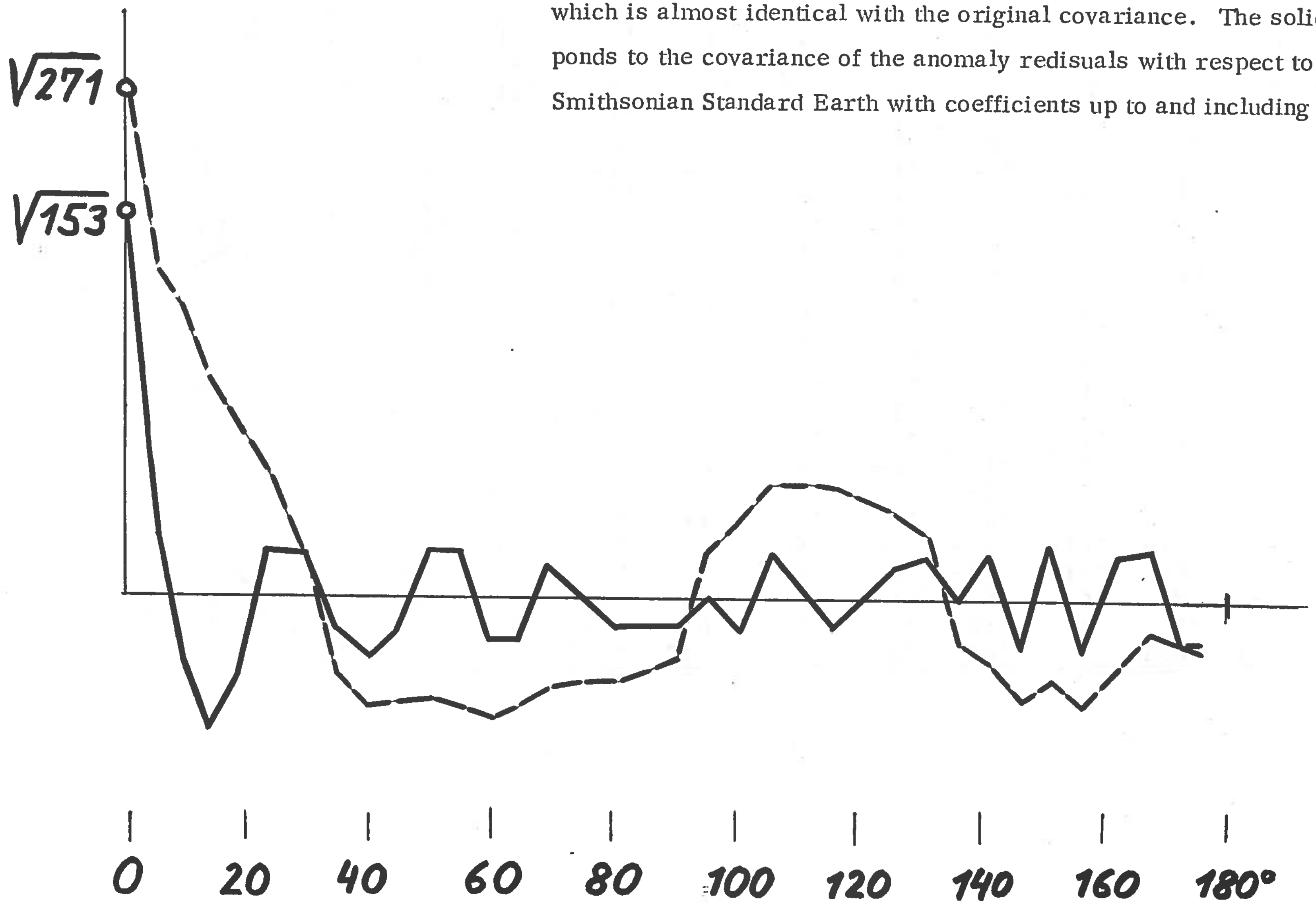
It is clear that in this case the ϵ_n^2 have to be greater. We try to account for that by multiplying the ϵ_n^2 in Table 11 with an appropriate factor. This is theoretically not satisfactory but may be practically feasible because of the smallness of the ϵ's.

The size of the factor will be determined from the $E(\varepsilon_s^2)$ values in Table 20 of Gaposchkin-Lambeck (1970). We have already used $E(\varepsilon_s^2) = 9$ in the combination case with $N = 12$. The corresponding value for the pure satellite solution is $E(\varepsilon_s^2) = 17$. Thus we use a factor of two. The corresponding results are listed in Table 13 on page 85.

