Reports of the Department of Geodetic Science

Report No. 150

INVESTIGATIONS OF CRITICAL CONFIGURATIONS FOR FUNDAMENTAL RANGE NETWORKS

by

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Prepared for

National Aeronautics and Space Administration
Washington, D. C.

Contract No. NGL 36-008-093
OSURF Project No. 2514

The Ohio State University
Research Foundation
Columbus, Ohio 43212

March, 1971

PREFACE

This project is under the supervision of Ivan I. Mueller, Professor of the Department of Geodetic Science at The Ohio State University, and it is under the technical direction of Jerome D. Rosenberg, Project Manager, Geodetic Satellites Program, NASA Headquarters, Washington, D. C. The contract is adminstered by the Office of University Affairs, NASA, Washington, D. C. 20546.

This report is a part of a dissertation submitted to the Graduate School of The Ohio State University as a partial fulfillment of the requirements for Doctor of Phiosophy degree.

ABSTRACT

A range network is defined to consist of ground stations and targets where only distances between the two sets of points are observed. Such a network is said to be fundamental when only those six constraints are used which are needed to define the coordinate system for an adjustment. In some cases when ground stations and/or targets have certain configurations, a unique adjustment in terms of coordinates may be impossible, even when the number of observations is sufficient and the coordinate system is uniquely defined. Such configurations are said to be critical.

In this study, critical configurations are investigated in two separate chapters. The first deals with ground stations lying all in a plane and the second deals with ground stations generally distributed. The two kinds of problems require different mathematical treatment and lead to quite different conclusions.

A typical critical configuration when all ground stations are in a plane arises when they all lie on one second order curve. When ground stations are generally distributed a typical critical configuration may be represented by all points of a network (ground stations and targets) lying on one second order surface. If these and some other more complex distributions of points are avoided, an adjustment of range networks yields a unique solution.

ACKNOWLEDGEMENTS

The author wishes to acknowledge most deeply the help, encouragement, and understanding of his advisor, Professor Ivan I. Mueller. He gave the author all the support he could and sacrificed any amount of time to help to clarify and improve different parts of this work.

The author is greatly indebted to Dr. Peter Meissl for his time and a number of most valuable suggestions offered during many hours of discussions. These discussions helped significantly in improving the quality of this work. Dr. Peter Meissl also accepted to serve as a guest member on the author's reading committee. Professors Urho A. Uotila and Richard H. Rapp served on the author's reading committee and offered many valuable suggestions and comments which helped to clarify several points.

A thorough proofreading of several parts of this work as well as valuable help in editing at the final stages were offered to the author by his colleague, Dr. Charles R. Schwarz.

A large amount of the computer time was furnished to the author by the Instruction and Research Computer Center of The Ohio State University.

The financial support was granted to the author during his study at The Ohio State University by the National Research Council of Canada and by the National Aeronautics and Space Administration under Research Grant No. NGL 36-008-093.

The author wishes to thank most heartily Miss Barbara Beer who typed the whole work with more than professional patience and skills. The author is indebted to her for the typing of long and extremely difficult text during the working hours as well as in the evenings with patience, devotion and ingenuity. He also wishes to thank Mrs. Evelyn Rist who outside her regular work, offered to help in typing of some additional parts of this presentation.

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1. TREATMENT OF RANGE OBSERVATIONS WITH ALL GROUND STATIONS IN PLANE

1.1 Introduction

The goal of the present study is to investigate the possibility to use the range observations between a set of ground stations and a set of targets (satellite points) which together are said to form a range network. As in most geodetic adjustments, the mathematical model for range observations is treated in a linearized form. The adjustment procedure applied to this model is the least squares method.

The only constraints necessary in range networks are the ones needed for defining the coordinate system; three to define its position and three to define its orientation, i.e., six constraints. Range observations being invariant with respect to the coordinate system, they do not offer information about it; thus, when an adjustment is performed in terms of coordinates a certain coordinate system has to be defined. Any coordinate system thus defined yields theoretically the same adjusted values of distances. In the theoretical part of this investigation, a coordinate system is chosen such that the first ground station is at its origin, the second one on its x axis and the third one in its xy plane. For practical computations, that coordinate system may be the most advantageous which renders the trace of the variance-covariance matrix for the coordinates of all or certain selected points a minimum. The constraints defining the coordinate system in this manner are called inner adjustment constraints. The idea of using inner adjustment constraints was first presented [13], then in [14] and [15], and recently in [1], Annex F and in [9]. The problem of inner adjustment constraints is treated in great detail in [16]. Their application in connection with an actual adjustment appeared in [2] and in [3].

When only six coordinate-system-defining constraints are used, the network is said to be fundamental. In this study, only fundamental networks are investigated.

The type of observations considered in such networks are ranges from ground stations to satellite points. No further distances or any other type of measurements are used. In certain cases when ground stations and/or targets are situated in special configurations, a unique adjustment is impossible even if the number of observations is sufficient. Such critical configurations result in singular solutions and their investigation is the subject of this work.

In this chapter, the ground stations are considered to be in one plane. In sections 1.2 and 1.3, three ground stations are considered to observe ranges to all the satellite points. In section 1.4, the principle of replacing of stations is introduced, which allows to use the derivations made in previous sections for a more general case, when not all satellite positions are observed by three stations. This leads to a case known in practice as "leapfrogging".

The basic idea used in treating the networks where the ground stations are approximately in one plane is to stipulate that theoretically all ground stations are exactly in the plane and to find the critical loci of the points in the network which will result in a singular solution. Applications for practical cases (where the condition of coplanarity is only approximately fulfilled) follows from the fact that conditions leading to singularity in theory lead to near-singularity in practice. Examples of the correspondence between such theoretical and practical configuration-conditions are the following:

(a) Targets on a straight line in theory correspond to satellite positions on a relatively short pass in practice.

(b) Ground stations lying on a second order (plane) curve in theory correspond in practice to ground stations in projection on the (best fitting) plane lying on or near a second order curve.

(c) A satellite group lying theoretically in a plane corresponds in practice to short satellite passes of approximately the same altitude above the ground network. This situation can arise when the same satellite is observed on different passes. The plane fitted to such targets is approximately parallel to the plane fitted to the ground stations.

The main outcome of this investigation will be the detection of singularity for the theoretical cases and the establishment of rules to avoid it. This is equivalent to having range measurements alone as a working observational mode for fundamental range networks with ground stations lying in or near a plane.

1.2 Basic Principle to Detect Singularity When Using Range Observational Mode with Three Ground Stations Observing All Targets

First a coordinate system will be defined, in which a cluster of points .l be adjusted so as to yield the relative position of all its points. Whenever :ra observations are used, the least squares adjustment will be applied. When s relative position with the same adjusted ranges is unique or not unique, the blem is said to be non-singular or singular respectively. For the sake of plicity in the derivations (range observations are invariant with respect to coordinate system), the origin of the coordinate system will be chosen to ncide with one ground station, the x axis will pass through another, and xy plane will contain a third ground station. These will also be the three tions observing all the satellite points and will be numbered for all derivations and discussions as 1, 2, and 3, respectively. The six conaints defining this coordinate system, called LOCAL COORDINATE TEM are the only constraints to be used and thus such cluster of points stitutes a fundamental network. The points 1, 2, and 3 before the adjustment in the above coordinate system have the coordinates (0, 0, 0), $(x_2^o, 0, 0)$, $(x_3^o, y_3^o, 0)$ and after the adjustment the coordinates (0, 0, 0), $(x_2^a, 0, 0)$, $(x_3^a, y_3^a, 0)$ respectively (in order that x y plane of the local coordinate system be well defined, $y_3 \neq 0$ has to hold; similarly for x axis $x_2 \neq 0$).

Considering $x^a = x^o + dx$, $y^a = y^o + dy$, $z^a = z^o + dz$, the above definition of the coordinate system corresponds to six constraints for the parameters, namely:

$$dx_1 = 0, \quad dy_1 = 0, \quad dz_1 = 0 \; ; \quad dy_2 = 0, \quad dz_2 = 0; \quad dz_3 = 0. \tag{1.2-1}$$

In the following, the coordinates of the ground stations will be denoted by small letters and those of the satellite points by capital letters, as well as the corresponding corrections; thus dx_i, dy_i, dz_i, denote corrections to the ith station and dX_j, dY_j, dZ_j, denote corrections to the jth satellite.

Next, the observation equations for the least squares adjustment will be formed. Let a_{ij}^x, a_{ij}^y, a_{ij}^z, denote directional cosines of the line connecting station "i" with satellite "j" with respect to the x, y, z axes of the local coordinate system. With v_{ij} denoting the residual to L_{ij}^b, observed distance between i and j, where $L_{ij}^b + v_{ij} = L_{ij}^a$ (adjusted distance), the observation equation for the distance i - j is of the form

$$v_{ij} = a_{ij}^x \; (dX_j - dx_i) + a_{ij}^y \; (dY_j - dy_i) + a_{ij}^z \; (dZ_j - dz_i) + L_{ij} \, . \tag{1.2-2}$$

Here $L_{ij} = L_{ij}^o - L_{ij}^b$ where L_{ij}^o is the distance i - j computed from preliminary (approximate) coordinates.

In matrix form, the observation equations are written as (writing X vector instead of dX):

$$V = AX + L$$
$$L^a = L^b + V.$$

From the least squares adjustment it is obtained

$$X = -\ (A^T PA)^{-1}\ (A^T PL).$$

Positive definite P matrix contains weights of the observed quantities; when the observations are uncorrelated and of equal precision (i.e., coming from the same population of random errors), then $P = I$ (identity matrix) can be used; this will be assumed throughout in this study.

The problem will then be singular if for the same set of L^a or V there are different solution sets X possible, or, which is the same, the solution vector X (and thus the relative position of the cluster points) corresponding to the same residual vector V is not unique.

Otherwise the problem is non-singular. If X denotes one solution, $\overline{X}$ any other solution and if $\partial X = X - \overline{X}$, then $\partial X = 0$ as the only possibility characterizes a non-singular problem.

Suppose

$$V = AX + L,$$

$$V = A\overline{X} + L;$$

then

$$A(X - \overline{X}) = 0,$$

or

$$\underset{(o \times u)}{A}\ \underset{(u \times 1)}{\partial X} = \underset{(o \times 1)}{0} \qquad (1.2\text{-}3)$$

where $o \geq u$; o is the number of all observations, while $u = 3 \times$ (number of all points) - 6, i.e., the number of all unknown parameters.

Whenever extra observations are used, $o > u$ and A is not a square matrix.

Matrix equation (1.2-3) represents a homogenous system of "o" equations in the "u" inknowns, which has always a solution, namely the trivial solution. If rank $A = u$, then only the trivial solution of (1.2-3) is possible and thus the problem is non-singular. Correspondingly, such $\underset{(o \times u)}{A}$ matrix will be called here "non-singular A". On the other hand, if rank $A < u$, the non-trivial solution of (1.2-3) also exists and the problem is singular, with $\underset{(\sigma \times u)}{A}$ matrix being called "singular A". Thus, in the following study, the column space of A will be dealt with.

Now from (1.2-3) it can be written for a typical row:

$$a^x_{ij}\,(\partial X_j - \partial x_i) + a^y_{ij}\,(\partial Y_j - \partial y_i) + a^z_{ij}\,(\partial Z_j - \partial z_i) = 0 \qquad (1.2\text{-}4a)$$

or, using vector notation

$$n^T_{ij}\,\partial X_j - n^T_{ij}\,\partial x_i = 0. \qquad (1.2\text{-}4b)$$

In this notation

$$n_{ij} = \begin{bmatrix} a^x_{ij} \\ a^y_{ij} \\ a^z_{ij} \end{bmatrix}, \quad \partial X_j = \begin{bmatrix} \partial X_j \\ \partial Y_j \\ \partial Z_j \end{bmatrix}, \quad \partial x_i = \begin{bmatrix} \partial x_i \\ \partial y_i \\ \partial z_i \end{bmatrix} \qquad (1.2\text{-}4c)$$

where, using preliminary coordinates,

$$a^x_{ij} = \frac{X_j - x_i}{s_{ij}}, \quad a^y_{ij} = \frac{Y_j - y_i}{s_{ij}}, \quad a^z_{ij} = \frac{Z_j - z_i}{s_{ij}} \qquad (1.2\text{-}4d)$$

with

$$s_{ij} = \left[(X_j - x_i)^2 + (Y_j - y_i)^2 + (Z_j - z_i)^2\right]^{\frac{1}{2}}$$

and where

$$\partial X_j = dX_j - \overline{dX_j}, \quad \partial x_i = dx_i - \overline{dx_i}. \qquad (1.2\text{-}4e)$$

Due to the chosen coordinate system, which is the same for any solution X, it must also hold that

$$\overline{dx_1} = 0, \quad \overline{dy_1} = 0, \quad \overline{dz_1} = 0; \quad \overline{dy_2} = 0, \quad \overline{dz_2} = 0; \quad \overline{dz_3} = 0 \qquad (1.2\text{-}5a)$$

and consequently, using (1.2-1) and (1.2-4e), that

$$\partial x_1 = 0, \ \partial y_1 = 0, \ \partial z_1 = 0; \quad \partial y_2 = 0, \ \partial z_2 = 0; \quad \partial z_3 = 0. \qquad (1.2\text{-}5b)$$

An example of a coefficient matrix A of observation equations such as used in (1.2-3) is given in Table (1.2-1); there only the necessary number of observations is present, i.e., $o = u$. The notation

$$n_{ij} = [a_{ij}^{x} \ a_{ij}^{y} \ a_{ij}^{z}]^{T}$$

represents the unit vector in the i - j direction; it was first introduced in (1.2-4c). The number of ground stations used in this example is seven and the corresponding necessary number of targets (satellite points) is fifteen. The stations observing all the targets are numbered as 1, 2, 3; the stations denoted as 7, 5, 6 observe four targets each; these targets are numbered as $j_{11} - j_{14}$, $j_{21} - j_{24}$, and $j_{31} - j_{34}$, respectively. Station 4 in this particular example is assumed to observe targets j_{41}, j_{42}, and j_{43}. The coordinate system is defined as before, so that (1.2-5b) is fulfilled. In the headings of Table (1.2-1), ∂x_2, ∂x_3, and ∂y_3 pertain to the x coordinate of station 2, x coordinate of station 3, and y coordinate of station 3 respectively. The notation ∂gr_i is designed to represent three columns for station i, i.e., ∂x_i, ∂y_i, and ∂z_i; the same holds also for $\partial X_{j_{mn}}$ with respect to the satellite point j_{mn}. If more than a necessary number of observations were used, the table would be expanded in an analogous manner: for each new station a three column block ∂gr, and for each new target a new three column block ∂X would be added; for new observations between existing stations and targets, rows in the corresponding row block "From i" would be added.

The satellite parameters will be now eliminated using stations 1, 2, 3 for which the relation (1.2-5b) holds. Since 1, 2, 3 observe all the satellites, the following equations will hold for any satellite point j :

$$i = 1 \ldots \ a_{ij}^{x} \, \partial X_j + a_{ij}^{y} \, \partial Y_j + a_{ij}^{z} \, \partial Z_j = 0, \qquad (1.2\text{-}6a)$$

Table (1.2-1)

The A matrix for seven ground stations and fifteen satellite points

	∂x_2	∂x_3	∂y_3	∂gr_4	∂gr_5	∂gr_6	∂gr_7	$\partial X_{j_{11}}$	$\partial X_{j_{12}}$	$\partial X_{j_{13}}$	$\partial X_{j_{14}}$	$\partial X_{j_{21}}$	$\partial X_{j_{22}}$	$\partial X_{j_{23}}$	$\partial X_{j_{24}}$	$\partial X_{j_{31}}$	$\partial X_{j_{32}}$	$\partial X_{j_{33}}$	$\partial X_{j_{34}}$	$\partial X_{j_{41}}$	$\partial X_{j_{42}}$	$\partial X_{j_{43}}$
From station 1								$n^T_{1j_{11}}$														
									$n^T_{1j_{12}}$													
										$n^T_{1j_{13}}$												
											$n^T_{1j_{14}}$											
												$n^T_{1j_{21}}$										
													$n^T_{1j_{22}}$									
														$n^T_{1j_{23}}$								
															$n^T_{1j_{24}}$							
																$n^T_{1j_{31}}$						
																	$n^T_{1j_{32}}$					
																		$n^T_{1j_{33}}$				
																			$n^T_{1j_{34}}$			
																				$n^T_{1j_{41}}$		
																					$n^T_{1j_{42}}$	
																						$n^T_{1j_{43}}$

Table (1,2-1)
(continued)

	∂x_2	∂x_3	∂y_3	∂gr_4	∂gr_5	∂gr_6	∂gr_7	$\partial X_{J_{11}}$	$\partial X_{J_{12}}$	$\partial X_{J_{13}}$	$\partial X_{J_{14}}$	$\partial X_{J_{21}}$	$\partial X_{J_{22}}$	$\partial X_{J_{23}}$	$\partial X_{J_{24}}$	$\partial X_{J_{31}}$	$\partial X_{J_{32}}$	$\partial X_{J_{33}}$	$\partial X_{J_{34}}$	$\partial X_{J_{41}}$	$\partial X_{J_{42}}$	$\partial X_{J_{43}}$
From station 2	$-a^x_{2J_{11}}$							$n^T_{2J_{11}}$														
	$-a^x_{2J_{12}}$								$n^T_{2J_{12}}$													
	$-a^x_{2J_{13}}$									$n^T_{2J_{13}}$												
	$-a^x_{2J_{14}}$										$n^T_{2J_{14}}$											
	$-a^x_{2J_{21}}$											$n^T_{2J_{21}}$										
	$-a^x_{2J_{22}}$												$n^T_{2J_{22}}$									
	$-a^x_{2J_{23}}$													$n^T_{2J_{23}}$								
	$-a^x_{2J_{24}}$														$n^T_{2J_{24}}$							
	$-a^x_{2J_{31}}$															$n^T_{2J_{31}}$						
	$-a^x_{2J_{32}}$																$n^T_{2J_{32}}$					
	$-a^x_{2J_{33}}$																	$n^T_{2J_{33}}$				
	$-a^x_{2J_{34}}$																		$n^T_{2J_{34}}$			
	$-a^x_{2J_{41}}$																			$n^T_{2J_{41}}$		
	$-a^x_{2J_{42}}$																				$n^T_{2J_{42}}$	
	$-a^x_{2J_{43}}$																					$n^T_{2J_{43}}$

Table (1.2-1)
(continued)

	∂x_2	∂x_3	∂y_3	∂gr_4	∂gr_5	∂gr_6	∂gr_7	$\partial X_{J_{11}}$	$\partial X_{J_{12}}$	$\partial X_{J_{13}}$	$\partial X_{J_{14}}$	$\partial X_{J_{21}}$	$\partial X_{J_{22}}$	$\partial X_{J_{23}}$	$\partial X_{J_{24}}$	$\partial X_{J_{31}}$	$\partial X_{J_{32}}$	$\partial X_{J_{33}}$	$\partial X_{J_{34}}$	$\partial X_{J_{41}}$	$\partial X_{J_{42}}$	$\partial X_{J_{43}}$
From station 3		$-a^x_{3J_{11}}$	$-a^y_{3J_{11}}$					$n^T_{3J_{11}}$														
		$-a^x_{3J_{12}}$	$-a^y_{3J_{12}}$						$n^T_{3J_{12}}$													
		$-a^x_{3J_{13}}$	$-a^y_{3J_{13}}$							$n^T_{3J_{13}}$												
		$-a^x_{3J_{14}}$	$-a^y_{3J_{14}}$								$n^T_{3J_{14}}$											
		$-a^x_{3J_{21}}$	$-a^y_{3J_{21}}$									$n^T_{3J_{21}}$										
		$-a^x_{3J_{22}}$	$-a^y_{3J_{22}}$										$n^T_{3J_{22}}$									
		$-a^x_{3J_{23}}$	$-a^y_{3J_{23}}$											$n^T_{3J_{23}}$								
		$-a^x_{3J_{24}}$	$-a^y_{3J_{24}}$												$n^T_{3J_{24}}$							
		$-a^x_{3J_{31}}$	$-a^y_{3J_{31}}$													$n^T_{3J_{31}}$						
		$-a^x_{3J_{32}}$	$-a^y_{3J_{32}}$														$n^T_{3J_{32}}$					
		$-a^x_{3J_{33}}$	$-a^y_{3J_{33}}$															$n^T_{3J_{33}}$				
		$-a^x_{3J_{34}}$	$-a^y_{3J_{34}}$																$n^T_{3J_{34}}$			
		$-a^x_{3J_{41}}$	$-a^y_{3J_{41}}$																	$n^T_{3J_{41}}$		
		$-a^x_{3J_{42}}$	$-a^y_{3J_{42}}$																		$n^T_{3J_{42}}$	
		$-a^x_{3J_{43}}$	$-a^y_{3J_{43}}$																			$n^T_{3J_{43}}$

Table (1.2-1)
(continued)