ψ	original	0	2	3	4	5	6	7	8	9	10	11	12
0.0	274.	271.	265.	234.	216.	209.	188.	179.	173.	165.	157.	154.	153.
5.0	116.	113.	107.	77.	59.	53.	34.	26.	20.	14.	7.	5.	4.
9.0	89.	86.	80.	52.	36.	30.	14.	8.	4.	0.	-3.	-4.	-5.
13.0	51.	48.	43.	16.	2.	-2.	-13.	-17.	-18.	-19.	-20.	-19.	-19.
18.0	34.	31.	26.	4.	-7.	-9.	-14.	-14.	-13.	-11.	-8.	-7.	-7.
23.0	20.	17.	13.	-5.	-11.	-12.	-10.	-8.	-5.	-2.	1.	2.	2.
29.0	5.	2.	-2.	-13.	-14.	-13.	-6.	-2.	1.	2.	3.	3.	2.
34.0	-4.	-7.	-10.	-16.	-13.	-11.	-2.	1.	2.	2.	-0.	-1.	-1.
39.0	-10.	-13.	-15.	-16.	-10.	-7.	0.	1.	1.	-1.	-4.	-4.	-4.
44.0	-9.	-12.	-14.	-9.	-2.	1.	5.	4.	2.	0.	-1.	-1.	-1.
49.0	-9.	-12.	-13.	-4.	4.	6.	5.	3.	1.	-0.	1.	1.	2.
54.0	-10.	-13.	-13.	-1.	6.	7.	3.	-0.	-1.	-1.	1.	2.	2.
59.0	-13.	-16.	-15.	-2.	4.	3.	-3.	-5.	-5.	-3.	-1.	-1.	-2.
64.0	-11.	-14.	-13.	1.	5.	3.	-4.	-4.	-3.	-1.	-1.	-2.	-2.
69.0	-6.	-9.	-7.	6.	7.	4.	-1.	1.	2.	3.	2.	1.	1.
74.0	-5.	-8.	-6.	6.	3.	1.	-1.	2.	3.	2.	0.	0.	0.
80.0	-5.	-8.	-5.	2.	-2.	-4.	-2.	1.	1.	-1.	-2.	-1.	-1.
85.0	-3.	-6.	-3.	1.	-5.	-6.	-1.	1.	-1.	-2.	-1.	-1.	-1.
90.0	-1.	-4.	-1.	-1.	-8.	-8.	-1.	-1.	-3.	-3.	-1.	-1.	-1.
95.0	5.	2.	5.	1.	-5.	-4.	1.	-0.	-2.	-0.	1.	0.	0.
100.0	9.	6.	9.	1.	-4.	-2.	1.	-1.	-2.	0.	-0.	-1.	-1.
105.0	16.	13.	15.	5.	2.	5.	4.	1.	2.	4.	2.	2.	2.
111.0	17.	14.	16.	3.	3.	6.	1.	-0.	2.	1.	-0.	0.	0.
116.0	16.	13.	14.	0.	4.	5.	-1.	-1.	1.	-1.	-1.	-0.	-1.
121.0	14.	11.	12.	-2.	4.	4.	-2.	0.	0.	-1.	0.	0.	0.
126.0	11.	8.	8.	-4.	4.	3.	-1.	2.	0.	-0.	2.	1.	1.
131.0	7.	4.	3.	-5.	2.	0.	0.	2.	0.	1.	2.	1.	2.
136.0	1.	-2.	-4.	-8.	-1.	-4.	0.	1.	-1.	1.	-0.	-0.	0.
141.0	-1.	-4.	-6.	-6.	-1.	-4.	3.	2.	2.	4.	1.	2.	2.
146.0	-8.	-11.	-14.	-8.	-6.	-8.	0.	-3.	-2.	-2.	-3.	-2.	-3.
151.0	-4.	-7.	-11.	0.	-1.	-2.	5.	1.	4.	2.	3.	3.	3.
156.0	-9.	-12.	-16.	0.	-5.	-5.	-1.	-4.	-2.	-5.	-2.	-3.	-3.
162.0	-2.	-5.	-10.	12.	2.	5.	1.	1.	2.	0.	3.	2.	2.
167.0	2.	-1.	-6.	20.	6.	11.	0.	4.	2.	3.	3.	3.	3.
172.0	1.	-2.	-8.	22.	5.	11.	-5.	2.	-3.	2.	-2.	-1.	-2.
175.0	1.	-2.	-8.	23.	5.	12.	-7.	1.	-5.	2.	-5.	-2.	-3.