	∂x_2	∂x_3	∂y_3	∂gr_4	∂gr_5	∂gr_6	∂gr_7	$\partial X_{J_{11}}$	$\partial X_{J_{12}}$	$\partial X_{J_{13}}$	$\partial X_{J_{14}}$	$\partial X_{J_{21}}$	$\partial X_{J_{22}}$	$\partial X_{J_{23}}$	$\partial X_{J_{24}}$	$\partial X_{J_{31}}$	$\partial X_{J_{32}}$	$\partial X_{J_{33}}$	$\partial X_{J_{34}}$	$\partial X_{J_{41}}$	$\partial X_{J_{42}}$	$\partial X_{J_{43}}$
From 7							$-n_{7J_{11}}^T$	$n_{7J_{11}}^T$														
							$-n_{7J_{12}}^T$		$n_{7J_{12}}^T$													
							$-n_{7J_{13}}^T$			$n_{7J_{13}}^T$												
							$-n_{7J_{14}}^T$				$n_{7J_{14}}^T$											
From 5					$-n_{5J_{21}}^T$							$n_{5J_{21}}^T$										
					$-n_{5J_{22}}^T$								$n_{5J_{22}}^T$									
					$-n_{5J_{23}}^T$									$n_{5J_{23}}^T$								
					$-n_{5J_{24}}^T$										$n_{5J_{24}}^T$							
From 6						$-n_{6J_{31}}^T$										$n_{6J_{31}}^T$						
						$-n_{6J_{32}}^T$											$n_{6J_{32}}^T$					
						$-n_{6J_{33}}^T$												$n_{6J_{33}}^T$				
						$-n_{6J_{34}}^T$													$n_{6J_{34}}^T$			
From 4				$-n_{4J_{41}}^T$																$n_{4J_{41}}^T$		
				$-n_{4J_{42}}^T$																	$n_{4J_{42}}^T$	
				$-n_{4J_{43}}^T$																		$n_{4J_{43}}^T$

$$i = 2 \ldots \; a^x_{2j}(\partial X_j - \partial x_2) + a^y_{2j}\,\partial Y_j + a^z_{2j}\,\partial Z_j = 0, \tag{1.2-6b}$$

$$i = 3 \ldots \; a^x_{3j}(\partial X_j - \partial x_3) + a^y_{3j}(\partial Y_j - \partial y_3) + a^z_{3j}\,\partial Z_j = 0. \tag{1.2-6c}$$

Simplifications $x_1 = y_1 = z_1 = 0$, $y_2 = z_2 = 0$, $z_3 = 0$ due to the chosen coordinate system yield in (1.2-6a):

$$\partial Z_j = -\frac{1}{Z_j}(X_j\,\partial X_j + Y_j\,\partial Y_j). \tag{1.2-7a}$$

Upon multiplication by s_{1j} and s_{2j} of (1.2-6a) and (1.2-6b) and taking the difference, it follows:

$$\partial X_j = (x_2 - X_j)\frac{\partial x_2}{x_2}. \tag{1.2-7b}$$

Similarly for (1.2-6a) and (1.2-6c) it is obtained:

$$\partial Y_j = -\frac{x_3}{y_3}(x_2 - X_j)\frac{\partial x_2}{x_2} + (x_3 - X_j)\frac{\partial x_3}{y_3} + (y_3 - Y_j)\frac{\partial y_3}{y_3}. \tag{1.2-7c}$$

In these equations it is necessary that

$$x_2 \neq 0, \quad y_3 \neq 0, \quad Z_j \neq 0 \text{ for any } j. \tag{1.2-8}$$

The first two relations were already used in the definition of the coordinate system; the last one means that the following derivations will hold only if no satellite is in the plane of the ground stations 1, 2, and 3. Otherwise such a point could not be determined from stations 1, 2, and 3, even if these were all known; in the linearized form, the point could freely move in the direction perpendicular to the plane of stations 1, 2, and 3. Upon plugging the expressions (1.2-7b) and (1.2-7c) into (1.2-7a), it is obtained:

$$\partial Z_j = -\frac{1}{Z_j}(x_2 - X_j)\left(X_j - Y_j\frac{x_3}{y_3}\right)\frac{\partial x_2}{x_2} - \frac{Y_j}{Z_j}(x_3 - X_j)\frac{\partial x_3}{y_3} - \frac{Y_j}{Z_j}(y_3 - Y_j)\frac{\partial y_3}{y_3}. \tag{1.2-7a'}$$

The relations (1.2-7a′), (1.2-7b) and (1.2-7c) express the variation of satellite parameters in terms of $\frac{\partial x_2}{x_2}$, $\frac{\partial x_3}{y_3}$, and $\frac{\partial y_3}{y_3}$, and thus make possible their elimination in (1.2-4a) for an arbitrary station i. After multiplying (1.2-4a) by s_{ij}, then by $(-Z_j)$ and carrying out some algebraic operation, the following relation is obtained for any j:

$$
\begin{aligned}
&Z_j (X_j - x_1)\partial x_1 + Z_j (Y_j - y_1)\partial y_1 + Z_j (Z_j - z_1)\partial z_1 + \\
&(z_1 Y_j - y_1 Z_j)(X_j - x_3)\frac{\partial x_3}{y_3} + (z_1 Y_j - y_1 Z_j)(Y_j - y_3)\frac{\partial y_3}{y_3} + \\
&\left[z_1\left(X_j - Y_j \frac{x_3}{y_3}\right) - Z_j\left(x_1 - y_1 \frac{x_3}{y_3}\right)\right](X_j - x_2)\frac{\partial x_2}{x_2} = 0.
\end{aligned}
\qquad (1.2\text{-}9)
$$

Similar approach was used in [1], Annex A and in [10] with these two main differences: first, only four ground stations, forming the ground network observing simultaneously were considered, and second, only six satellite points were used, which means that no extra observations were considered in that derivation.

The result (1.2-9) for all stations i can be written in a matrix form as

$$\widetilde{A}\,\partial\widetilde{X} = 0 . \qquad (1.2\text{-}10)$$

$\widetilde{A}$ matrix is presented in Table (1.2-2). It is of dimension $(\widetilde{o} \times \widetilde{u})$

where

$$\widetilde{o} = o - 3 \times (\text{number of all satellite points})$$

and

$$\widetilde{u} = u - 3 \times (\text{number of all satellite points}).$$

What was said for the matrix A and ∂X of (1.2-3) applies also for the matrix $\widetilde{A}$ and $\partial\widetilde{X}$ of (1.2-10). In particular, when dealing with "non-singular $\widetilde{A}$", only trivial solution for $\partial\widetilde{X}$ is possible to fulfill (1.2-10).

The expression $\widetilde{A}\,\partial\widetilde{X} = 0$ (pertaining to ground stations) together with $\partial X^s = S \begin{bmatrix} \partial x_2 \\ \partial x_3 \\ \partial y_3 \end{bmatrix}$ where s denotes any satellite point and S is the corresponding

Table (1.2-2)

Explicit Expression for $\widetilde{A}$ Matrix

	∂x_4	∂y_4	∂z_4	∂x_5	∂y_5	∂z_5	∂x_1	∂y_1	∂z_1	$\partial x_3/y_3$	$\partial y_3/y_3$	$\partial x_2/x_2$
station 4	$Z_1(X_1-x_4)$	$Z_1(Y_1-y_4)$	$Z_1(Z_1-z_4)$							$(z_4Y_1-y_4Z_1)(X_1-x_3)$	$(z_4Y_1-y_4Z_1)(Y_1-y_3)$	$[z_4(X_1-Y_1\frac{x_3}{y_3})-Z_1(x_4-y_4\frac{x_3}{y_3})](X_1-x_2)$
	$\vdots$	$\vdots$	$\vdots$							$\vdots$	$\vdots$	$\vdots$
	$Z_j(X_j-x_4)$	$Z_j(Y_j-y_4)$	$Z_j(Z_j-z_4)$							$(z_4Y_j-y_4Z_j)(X_j-x_3)$	$(z_4Y_j-y_4Z_j)(Y_j-y_3)$	$[z_4(X_j-Y_j\frac{x_3}{y_3})-Z_j(x_4-y_4\frac{x_3}{y_3})](X_j-x_2)$
	$\vdots$	$\vdots$	$\vdots$							$\vdots$	$\vdots$	$\vdots$
station 5				$Z_1'(X_1'-x_5)$	$Z_1'(Y_1'-y_5)$	$Z_1'(Z_1'-z_5)$				$(z_5Y_1'-y_5Z_1')(X_1'-x_3)$	$(z_5Y_1'-y_5Z_1')(Y_1'-y_3)$	$[z_5(X_1'-Y_1'\frac{x_3}{y_3})-Z_1'(x_5-y_5\frac{x_3}{y_3})](X_1'-x_2)$
				$\vdots$	$\vdots$	$\vdots$				$\vdots$	$\vdots$	$\vdots$
				$Z_j'(X_j'-x_5)$	$Z_j'(Y_j'-y_5)$	$Z_j'(Z_j'-z_5)$				$(z_5Y_j'-y_5Z_j')(X_j'-x_3)$	$(z_5Y_j'-y_5Z_j')(Y_j'-y_3)$	$[z_5(X_j'-Y_j'\frac{x_3}{y_3})-Z_j'(x_5-y_5\frac{x_3}{y_3})](X_j'-x_2)$
				$\vdots$	$\vdots$	$\vdots$				$\vdots$	$\vdots$	$\vdots$
station i							$Z_1^i(X_1^i-x_1)$	$Z_1^i(Y_1^i-y_1)$	$Z_1^i(Z_1^i-z_1)$	$(z_1Y_1^i-y_1Z_1^i)(X_1^i-x_3)$	$(z_1Y_1^i-y_1Z_1^i)(Y_1^i-y_3)$	$[z_1(X_1^i-Y_1^i\frac{x_3}{y_3})-Z_1^i(x_1-y_1\frac{x_3}{y_3})](X_1^i-x_2)$
							$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
							$Z_j^i(X_j^i-x_1)$	$Z_j^i(Y_j^i-y_1)$	$Z_j^i(Z_j^i-z_1)$	$(z_1Y_j^i-y_1Z_j^i)(X_j^i-x_3)$	$(z_1Y_j^i-y_1Z_j^i)(Y_j^i-y_3)$	$[z_1(X_j^i-Y_j^i\frac{x_3}{y_3})-Z_j^i(x_1-y_1\frac{x_3}{y_3})](X_j^i-x_2)$
							$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(3 x 3) matrix obtained from (1.2-7b), (1.2-7c), and (1.2-7a′) were derived from the equation $A\partial X = 0$. Therefore, any $\partial X = \begin{bmatrix} \partial X^s \\ \partial \widetilde{X} \end{bmatrix}$ fulfilling this equation will also yield $\widetilde{A}\partial\widetilde{X} = 0$. Thus whenever $\partial X \neq 0$ fulfills $A\partial X = 0$, it must also hold that $\partial\widetilde{X} \neq 0$, since $\partial\widetilde{X} = 0$ would imply that $\begin{bmatrix} \partial x_2 \\ \partial x_3 \\ \partial y_3 \end{bmatrix} = 0$ and consequently $\partial X^s = 0$ using (1.2-7a′) to (1.2-7c) for each j; this, however, would be a contradiction to $\partial X \neq 0$. Thus $\partial X \neq 0$, fulfilling $A\partial X = 0$ implies $\partial\widetilde{X} \neq 0$, fulfilling $\widetilde{A}\partial\widetilde{X} = 0$. Conversely, whenever $\partial\widetilde{X} \neq 0$ fulfils $\widetilde{A}\partial\widetilde{X} = 0$, it must also hold that $\partial X \neq 0$ fulfils $A\partial X = 0$, since $\partial\widetilde{X}$ is a subset of ∂X. Clearly, trivial solution for $\partial\widetilde{X}$ corresponds to trivial solution for ∂X (due to $\partial X^s = 0$), and trivial solution for ∂X corresponds to trivial solution for $\partial\widetilde{X}$ ($\partial\widetilde{X}$ is a subset of ∂X). It can be concluded that whenever A is singular or non-singular, its corresponding $\widetilde{A}$ is also singular or non-singular. Consequently investigations pertaining to the column space of $\widetilde{A}$ will be used rather than dealing with the column space of A. *

It is noteworthy that in Table (1.2-1) the satellites observed from stations 4, 5, . . . i (together with the stations 1, 2, 3) may or may not be the same. For that reason they were denoted as j, j′, ĵ. Neither does their number in different groups have to be the same. It can be observed that when the mentioned satellite points are the same in all the groups, it is the case when all ground stations are observing simultaneously. When none of the satellites in different groups are the same, it is the case when only four ground stations observe simultaneously (stations 1, 2, 3, and i, for instance). This occurs in practice with SECOR observations.