Table 12

This figure illustrates column N = 0 and N = 12 of Table 12 on the previous page. The square root of the absolute values in the columns has been taken and the sign has been retained. The intermittent line corresponds to N = 0 which is almost identical with the original covariance. The solid line corresponds to the covariance of the anomaly redisuals with respect to the 1969 Smithsonian Standard Earth with coefficients up to and including N = 12.



ψ	original	0	2	3	4	5	6	7	8	9	10	11	12
0.0	274.	271.	265.	234.	216.	209.	189.	180.	174.	168.	162.	162.	162.
5.0	116.	113.	107.	77.	59.	53.	34.	27.	21.	16.	11.	11.	11.
9.0	89.	86.	80.	52.	36.	30.	14.	9.	5.	2.	-1.	-1.	-1.
13.0	51.	48.	43.	16.	2.	-2.	-13.	-16.	-18.	-19.	-19.	-19.	-19.
18.0	34.	31.	26.	4.	-7.	-9.	-13.	-14.	-13.	-11.	-9.	-9.	-9.
23.0	20.	17.	13.	-5.	-11.	-12.	-10.	-8.	-5.	-3.	-0.	-0.	-0.
29.0	5.	2.	-2.	-13.	-14.	-13.	-6.	-2.	-0.	2.	2.	2.	2.
34.0	-4.	-7.	-10.	-16.	-13.	-11.	-2.	1.	2.	1.	-0.	-0.	-0.
39.0	-10.	-13.	-15.	-16.	-10.	-7.	-0.	1.	1.	-1.	-3.	-3.	-3.
44.0	-9.	-12.	-14.	-9.	-2.	1.	5.	4.	2.	1.	-0.	-0.	-0.
49.0	-9.	-12.	-13.	-4.	4.	6.	5.	3.	1.	0.	1.	1.	1.
54.0	-10.	-13.	-13.	-1.	6.	7.	3.	-0.	-1.	-1.	1.	1.	1.
59.0	-13.	-16.	-15.	-2.	4.	3.	-3.	-5.	-5.	-3.	-2.	-2.	-2.
64.0	-11.	-14.	-13.	1.	5.	3.	-4.	-4.	-3.	-1.	-1.	-1.	-1.
69.0	-6.	-9.	-7.	6.	7.	4.	-0.	1.	2.	3.	2.	2.	2.
74.0	-5.	-8.	-6.	6.	3.	1.	-1.	2.	3.	2.	0.	0.	0.
80.0	-5.	-8.	-5.	2.	-2.	-4.	-2.	1.	0.	-1.	-2.	-2.	-2.
85.0	-3.	-6.	-3.	1.	-5.	-6.	-1.	0.	-1.	-2.	-1.	-1.	-1.
90.0	-1.	-4.	-1.	-1.	-8.	-8.	-1.	-1.	-3.	-3.	-2.	-2.	-2.
95.0	5.	2.	5.	1.	-5.	-4.	1.	-0.	-2.	-0.	1.	1.	1.
100.0	9.	6.	9.	1.	-4.	-2.	1.	-1.	-2.	0.	-0.	-0.	-0.
105.0	16.	13.	15.	5.	2.	5.	4.	1.	3.	4.	2.	2.	2.
111.0	17.	14.	16.	3.	3.	6.	1.	0.	2.	1.	-0.	-0.	-0.
116.0	16.	13.	14.	0.	4.	5.	-1.	-0.	1.	-1.	-1.	-1.	-1.
121.0	14.	11.	12.	-2.	4.	4.	-2.	0.	0.	-1.	0.	0.	0.
126.0	11.	8.	8.	-4.	4.	3.	-1.	1.	0.	-0.	1.	1.	1.
131.0	7.	4.	3.	-5.	2.	0.	0.	2.	0.	1.	2.	2.	2.
136.0	1.	-2.	-4.	-8.	-1.	-4.	-0.	1.	-1.	1.	-0.	-0.	-0.
141.0	-1.	-4.	-6.	-6.	-1.	-4.	3.	2.	1.	3.	1.	1.	1.
146.0	-8.	-11.	-14.	-8.	-6.	-8.	-0.	-3.	-2.	-2.	-3.	-3.	-3.
151.0	-4.	-7.	-11.	0.	-1.	-2.	5.	1.	4.	2.	3.	3.	3.
156.0	-9.	-12.	-16.	0.	-5.	-5.	-2.	-4.	-2.	-4.	-2.	-2.	-2.
162.0	-2.	-5.	-10.	12.	2.	5.	1.	1.	2.	0.	3.	3.	3.
167.0	2.	-1.	-6.	20.	6.	11.	0.	4.	2.	3.	3.	3.	3.
172.0	1.	-2.	-8.	22.	5.	11.	-5.	1.	-3.	1.	-2.	-2.	-2.
175.0	1.	-2.	-8.	23.	5.	12.	-7.	1.	-4.	1.	-4.	-4.	-4.