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* This can also be illustrated by the following argument: If $\widetilde{A}\partial\widetilde{X} = 0$ (which followed from $A\partial X = 0$) has $\partial\widetilde{X} = 0$ as the only possible solution, then all the ground stations are uniquely determined. But every satellite point was observed from stations 1, 2, 3, and did not lie in a plane with them. Thus these three stations alone (uniquely determined) would be sufficient for the unique determination of all the satellites, which would then mean that the whole cluster of points is uniquely determined.

1.3 Range Observational Mode Investigations with Observing Stations in Plane

The use of range observations when the observing ground stations lie exactly in a plane is possible whenever $\widetilde{A}$ matrix is non-singular with the z-coordinates of all ground stations equal to zero. First, it is necessary to plug for local coordinates: $z_4 = z_5 = \ldots = z_i = 0$ in the expression for $\widetilde{A}$ matrix as seen in Table (1.2-2). Since the equivalence operations do not alter the rank of a matrix, it is now possible to divide a row pertaining to jth satellite by $Z_j \neq 0$, for all j. Further, each of the last three columns will be multiplied by -1. Finally, if all the columns are identified by their headings in Table (1.2-2), these further equivalence operations will be performed:

$$\partial x_3/y_3 \rightarrow \partial x_3/y_3 - y_4\, \partial x_4 - y_5\, \partial x_5 - \ldots - y_i\, \partial x_i,$$

$$\partial y_3/y_3 \rightarrow \partial y_3/y_3 - y_3\, \partial y_4 - y_5\, \partial y_5 - \ldots - y_i\, \partial y_i,$$

$$\partial x_2/x_2 \rightarrow \partial x_2/x_2 - (x_4 - y_4 \frac{x_3}{y_3})\, \partial x_4 - (x_5 - y_5 \frac{x_3}{y_3})\, \partial x_5 - \ldots - (x_i - y_i \frac{x_3}{y_3})\, \partial x_i.$$

The resulting matrix, whose rank is the same as that of $\widetilde{A}$, is denoted as $\widetilde{\widetilde{A}}$ and its form is shown in Table (1.3-1).

If only one quad of stations is observing, e.g. quad consisting of stations 1, 2, 3, i, all in a plane, then only "station i" submatrix of $\widetilde{\widetilde{A}}$ is to be considered. It is seen that no more than four columns of this submatrix are independent. To avoid singularity two more coordinates (out of three: x_2, x_3, y_3) would have to be held fixed. If in addition all the satellite points for this quad were lying in one plane parallel to the plane of ground stations (which could be approximately fulfilled in practice when the same satellite is observed when passing above the ground stations), no more than three columns of the submatrix could be independent. The columns ∂z_i, $\partial x_3/y_3$, $\partial y_3/y_3$, $\partial x_2/x_2$ would be all constant. To avoid singularity in this case three more coordinates (out of four: z_i, x_2, x_3, y_3) would have to be fixed. An example for avoiding singularity in this case could be holding of nine coordinate fixed such that the stations 1, 2, 3 would be com-

Table (1.3-1)

Explicit expression for $\widetilde{\widetilde{A}}$ matrix

	∂x_4	∂y_4	∂z_4	∂x_5	∂y_5	∂z_5	∂x_1	∂y_1	∂z_1	$\partial x_3/y_3$	$\partial y_3/y_3$	$\partial x_2/x_2$
station 4	X_1-x_4	Y_1-y_4	Z_1							$y_4(x_4-x_3)$	$y_4(y_4-y_3)$	$(x_4-y_4\frac{x_3}{y_3})\ (x_4-x_2)$
	$\vdots$	$\vdots$	$\vdots$							$\vdots$	$\vdots$	$\vdots$
	X_j-x_4	Y_j-y_4	Z_j							$y_4(x_4-x_3)$	$y_4(y_4-y_3)$	$(x_4-y_4\frac{x_3}{y_3})\ (x_4-x_2)$
	$\vdots$	$\vdots$	$\vdots$							$\vdots$	$\vdots$	$\vdots$
station 5				$X_1'-x_5$	$Y_1'-y_5$	Z_1'				$y_5(x_5-x_3)$	$y_5(y_5-y_3)$	$(x_5-y_5\frac{x_3}{y_3})\ (x_5-x_2)$
				$\vdots$	$\vdots$	$\vdots$				$\vdots$	$\vdots$	$\vdots$
				$X_j'-x_5$	$Y_j'-y_5$	Z_j'				$y_5(x_5-x_3)$	$y_5(y_5-y_3)$	$(x_5-y_5\frac{x_3}{y_3})\ (x_5-x_2)$
				$\vdots$	$\vdots$	$\vdots$				$\vdots$	$\vdots$	$\vdots$
station i							$X_{\hat{1}}-x_1$	$Y_{\hat{1}}-y_1$	$Z_{\hat{1}}$	$y_1(x_1-x_3)$	$y_1(y_1-y_3)$	$(x_1-y_1\frac{x_3}{y_3})\ (x_1-x_2)$
							$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
							$X_{\hat{j}}-x_1$	$Y_{\hat{j}}-y_1$	$Z_{\hat{j}}$	$y_1(x_1-x_3)$	$y_1(y_1-y_3)$	$(x_1-y_1\frac{x_3}{y_3})\ (x_1-x_2)$
							$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

pletely fixed.

From the above illustrations it is immediately clear that four ground stations lying in a plane and observing only ranges to satellites can never form a fundamental network, no matter how many satellite points are used and what is their distribution. The fact that this particular case (four ground stations lying in a plane) is singular was also shown in [1], Annex A and in [10], where only six satellite points were considered.

When more than five ground stations observe ranges to satellites the system does not have to be singular even if all the stations lie in a plane as being investigated in this section. The singularity when it does occur can be conveniently divided into three categories.

1. Matrix $\tilde{\tilde{A}}$ is singular if any block of three consecutive columns except the last one is singular (i.e. its rank is less than three in this context). Since these blocks are mutually orthogonal, this is the only way for matrix $\tilde{\tilde{A}}$ minus the last three columns to be singular. This type of singularity will be called singularity A).

2. Matrix $\tilde{\tilde{A}}$ is singular if the block consisting of the last three columns is singular. This type of singularity will be called singularity B).

3. Matrix $\tilde{\tilde{A}}$ is singular if all its columns together are linearly dependent, in absence of singularity A) or B). This singularity, involving the last three columns together with the other columns of $\tilde{\tilde{A}}$ will be called global singularity or singularity C).

1.31 Singularity A)

Three column block for any station (4, 5, ... i) is singular, i.e. its rank is less than three, if the determinant of any (3 x 3) submatrix of this block is equal to zero. Let (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X, Y, Z) denote the coordinates of the first, second, and any further satellite point respectively observed by a particular ground station, and $(x, y, z = 0)$ the coordinates of this ground station. Then for singularity A) it holds

$$\begin{vmatrix} X_1 - x & Y_1 - y & Z_1 \\ X_2 - x & Y_2 - y & Z_2 \\ X - x & Y - y & Z \end{vmatrix} = 0. \qquad (1.3\text{-}1)$$

This equation represents a surface of the first order in (X, Y, Z), a plane. The plane passes through satellites 1, 2, and the ground station as it is seen by plugging the coordinates of the above mentioned points for (X, Y, Z). Obviously, any such three column block is always singular if less than three distinct satellite points are observed by any ground station. From the above derivation it follows that such a block may be singular even when more than three satellite points are present, namely, when they all lie in one plane. This can be also easily illustrated geometrically. Even if all the targets observed by certain station i and lying in one plane with it were known, this station could not be determined from them. In the linearized from, it could freely move in the direction perpendicular to the plane of the targets. An illustration of this configuration is presented in Figure 1.

In Appendix 2, Best Fitting Plane, a procedure is outlined in order to determine effectively how a set of given points is close to a plane, which could serve to detect the above singularity. The coordinates of a ground station and of respective satellite points are used to fit a plane by the least squares method to these points; subsequently, an average distance of these points from the plane is computed.

1.32 Singularity B)

1.321 General Considerations

The last three column block in $\tilde{A}$ matrix is singular, i.e. its rank is less than three, if the determinant of any (3 x 3) submatrix of this block is equal to zero; a condition, similar to that for singularity A). Here again, such two rows that are linearly independent in the three column block may be held fixed and any row other than these two rows may gradually occupy the third row's position, thus creating (3 x 3) submatrices. If the determinants

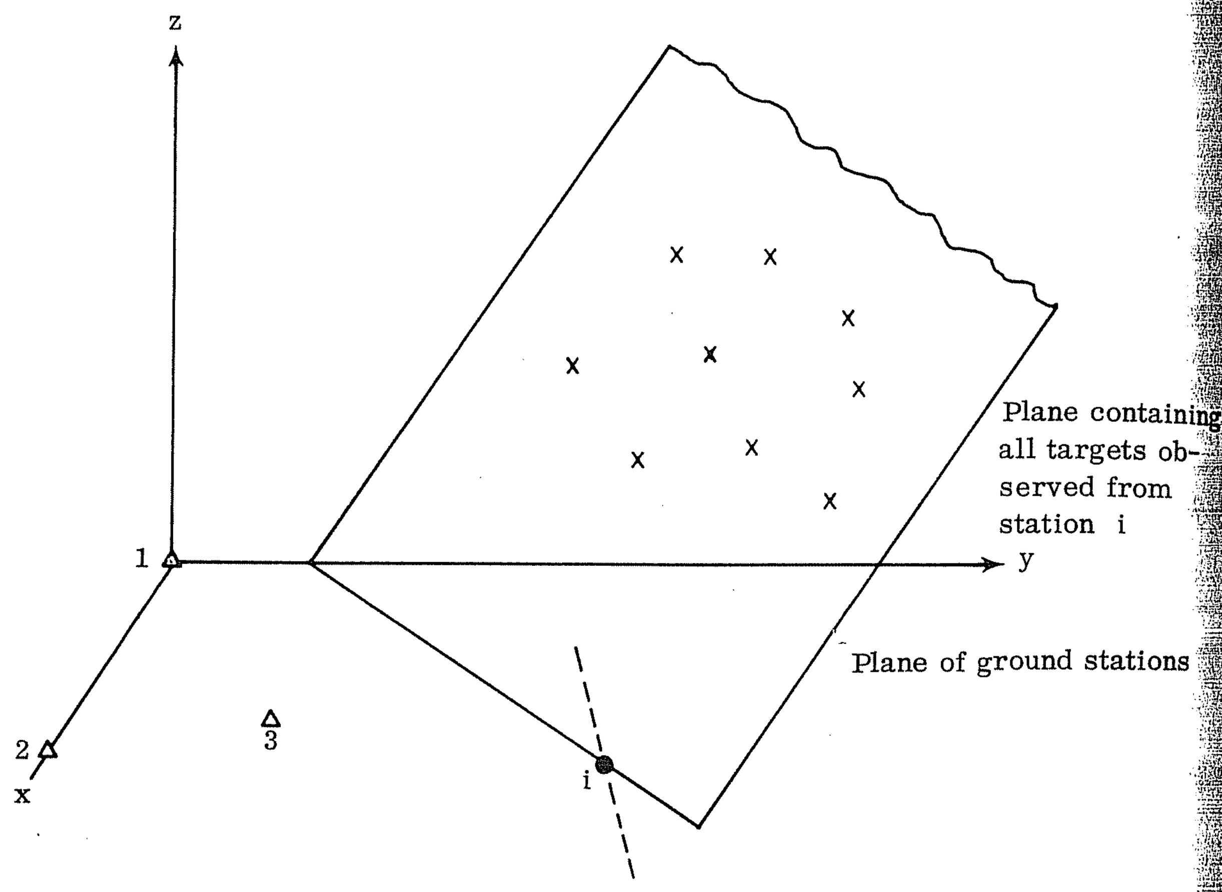


Figure 1

ILLUSTRATION OF SINGULARITY A): Station i is in the plane of its observed targets.

computed for these (3 x 3) submatrices are all zeroes, then the determinants of all (3 x 3) submatrices are zero since any row is contained in the row space of the chosen two rows and so the row space or column space of the above three column block is two (if all the rows were multiples of each other then the row or column space of the block would be one).

From Table (1.3-1) it is seen that in the last three columns all the rows pertaining to one ground station (other than 1, 2, 3) are the same. Thus the determinant of a (3 x 3) submatrix is zero whenever any two rows belonging to the same ground station are used to form such submatrix. In a network consisting of four ground stations in a plane all three rows in any such submatrix are the same and in a network consisting of five ground stations in a plane at least two of the three rows in any such submatrix are the same. Thus not only with four, but also with five ground stations lying in a plane the problem is always singular, namely singularity B) occurs.

Having more than five ground stations, all in one plane, singularity B) occurs only as a special case which will be treated now. First, one row belonging to station 4 and one row belonging to station 5 will be chosen to be the two fixed rows when forming the (3 x 3) submatrices in the last three column block, whose determinants will be examined. The third row will be gradually taken as being the row belonging to any ground station beyond 4 and 5. If the determinants of all such submatrices are zero then singularity B) is taking place, in which case it holds that

$$\begin{vmatrix} y_4\,(x_4-x_3) & y_4(y_4-y_3) & (x_4-y_4\,\dfrac{x_3}{y_3})\,(x_4-x_2) \\ y_5\,(x_5-x_3) & y_5(y_5-y_3) & (x_5-y_5\,\dfrac{x_3}{y_3})\,(x_5-x_2) \\ y\ \ (x\ -x_3) & y\ (y\ -y_3) & (x\ -y\ \ \dfrac{x_3}{y_3})\,(x\ -x_2) \end{vmatrix} = 0 \qquad (1.3\text{-}2)$$

where (x, y) are the coordinates of any ground station beyond 4 and 5, the z coordinates of all the stations being zero.

It can be seen that (1.3-2) represents an equation of a second order curve in xy plane. The curve passes through the ground stations 1, 2, 3, 4, 5, since when plugging for (x, y) any of $(0, 0)$, $(x_2, 0)$, (x_3, y_3), (x_4, y_4), (x_5, y_5), the equation (1.3-2) is satisfied (in the first three cases the third row of the determinant in (1.3-2) contains zeroes and in the last two cases this row is equal to the first and second row respectively).

In the above, the assumption was made that there existed two rows in (1.3-2) which were linearly independent, taken there as the rows corresponding to ground stations 4 and 5. If these two rows happened to be multiples of each other, different rows would have to be used, otherwise the determinant (1.3-2) would always be zero regardless of the third row and so regardless of the rank of the last three column block in $\tilde{\tilde{A}}$. This is fulfilled in most practical cases and so the following only completes the theoretical discussion. The situation with any two rows in this column block being linearly dependent can be easily analyzed. If a row in the last three column block of $\tilde{\tilde{A}}$ matrix corresponding to any station beyond station 4 is linearly dependent on the row of station 4, then it holds simultaneously that

$$\begin{vmatrix} y_4(x_4 - x_3) & y_4(y_4 - y_3) \\ y(x - x_3) & y(y - y_3) \end{vmatrix} = 0, \qquad (1.3\text{-}2a)$$

$$\begin{vmatrix} y_4(x_4 - x_3) & (x_4 - y_4 \frac{x_3}{y_3})(x_4 - x_2) \\ y(x - x_3) & (x - y \frac{x_3}{y_3})(x - x_2) \end{vmatrix} = 0, \qquad (1.3\text{-}2b)$$

and

$$\begin{vmatrix} y_4(y_4 - y_3) & (x_4 - y_4 \frac{x_3}{y_3})(x_4 - x_2) \\ y(y - y_3) & (x - y \frac{x_3}{y_3})(x - x_2) \end{vmatrix} = 0 \qquad (1.3\text{-}2c)$$

where (x, y) are now the coordinates of any such station. If station 4 lies on a straight line with stations 1 and 2, i.e., if

$$y_4 = 0,$$

then it is seen that in order to fulfill (1.3-2a) - (1.3-2c), any station beyond station 4 must also lie on the same straight line, i.e.,

$$y = 0$$

must hold (assuming that no two stations can coincide). This result stipulates that all stations with exception of station 3 would have to lie in a straight line. Of the other cases with restricted location of station 4, only

$$x_4 = x_3$$

can yield meaningful results in order to satisfy (1.3-2a) - (1.3-2c), namely

$$x = x_4 = x_3 = 0, \text{ any } y$$

and

$$x = x_4 = x_3 = x_2, \text{ any } y.$$

These results again indicate that all stations except one lie in a straight line (in a specific position). Considering the cases with station 4 in general position, i.e.,

$$y_4 \neq 0, \quad y_4 \neq y_3, \quad x_4 \neq x_3,$$

the following situation is obtained: (1.3-2a) results in a straight line in general position through station 3 and 4, while (1.3-2b) and (1.3-2c) represents each a second order curve (with non-zero coefficient for x^2), passing through stations 1, 2, 3, and 4. Both these curves necessarily intersect the above straight line at two locations, corresponding to station 3 and station 4. For (x, y) in such locations the equations (1.3-2a) - (1.3-2c) would be satisfied; however, this implies that further stations would coincide with either of stations 3, 4, which is not true. Therefore, the only further realistic configuration which would cause linear dependence of any two rows in the last three column block of $\tilde{\tilde{A}}$ matrix is such that each of (1.3-2b) and (1.3-2c) represents a straight line coinciding with the above line through stations 3 and 4. This can hold only if

$$x_4 - y_4 \frac{x_3}{y_3} = 0$$

and

$$x - y \frac{x_3}{y_3} = 0,$$

which indicates that stations 1, 3, and 4 would have to lie in a straight line which would also contain all stations beyond station 4. Station 2 would be the only one not lying on this line. Consequently, any two rows in the last three column block of $\tilde{\tilde{A}}$ matrix would be linearly dependent if and only if all the stations except one lay in a straight line. Clearly, adding one more station could not remove singularity B) in such cases, since the above block would contain at most two independent rows. Therefore, at least two additional stations would be needed. This can be easily interpreted geometrically, since by adding one more point to the configuration of a straight line and an isolated point, one could not avoid having all the points on one second order curve (in this case degenerated into two lines). Since the cases with all stations except one lying in a straight line can be immediately detected by inspection, such configurations (leading to only one independent row in the last three column block of $\tilde{\tilde{A}}$ matrix) will be always discarded. Consequently, it is assumed that station 5 exists such that it does not lie in a line with all except one of stations 1, 2, 3, and 4.

All real solution (x, y) satisfying (1.3-2) represent the critical loci leading to singularity B). General equation of second degree has the form:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \qquad (1.3\text{-}3a)$$

Denoting $$J = \begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} \qquad (1.3\text{-}3b)$$

as the determinant of the A matrix presented later in (1.3-5) - (1.3-5b), and

$$\Delta = \begin{vmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{vmatrix} \qquad (1.3\text{-}3c)$$

as the determinant of the "augmented A matrix" according to [4], p. 352, the real solutions of (1.3-3a), according to [4], p. 353, are presented in Table (1.3-2).

Table (1.3-2)
Representation of Real Solutions for Second Degree Equations in Two Variables

Δ	J	Description
$\neq 0$	> 0	ellipse
$\neq 0$	< 0	hyperbola
$\neq 0$	0	parabola
0	< 0	intersecting lines
0	0	parallel lines (distict or coincidental)

It was shown that the stations 1 through 5 determine the second degree curve representing the critical loci. This curve can be computed, drawn, and visual inspection made as to whether all the remaining stations lie on (or near) it, in which case the singularity (or near-singularity) B) would occur. This is a more practical procedure than to examine (1.3-2) for each station beyond 5 separately. In practical computations $J = 0$ or $\Delta = 0$ would never occur exactly so that a computer program can be written limited to the determination of an ellipse or hyperbola; the other cases, above all the intersecting or parallel lines can be detected beforehand by visual inspection.