Table 13

10. Comparison with uncorrelated block errors.

In this section we try to answer the following question. How do the 12 mg residual standard deviation of $5^\circ \times 5^\circ$ anomalies which are obtained after the removal of the estimated harmonics up to order 12 compare with the residual errors of $5^\circ \times 5^\circ$ block means which are derived by gravimetric methods. Note that in the combination procedure gravimetric and satellite information has been combined. In areas with good gravity coverage the accuracy of the final (combined) estimate of a $5^\circ \times 5^\circ$ block is then certainly not less than it was before the combination with the satellite data. Our question pertains then mainly to areas with poor gravimetric coverage. On the other hand, if we deal with a pure satellite solution then our question concerns the residual anomalies all over the earth.

First we clarify the following: What is the standard error of the geoid which is caused by the residual gravity anomalies (which in unsurveyed areas have to be regarded as errors). It will be convenient to discuss this question in terms of spherical harmonics.

It will be necessary to estimate the full spherical harmonics expansion of the covariance $C(t)$ of the original anomalies. Table 10 (first column) gives the degree variances up to $n = 12$. Summed they account only for 130 mg^2 of a total of $C(0) = 274$ (Table 6). We dispose of the remaining 144 mg^2 in the following way:

$$\sigma_n^2(\Delta g) = \frac{c}{(n+d)^2} \beta_n^2 , \quad n \geq 13. \tag{10 - 1}$$

This formula is motivated as follows: $c/(n+d)^2$ is assumed to constitute the degree variance of the unaveraged gravity anomalies for $n \geq 13$. The denominator assures that the corresponding covariance function is continuous. β_n is chosen as in formula (3 - 14a) whereby ψ_o is assumed to be $2^\circ 8$. β_n are thus the eigen-values of the averaging operator over circular disks of spherical radius $\psi_o = 2^\circ 8$. The area of these disks is approximately equal to that of $5^\circ \times 5^\circ$ blocks (at the equator). Hence $(c/(n+d)^2)\beta_n^2$ are the degree variances of the averaged anomalies for $n \geq 13$. c and d are chosen in a way that for the sum over the $\sigma_n^2(\Delta g)$, $n \geq 13$ accounts for 144 mg and that $\sigma_{13}^2(\Delta g)$ is roughly $5 mg^2$, i.e. close to $\sigma_{11}^2(\Delta g) = 5.7\ mg^2$ and

$\sigma_{12}^2(\Delta g) = 3.5 \text{ mg}^2$ in Table 10, column 1. After some trial calculations the values $c = 5.10^5$, $d = 250$ have been adopted. They gave the desired sum $\sum_{n=13}^{\infty} \sigma_n^2(\Delta g) =$ 144 mg^2 and σ_{13}^2 is about 6 mg.

According to formula (6 - 3) and the discussion on error variances at the end of section 6, we get the error variance of N in the form

$$\sigma^2(\delta N) = \left(\frac{R}{G}\right)^2 \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} \sigma_n^2(\delta\Delta g) \tag{10-2}$$

The $\sigma_n^2(\delta\Delta g)$ are the error-degree variances of Δg. In our case we have to put them equal to the ϵ_n^2 in Table 11 for $n \leq 12$ and equal to the above estimated values for $n > 12$.