The equation (1.3-2) can be expressed as

$$x^2 C + y^2 B + xy\left(A - \frac{x_3}{y_3} C\right) + x(-x_2 C) + y\left(-x_3 A - y_3 B + x_2 \frac{x_3}{y_3} C\right) = 0 \quad (1.3\text{-}4)$$

It is easily verified that $A = B = C = 0$ would occur if and only if the first two rows of the determinant in (1.3-2) were linearly dependent, the case which was treated separately and discarded. The whole equation can be divided by C, which is non-zero in general, since of the stations 1, 2, 3, any one can be chosen as the origin of the local coordinate system and any one as determining the direction of the x-axis. The coefficients A, B, C in the above equation are obtained from (1.3-2) as

$$A = y_4(y_4-y_3)(x_5-y_5\frac{x_3}{y_3})(x_5-x_2) - y_5(y_5-y_3)(x_4-y_4\frac{x_3}{y_3})(x_4-x_2), \quad (1.3\text{-}4a)$$

$$B = y_5(x_5-x_3)(x_4-y_4\frac{x_3}{y_3})(x_4-x_2) - y_4(x_4-x_3)(x_5-y_5\frac{x_3}{y_3})(x_5-x_2), \quad (1.3\text{-}4b)$$

$$C = y_4 y_5[(x_4-x_3)(y_5-y_3) - (y_4-y_3)(x_5-x_3)]. \quad (1.3\text{-}4c)$$

The second order equation (1.3-4) in local coordinates (x, y) can be expressed in matrix form as

$$x^T A x + x^T a = 0 \quad (1.3\text{-}5)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad (1.3\text{-}5a)$$

with

$$a_{11} = 1,$$

$$a_{12} = a_{21} = \frac{1}{2}\left(\frac{A}{C} - \frac{x_3}{y_3}\right), \quad (1.3\text{-}5b)$$

$$a_{22} = \frac{B}{C},$$

and

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tag{1.3-5c}$$

with

$$a_1 = -x_2,$$

$$a_2 = x_2 \frac{x_3}{y_3} - x_3 \frac{A}{C} - y_3 \frac{B}{C}, \tag{1.3-5d}$$

while

$$x = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{1.3-5e}$$

Since five distinct points are sufficient to determine the equation of second order curve, A matrix and a vector could be also determined by fitting second order curve to stations 1 through 5. This is done in Appendix 3 and the same expressions as in (1.3-5b) and (1.3-5d) are obtained for A, a. An illustration of singularity B) is presented in Figure 2.

1.322 Computation of Critical Curve.

Practical computations pertaining to the critical curve are made in four steps:

(1) Transformation of the coordinates of all stations and satellites from the basic coordinate system to which all the points refer into the local coordinate system.

(2) Computation of the curve in canonical form.

(3) Transformation of all the points of interest from the canonical to the local coordinate system.

(4) Transformation of these points from the local to the basic coordinate system. With their aid the critical curve may be easily drawn and conclusions made as to the position of the ground stations with respect to it.

For the transformation of coordinates set forth in (1), the following notations will be introduced:

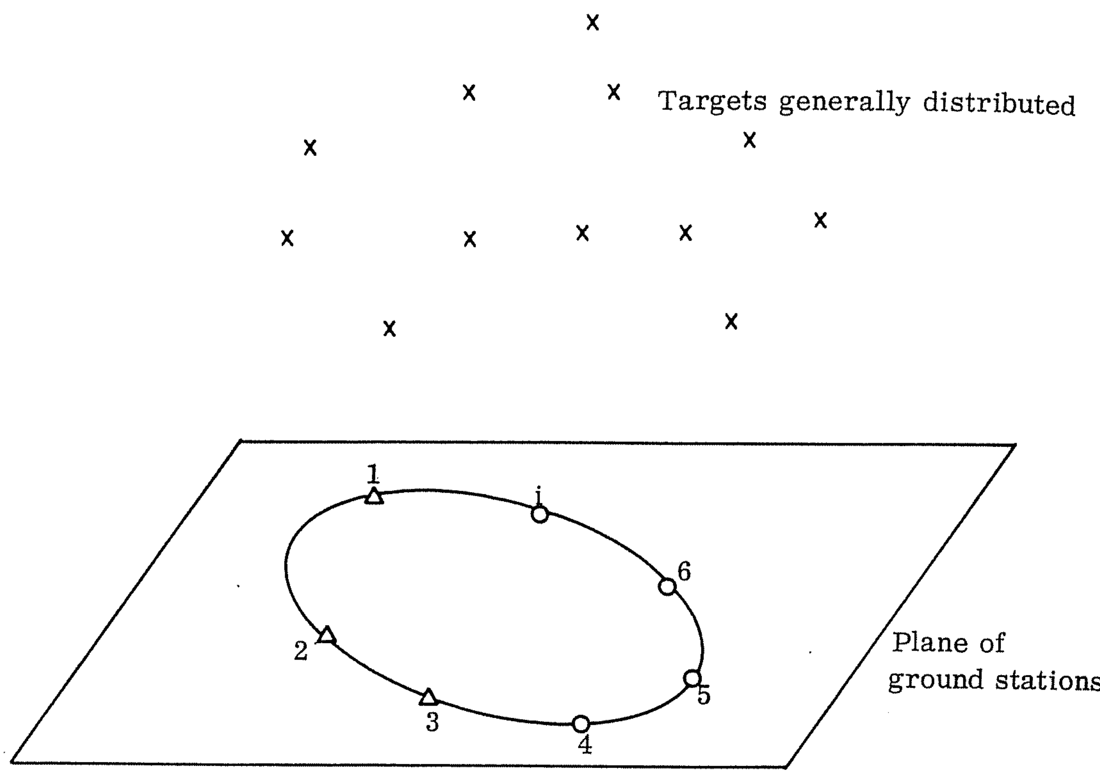


Figure 2

ILLUSTRATION OF SINGULARITY B): Stations 1, 2, 3 observe all targets; all stations are on a second order curve.

$$X = \begin{bmatrix} X \\ Y \end{bmatrix}$$ . . . coordinates of a point in the basic coordinate system; thus in particular

$$X_0 = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$ . . . coordinates of the origin of the local coordinate system,

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$ . . . coordinates of a point in the local coordinate system.

According to (A4-5a) through (A4-5c) in Appendix 4,

$$x = P^T (X - X_0), \tag{1.3-6}$$

where x was identified with X' and P with R, and where

$$P = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \tag{1.3-6a}$$

α being the angle between the x and X axes, measured counterclockwise under the assumption that both coordinates systems are right handed. The origin of the local coordinate system is assumed to coincide with station 1 and its x-axis to pass through the station 2, giving

$$\cos\alpha = (X_2 - X_1) / S_{12}$$

and

$$\sin\alpha = (Y_1 - Y_2) / S_{12},$$

where

$$S_{12} = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}.$$

The matrix equation (1.3-6) can also be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X - X_1 \\ Y - Y_1 \end{bmatrix},$$

determining local coordinates of any point given in the basic coordinate system (here capital letters denote stations' coordinates).

Computation of the critical curve in canonical form, such as stipulated in (2), comprises computation of the parameters of this curve, its explicit equation in the local coordinate system being given by (1.3-5), namely,

$$x^T A x + x^T a = 0.$$

This form coincides with (A4-30) of Appendix 4 with $c = 0$, due to the fact that the critical curve passes through the origin of the local coordinate system (station 1). The procedure to compute the size, shape, center, and orientation of this curve was outlined in section A4.4 and also used in practical computations. (The local coordinate system is called in section A4.4 "original coordinate system".) In particular, the semi-axes a_1, a_2, and the kind of second order curve, found there, determine the size and shape of the critical curve, while x_0 and R determine the center and orientation of the critical curve with respect to the local coordinate system. Assumptions made in section A4.4 excluded special cases when singularity B) is caused by all stations lying on two straight lines (intersecting or parallel); these assumptions read as $J \neq 0$ and $\Delta \neq 0$, with J and Δ defined in (1.3-3b) and (1.3-3c). Some of the special cases were illustrated separately in Appendix 5.

The values of x_0 and R are then used to transform any point on the critical curve from the canonical to the local coordinate system, as required by (3). In particular, the points of interest are the center of the curve and four "main curve points", identified with the end points of the curve's axes (major and minor axes for ellipse and transverse and conjugate axes for hyperbola). In section A4.4, these points in canonical coordinates are presented in (A4-32). Their transformation into the local coordinate system is made according to (A4-8) as

$$x = x_0 + Rx'$$

where x′ refers to a coordinate vector in the canonical coordinate system.

Transformation of these (and any other) points from the local to the basic coordinate system set forth in (4) is then carried out according to (A4-5a) or (1.3-6), as

$$X = X_o + Px.$$

1.33 Singularity C)

When dealing with singularity C) in a range adjustment problem, also called global singularity, all of the columns in $\tilde{\tilde{A}}$ matrix (presented in Table (1.3-1), section 1.3) are taken into consideration. In the global analysis of the causes leading to singularity (in this context rank deficiency) of $\tilde{\tilde{A}}$ matrix, it will have to be assumed that no three column block is singular. Failure to fulfill this condition can be divided into two groups: first, in which anyone of the three column blocks except the last one is singular, in section 1.31 called singularity A), and second, in which the last three column block is singular, in section 1.32 called singularity B). Naturally, these two groups have to be treated separately in order to make the analysis of the global singularity complete. It was done in sections 1.31 and 1.32, although the above reason for such separate treatments was not given there. Elimination of singularity A) and singularity B) are necessary conditions for $\tilde{\tilde{A}}$ matrix to be non-singular. Further necessary conditions are presented in section 1.331.

1.331 Necessary Conditions to Avoid Singularity C).

As seen from Table (1.3-1), among all the rows pertaining to observations from one ground station in matrix A, at most four rows can be linearly independent. First, the condition guaranteeing that such a row block has indeed rank four will be formulated. Obviously, only one non-zero column of the last three column block (or one non-zero combination of the three columns) is to be considered here, since two of these columns can always be brought to zero by equivalence operations. Consequently, the rank of the following matrix M_k, corresponding to the kth row block will be considered:

$$M_k = \begin{bmatrix} X_1 - x_k & Y_1 - y_k & Z_1 & c \\ \vdots & \vdots & \vdots & \vdots \\ X_j - x_k & Y_j - x_k & Z_j & c \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{1.3-7}$$

where c is a constant and

$$c \neq 0. \tag{1.3-7a}$$

Using row and column equivalence operations it is seen that

$$M_k \sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & c \\ \hline X_2 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ X_j - X_1 & Y_j - Y_1 & Z_j - Z_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{array}\right] = \left[\begin{array}{c|c} 0 & c \\ \hline S_k & 0 \end{array}\right].$$

It holds that rank $M_k = 4$ if and only if rank $S_k = 3$. The first two rows in S_k are assumed independent (they would be dependent only if the satellite points 1, 2, 3 were lying on a straight line; in this case another row could replace row two; if no such row existed then all the satellite points observed by station k would lie on a straight line, rank S_k would be one and rank M_k two; but then singularity A) would occur contrary to the necessary assumptions of singularity A) and singularity B) eliminated). Let X, Y, Z denote coordinates of any satellite point beyond three observed from the k th ground station. Should the rank S_k be less than three it would have to hold for each such satellite point:

$$\begin{vmatrix} X_2 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 \\ X_3 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 \\ X - X_1 & Y - Y_1 & Z - Z_1 \end{vmatrix} = 0. \qquad (1.3\text{-}8)$$

This first degree equation in X, Y, Z represents a plane passing through points X_i, Y_i, Z_i, i = 1, 2, 3, i.e. passing through satellite points 1, 2, 3 (if any of these X_i, Y_i, Z_i are plugged for X, Y, Z, the equation (1.3-7) is satisfied). Consequently, the rank of S_k is less than three and the rank of M_k is less than four if all the satellite points observed from the kth ground station lie in one plane.