We get

$$\sum_{n=2}^{12} \frac{\epsilon_n^2}{(n-1)^2} \approx 0.11$$

and

$$\sum_{n=13}^{\infty} \frac{5.10^5}{(n+250)^2} \frac{\beta_n^2}{(n-1)^2} \approx 0.35$$

Hence

$$\sigma^2(\delta N) = \left(\frac{R}{G}\right)^2 \times 0.46$$

Thus $\sigma(\delta N) \approx 4.4$ m. This is close to estimates in Gaposchkin-Lambeck (1970). It should be pointed out that this is the error of the smoothed geoid, i.e. a geoid which is averaged over areas of about $5^\circ \times 5^\circ$ in the same way as the anomalies are. (Cf. the end of section 5).

Now we assume that we have $5^\circ \times 5^\circ$ means which are uncorrelated. Assume for the purpose of better comparison that their error is also 12 mg. What is the error in the undulations caused by them? The numerical results in Heiskanen-Moritz (1967), chapter 7, suggests an undulation error of about 7.5 m. Let us check this result in a different manner.

Assume for this purpose that averaging takes place not over $5^\circ \times 5^\circ$ blocks but again over a circular area of 2.8° spherical radius which is of about the same size.

To obtain a covariance of averages over these $\varphi_o = 2.8^o$ disks we take the process $z^{(N)}(\xi)$ from section 4, equation (4 - 29), for N = 500 and zero and first order harmonics omitted. We multiply the process by a suitable factor ρ^2 in order to obtain the desired variance 144.

The formula for the error variance of the undulations is then

$$\sigma^2(\delta N) = \rho^2 \left(\frac{R}{G}\right)^2 \sum_{n=2}^{500} \frac{2n+1}{(n-1)^2} \beta_n^2 = \dots (6.9\,m)^2$$

The value $\sigma(\delta N) = 6.9\,m$ is in good agreement with the 7.5 m from Heiskanen-Moritz (1967).

This shows the following: Due to their particular correlation the mean anomaly residuals after removal of the estimated harmonics up to order 12 yield a smaller error in the geoidal undulation than uncorrelated errors of mean anomalies with the same standard deviation. The factor is roughly 1/2.

If we have uncorrelated errors in mean anomalies of about 6 mg, then the error of the geoid is about the same as in the case of unknown mean anomaly residuals with respect to the Gaposchkin-Lambeck geoid.

The reason behind this is the following: The geoid is much smoother than the gravity anomalies. The smoothing properties of Stokes' operator have already been demonstrated in section 2. Stokes' operator damps the harmonics of degree n by a factor proportional to $\frac{1}{n-1}$. The geoid depends therefore mainly on the lower degree harmonic components of the gravity anomaly field. Satellite methods are able to provide this component with good accuracy. Even though the anomaly residuals with respect to a geoid determined by satellite methods (or by a combination procedure) are large, their correlation is in favor of smoothing out their impact upon geoid errors.

Mean anomalies determined by gravimetric methods may be regarded as having errors uncorrelated between neighboring blocks. This error is even with the use of airborne or shipborne methods smaller than the 12 mg standard deviation of the anomaly residuals with respect to a (combined) satellite geoid. The error may be around 6 mg. This error is not so favorable on smoothing out the impact upon geoid errors. There are low degree components inherent in them which are certainly small because of the zero correlation between blocks. Nevertheless, they are larger than in the other case where they are precisely counteracted by appropriate correlation (which must necessarily be negative somewhere).

If the undulation errors in the resulting geoid are the only basis for comparing satellite and gravimetric methods then we can only say that the Gaposchkin-Lambeck (1970) geoid is about as good as one determined from an overall coverage of the earth with mean $5^\circ \times 5^\circ$ anomalies having errors of about 6 mg. This seems to be true despite of the fact that the anomaly residuals with respect to the Gaposchkin-Lambeck geoid have standard deviation of about 12 mg.

Errors in the undulation of the geoid may however not be the only quantity of interest. In inertial navigation, for example, the deflections of the vertical play a decisive role. Let us turn to errors in the deflection of the vertical.

Using formula (6 - 4) we get for $\delta\Delta v$ instead of Δv:

$$\sigma^2(\delta\Delta v) = \frac{1}{G^2} \sum_{n=2}^{\infty} \frac{n(n+1)}{(n-1)^2} \sigma_n^2(\delta\Delta g)$$

If we turn first to the anomaly residuals with respect to the Gaposchkin-Lambeck (1970) geoid then we have again to put $\sigma_n^2(\delta\Delta g)$ equal to ϵ_n^2 (Table 11) for $n \leq 12$, whereas for $n > 12$ we put them equal to the estimates derived at the beginning of this section.

We get

$$\sum_{n=2}^{12} \frac{n(n+1)}{(n-1)^2} \epsilon_n^2 = 11.5$$

$$\sum_{n=13}^{\infty} \frac{5.10^5}{(n+250)^2} \beta_n^2 \frac{n(n+1)}{(n-1)^2} = 163$$

This gives

$$\sigma^2(\delta\Delta v) = \frac{1}{G^2} 175$$

which leads to a standard deviation of the deflection of the vertical amounting to 3".

We turn now toward uncorrelated $5^\circ \times 5^\circ$ mean anomaly errors which for reasons of better compairson shall for the first also amount to 12 mg. The formula in accordance with the similar $\sigma^2(\delta N)$ calculation is then

$$\sigma^2(\delta\Delta v) = \frac{\rho^2}{G^2} \sum_{n=2}^{500} \frac{2n+1}{(n-1)^2}\, n(n+1)\beta_n^2$$

and yields $\sigma^2(\delta\Delta v) = \frac{1}{G^2} 173$. This leads to the same standard deviation of 3".

This means however that with uncorrelated anomaly errors of about 6 mg we have an error in the (smoothed) deflections of 1"5.

The qualitative reason for this is also clear from our previous discussion. The transition Δg to Δv involves no smoothing or unsmoothing the higher harmonics of Δg are multiplied by a factor, which is almost constant for larger n. Hence the regional variation in Δv is up to a factor about the same as that of Δg. (Stochastic process models dealing with very local variations Δg and Δv show even a strict proportionality between $\sigma(\Delta g)$, $\sigma(\Delta v)$, cf. Shaw-Henrikson (1969). We should not wonder therefore that multiplying $\sigma(\Delta g)$ by a factor of $\frac{1}{2}$ yields a $\sigma(\Delta v)$ multiplied by $\frac{1}{2}$, irregardless of a change in the correlation pattern for distant points.

Thus we see that the geoid obtained from gravimetrically determined $5^\circ \times 5^\circ$ block means having uncorrelated errors of 6 mg is indeed better than that obtained by the harmonics up to degree 12 on the Gaposchkin-Lambeck solution. The improvement does not show up in the undulations. It becomes apparent in the quantities characterizing the slope of the geoid.

Recall that the deflections of the vertical in the above discussion are smoothed versions of the true deflections. The reason for this is that the gravity anomalies underlying the discussion are already smoothed versions; namely, averages over areas comparable to $5^\circ \times 5^\circ$ blocks (at the equator). Thus our standard deviations of 3" and 1"5 seconds refer to smoothed versions of residuals and errors, respectively. The errors of the corresponding quantities prior to averaging are certainly greater. A rough estimate for them is obtained by comparing the standard deviation of the (unaveraged) gravity anomalies in Table 2 which is $\sqrt{1201}$ = 35 mg with the standard deviation of $\sqrt{274}$ = 17 mg of the averaged anomalies in Table 6. The latter have been used in the estimates leading to 3" and 1"5. If we split the 35 mg in the following way

$$35^2 \;=\; 17^2 + 30^2,$$

we see that a 30 mg additional standard deviation is involved in the transition from the averaged to the unaveraged anomalies. If we add this to the 12 mg standard deviation of the anomaly residuals with respect to the Gaposchkin-Lambeck - $N \leq 12$ geoid we obtain $12^2 + 30^2 = 32$ mg. If we add it to the 6 mg error standard deviation of the block means we get $6^2 + 30^2 = 31$ mg. We know that the standard deviation ratio of anomalies and deflections is within relatively narrow bounds, especially if the low frequent portions are nearly removed. In that case the ratio is about $4 \cdot 5$ mg/arc sec, (Table 1a). Applying this ratio toward the above established estimates of 32 and 31 mg we find $32/4 \cdot 5 \approx 7''$ (Gaposchkin-Lambeck) and $31/4 \cdot 5 \approx 7''$ $5^\circ \times 5^\circ$ blocks. There is no significant difference in the standard deviations of the unaveraged deflections. The difference is only apparent in the averaged versions. This means that only the low to medium frequent portions of the deflections are better recovered by the $5^\circ \times 5^\circ$ averages with 6 mg error than by the Gaposchkin-Lambeck $N \leq 12$ geoid. This may, however, be of interest for inertial navigation over larger distances.

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