Each observing station beyond stations 1, 2, 3 contributes with three columns to matrix $\tilde{\tilde{A}}$. These three columns are assumed to be independent (due to singularity A) eliminated). Thus each row block has rank at least three and at most four. Since $\tilde{\tilde{A}}$ matrix contains three more columns (the last three columns) corresponding to stations 1, 2, 3 (actually only 2 and 3), there must be at least three such row blocks of rank four in order that $\tilde{\tilde{A}}$ be not necessarily singular. Otherwise ranks of individual row blocks added together would not even reach the number of columns in $\tilde{\tilde{A}}$. Thus, a further necessary condition for $\tilde{\tilde{A}}$ to be non-singular is that at least three row blocks have rank four. Defining a set of points which are not all lying in one plane as points "off-plane", the above conclusion may be restated as follows: in addition to the assumptions of singularity A) and singularity B) eliminated a further necessary condition for $\tilde{\tilde{A}}$ to be non-singular stipulates that at least three ground stations in addition to stations 1, 2, 3 must observe off-plane targets.

1.332 Necessary and Sufficient Conditions to Avoid Singularity C).

In section 1.331, the necessary conditions for avoiding singularity C) were presented. It will be shown that with some further specifications these are also the necessary and sufficient conditions for non-singular $\tilde{\tilde{A}}$ matrix.

Let the following notations be introduced pertaining to $\tilde{\tilde{A}}$ matrix of Table

(1.3-1): the columns with headings ∂x_k, ∂y_k, ∂z_k will be denoted respectively as (column) vectors v_k^x, v_k^y, v_k^z, $k = 4, 5, \ldots, g$. The letter s will refer to the number of ground stations beyond station 3, while g will stand for the total number of ground stations, namely,

$$g = s + 3.$$

The last three (column) vectors will be denoted as v^1, v^2, v^3, respectively. The condition for singularity of $\tilde{\tilde{A}}$ can be expressed as follows: $\tilde{\tilde{A}}$ is singular if there exists a set of coefficients, divided into a group containing coefficients a and a group containing coefficients b, such that this set is not necessarily a zero set and that the relation

$$a_4^x v_4^x + a_4^y v_4^y + a_4^z v_4^z + \ldots + a_g^x v_g^x + a_g^y v_g^y + a_g^z v_g^z + b_1 v^1 + b_2 v^2 + b_3 v^3 = 0 \tag{1.3-9}$$

holds, i.e., that (1.3-9) is consistent for this set of coefficients. Obviously, the above system of homogeneous equations can always be made consistent, namely, when all these coefficients are equal to zero. Thus $\tilde{\tilde{A}}$ is singular if (1.3-9) can be made consistent with some non-zero coefficients a or b. Otherwise $\tilde{\tilde{A}}$ is non-singular. It is seen from Table (1.3-1) that $\tilde{\tilde{A}}$ can be divided into s row blocks, associated with observations from station k, $k = 4, 5, \ldots, g$. With exception of the last three columns, the non-zero columns in the k th row block will be denoted as $\tilde{v}_k^x$, $\tilde{v}_k^y$, and $\tilde{v}_k^z$. Clearly, these (column) vectors represent the only non-zero elements in v_k^x, v_k^y, and v_k^z (column) vectors, respectively. Furthermore, the last three columns in this row block can be denoted as

$$f_k^1 \times \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad f_k^2 \times \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \text{and} \quad f_k^3 \times \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

where f_k^1, f_k^2, f_k^3 are terms appearing in each row of the last three columns of $\tilde{\tilde{A}}$ for this row block. The system of equations (1.3-9) is thus composed of smaller systems, corresponding to the above row blocks, which all have only the b-coefficients in common; (1.3-9) then corresponds to the totality of the systems such as

$$a_k^x \widetilde{v}_k^x + a_k^y \widetilde{v}_k^y + a_k^z \widetilde{v}_k^z + b_1 f_k^1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + b_2 f_k^2 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + b_3 f_k^3 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0,$$

$$k = 4, 5, \ldots, g,$$

or

$$a_k^x \widetilde{v}_k^x + a_k^y \widetilde{v}_k^y + a_k^z \widetilde{v}_k^z + c_k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0, \tag{1.3-10a}$$

$$k = 4, 5, \ldots, g \tag{1.3-10b}$$

where

$$b_1 f_k^1 + b_2 f_k^2 + b_3 f_k^3 = c_k. \tag{1.3-10c}$$

Consequently, the system of homogenous equations (1.3-9) can be written as composed of the systems (1.3-10a), with k and c_k such as in (1.3-10b) and (1.3-10c), respectively. It has the following form:

$$\begin{array}{lllll}
a_4^x \widetilde{v}_4^x + a_4^y \widetilde{v}_4^y + a_4^z \widetilde{v}_4^z & & & + c_4 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0 \\
& a_5^x \widetilde{v}_5^x + a_5^y \widetilde{v}_5^y + a_5^z \widetilde{v}_5^z & & + c_5 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0 \\
& & \vdots & \vdots & \vdots \\
& & a_g^x \widetilde{v}_g^x + a_g^y \widetilde{v}_g^y + a_g^z \widetilde{v}_g^z & + c_g \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0
\end{array} \tag{1.3-11}$$

where

$$\begin{aligned}
b_1 f_4^1 + b_2 f_4^2 + b_3 f_4^3 &= c_4 \\
b_1 f_5^1 + b_2 f_5^2 + b_3 f_5^3 &= c_5 \\
\vdots \qquad & \quad \vdots \\
b_1 f_g^1 + b_2 f_g^2 + b_3 f_g^3 &= c_g.
\end{aligned} \tag{1.3-12}$$

Necessarily, the system of homogeneous equations (1.3-9) is consistent for some set of coefficients a, b, if and only if the systems (1.3-11) and (1.3-12) are consistent for the same set of coefficients. If for any k in (1.3-10a) or (1.3-11) $\tilde{v}_k^x$, $\tilde{v}_k^y$, and $\tilde{v}_k^z$ were not linearly independent, then there would exist corresponding a-coefficients not all three equal to zero such that

$$a_k^x \tilde{v}_k^x + a_k^y \tilde{v}_k^y + a_k^z \tilde{v}_k^z = 0 \tag{1.3-13}$$

would hold. Accordingly, with all the remaining a-coefficients and all three b-coefficients equal to zero, or correspondingly with all the terms c equal to zero, the systems (1.3-11) and (1.3-12) would be consistent for such a non-zero set of coefficients and $\tilde{\tilde{A}}$ would be singular. However, these special conditions are assumed to be non-existent due to the necessary conditions regarding singularity A). It then follows from (1.3-10a) and (1.3-10b) that

$$a_k^x = a_k^y = a_k^z = 0 \text{ if and only if } c_k = 0 \tag{1.3-14}$$

in order that (1.3-11) be consistent. Thus, $c_k = 0$ for any k will guarantee that all three corresponding a-coefficients are equal to zero. However, even if this were the only possibility to make the system (1.3-11) consistent, $\tilde{\tilde{A}}$ would not be necessarily non-singular; namely, if the last three columns of $\tilde{\tilde{A}}$ were linearly dependent, then b_1, b_2, b_3 not all zero would exist such that the system (1.3-12) with the terms

$$c_4 = c_5 = \ldots = c_g = 0 \tag{1.3-15}$$

would be consistent. Then (1.3-9) would be fulfilled with not all the coefficients equal to zero (namely the coefficients b would be different from zero) and $\tilde{\tilde{A}}$ would be singular. However, due to the necessary conditions with respect to singularity B) it holds for all k's that

$$b_1 = b_2 = b_3 = 0 \text{ if and only if } \begin{bmatrix} \vdots \\ c_k \\ \vdots \end{bmatrix} = 0 \tag{1.3-16}$$

in order that (1. 3-12) be consistent. With the causes leading to singularity A) and singularity B) eliminated as the necessary conditions for non-singular $\widetilde{\widetilde{A}}$ matrix, the following definition can be formulated using (1. 3-14) and (1. 3-16): $\widetilde{\widetilde{A}}$ matrix is non-singular if and only if the only c-terms making both (1. 3-11) and (1. 3-12) consistent are those of (1. 3-15). Otherwise $\widetilde{\widetilde{A}}$ is singular.

As a natural consequence of the above definition, the c-terms which make (1. 3-11) and (1. 3-12) consistent will be analyzed. For any subset of (1. 3-11), such as (1. 3-10a) associated with the observations from station k, it holds that whenever

$$\text{rank}\ [\widetilde{v}_k^x,\ \widetilde{v}_k^y,\ \widetilde{v}_k^z,\ \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}] = 4, \qquad (1.3\text{-}17a)$$

then only the trivial solution, i. e.

$$a_k^x = a_k^y = a_k^z = \alpha_k = 0, \qquad (1.3\text{-}17b)$$

is possible. Otherwise an infinite number of solutions exists, including the one of (1. 3-17b). The expression (1. 3-17a) is true whenever

$$\text{rank}\ M_k = 4 \qquad (1.3\text{-}17c)$$

where M_k is given by (1. 3-7), since the matrix of (1. 3-17a) is exactly M_k with

$$c = 1 \neq 0,$$

in accordance with (1. 3-7a). The relation (1. 3-17c) holds if and only if the corresponding station k observed off-plane targets. Next, the system (1. 3-12) will be written in matrix form as

$$F\,B = C,$$

where

$$F = \begin{bmatrix} f_4^1 & f_4^2 & f_4^3 \\ \vdots & \vdots & \vdots \\ f_g^1 & f_g^2 & f_g^3 \end{bmatrix},$$

$$B = [b_1\ b_2\ b_3]^T,$$

and

$$C = [c_4 \ldots c_g]^T.$$

Should (1.3-12) represent a consistent system of equations, it must hold that

$$\text{rank}\,[F\ C] = \text{rank}\,F = 3, \tag{1.3-18}$$

since singularity B) was discarded. Thus the rows of $[F\ C]$ span a three-dimensional subspace W of V, where V is the space of all 4-vectors. Due to (1.3-18), three independent rows of F may be found; the same three rows of $[F\ C]$ span W. Finding three independent rows of F is equivalent to finding three stations beyond station 3 which do not all lie on a second order curve together with stations 1, 2, 3. This may be seen from (1.3-2) where the three rows inside the determinant, equivalent to the above three rows of F, are independent only if the corresponding stations do not lie on a second order curve with stations 1, 2, 3. Three or more stations with this property will be said to be off-curve, or equivalently, it will be stated that singularity B) was removed for these particular stations. Otherwise the stations will be said to be on-curve. Since the above three rows of $[F\ C]$ span W, then all the elements of C, or all the c-terms in (1.3-12), will be necessarily zero if the c-terms in all these three rows are zero (all c-terms are linear combinations of the above three c-terms corresponding to three off-curve stations). But these three c-terms will have to be zero with no other solution possible only if (1.3-17a) holds for the corresponding stations, which occurs only when these stations observe the satellite points which are off-plane, as it can be seen from the pertinent conclusion in section 1.331. Under such conditions, leading to all c-terms being zero, the relations (1.3-14) and (1.3-16) imply that all a-terms and b-terms must be zero should (1.3-11) and (1.3-12) be consistent. Consequently, these conditions stating that at least three off-curve stations observe off-plane targets imply that $\widetilde{\tilde{A}}$ is non-singular, provided singularity A) is discarded, and with this last statement represent the necessary and sufficient conditions for non-singular $\widetilde{\tilde{A}}$ matrix.

The requirement that the above three or more stations be off-curve necessarily implies that singularity B) cannot exist. As a matter of fact, it represents a stronger statement, which was conveniently worded as eliminating singularity B) for these particular stations. On the other hand, if such three or more stations did not lie off-curve, or if only two stations (they can never lie off-curve) observed off-plane targets, then the corresponding rows of [FC] would not span W. This means that when only on-curve stations observed satellite points which were off-plane, not all the c-terms would have to be zero to make (1.3-12) consistent. Only those c-terms would have to be zero, whose rows would be linear combinations of the above rows, i.e., all the c-terms corresponding to on-curve stations. For instance, suppose that the first two rows of (1.3-12) are independent with $c_4 = c_5 = 0$ as the only possibility and suppose that no further c-term has this property. Choosing a third independent row, corresponding now to an off-curve station (which did not observe off-plane targets) and choosing its c-terms different from zero, a unique non-trivial solution for b_1, b_2, and b_3 can be obtained. This will yield uniquely the other c-terms from (1.3-12); it is clear that not all c-terms are zero, while the system (1.3-12) is consistent. The non-zero c-terms are exactly those associated with stations off-curve. The fact that singularity B) for all the stations was discarded did not help here, since the stations observing off-plane satellites were not themselves off-curve stations. Furthermore, when the c-terms, different from zero (and necessarily corresponding to the stations which did not make off-plane observations) are used in (1.3-11), a unique non-zero solution can be found for the a-coefficients in a subsystem of (1.3-11), such as (1.3-10a), corresponding to any of these c-terms. This is true because for such a subsystem the relation (1.3-17a) does not hold: the rank of a matrix such as presented in (1.3-17a) with the last column present cannot be four since the corresponding station did not make off-plane observations. On the other hand the rank of the same matrix without the last column is three due to the fact that singularity A) was discarded. Consequently, the rank of this matrix is three, the same as the rank of this matrix augmented by a non-zero constant column $\begin{bmatrix} C_k \\ \vdots \\ C_k \end{bmatrix}$, which guarantees a unique non-zero solution for the three a-coefficients of the subsystem.

In conclusion, necessary and sufficient conditions for $\widetilde{\widetilde{A}}$ matrix to be non-singular will be restated: $\widetilde{\widetilde{A}}$ matrix is non-singular if singularity A) does not occur and if there exist off-curve stations (necessarily at least three) making off-plane observations. Otherwise $\widetilde{\widetilde{A}}$ is singular; in absence of singularity A) and singularity B) the singularity of $\widetilde{\widetilde{A}}$ was defined as being singularity C). It occurs when the stations making off-plane observations are not themselves off-curve stations. Singularity C) is illustrated in Figure 3.

1.333 Illustration that Discarding of Singularity A) and Singularity C) Yields Unique Solution in Adjustment.

First of all, singularity B) is discarded whenever singularity C) is eliminated as pointed out in section 1.332, since this implies that there exist some stations off-curve and, therefore, all the stations are necessarily off-curve. Further, the number of observations will be shown to be at least as large as the number of unknowns in an adjustment. Stations 1, 2, 3 are always assumed to observe all the satellite points. Due to the removal of singularity C), there are at least three more stations observing at least four satellite points each (less than four satellite points could always form a plane). Due to the removal of singularity A), all the remaining stations observe at least three satellite points each (less than three satellite points could always lie in a plane with the observing station).

Suppose there are g ground stations, of which three observe all the satellite points and three observe at least four satellite points each, thus making at least twelve observations. Further, suppose there are s satellite points. The

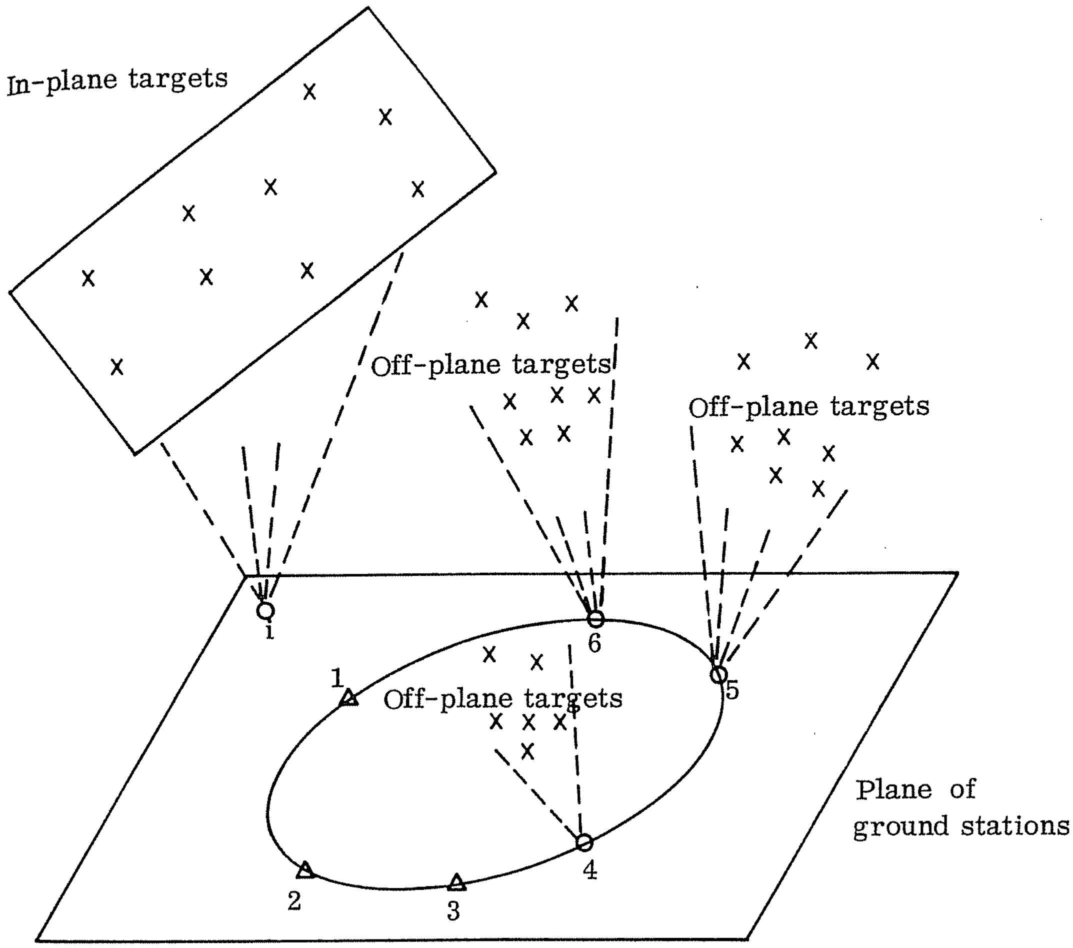


Figure 3

ILLUSTRATION OF SINGULARITY C): Stations 1, 2, 3 observe all targets; all stations observing off-plane targets are on a second order curve with stations 1, 2, 3.

number of unknowns in the whole system (three per stations or satellite point, minus six unknowns representing constraints or removed parameters) is equal to

$$3g + 3s - 6.$$

The number of observations in the whole system is composed of the following groups:

	$3s$	... due to stations 1, 2, 3 observing all satellite points
at least	12	... due to three further stations observing at least four satellite points each

and

at least	$(g-6)3$	... due to the rest of stations observing at least three satellite points each.

The total of all the observations is then at least

$$3g + 3s - 6,$$

which is as many as there are unknown parameters; this proves the asserted statement.

Whenever the words "at least" do not apply, the system has exactly the same number of unknowns as there are observations and a unique solution is possible without an adjustment.

It is clear that if singularity A) and singularity C) (consequently also singularity B)) are discarded for a network of six ground stations, the network will be non-singular no matter how many ground stations and corresponding satellites are added to it, provided that singularity A) is eliminated also for each of those new stations. This follows from the fact that all the c-terms are zeroes (i. e., also those corresponding to the new stations since all the b-terms were zeroes) and that the a-terms for the new stations must then also be zeroes due to (1. 3-14). It can be easily visualized in the following way: in a well-determined network of six ground stations, any number of satellites not lying in the plane of the ground stations can be determined using observations from any three stations (here 1, 2, 3). Any new station, co-observing these satellites, can be determined from them, provided it does not lie in one plane with them. But this condition is exactly that of singularity A) eliminated (for any such new station), which is therefore a necessary and sufficient condition for expanding of non-singular range networks beyond the non-singular networks of six ground stations.

1.34 Critical Configurations if All Ground Stations Co-observe.

When all stations observe all targets, any three stations can be considered to be stations 1, 2, and 3, used in previous derivations. For this reason singularity A) loses its original meaning: if all the targets lie in a plane through a certain ground station, then such a station can be taken for instance as station 1; with such numbering of stations singularity A) does not occur. This can be seen from section 1.31 where it was shown that singularity A) occurs if all satellite points observed by a particular ground station beyond stations 1, 2, 3, lie in a plane containing that ground station; if it contains any of stations 1, 2, 3, instead, singularity A) does not occur. Nevertheless, the above configuration results in a singular network, since it is a special case of singularity C) described below.

Singularity A) could occur in one case only, namely if all targets lay in a straight line. Planes through such targets would contain any ground station. This, however, is also a special case of singularity C) described below.

When all stations co-observe, singularity C) could occur only in two instances as follows:

(a) If the targets are not all lying in one plane, all stations would have to be on one second order curve in order that singularity C) occur. This cooresponds to singularity B) of section 1.32.

(b) If the stations are not all lying on one second order curve, all targets would have to be in one plane for singularity C) to occur. The distribution of targets in this plane is irrelevant. As a special case, such a plane would pass through a certain station. When all targets lie in a straight line, which is another special case, this type of singularity always occurs.

Illustrations of singularity C) for parts (a) and (b) appear in Figures 4 and 5, respectively.

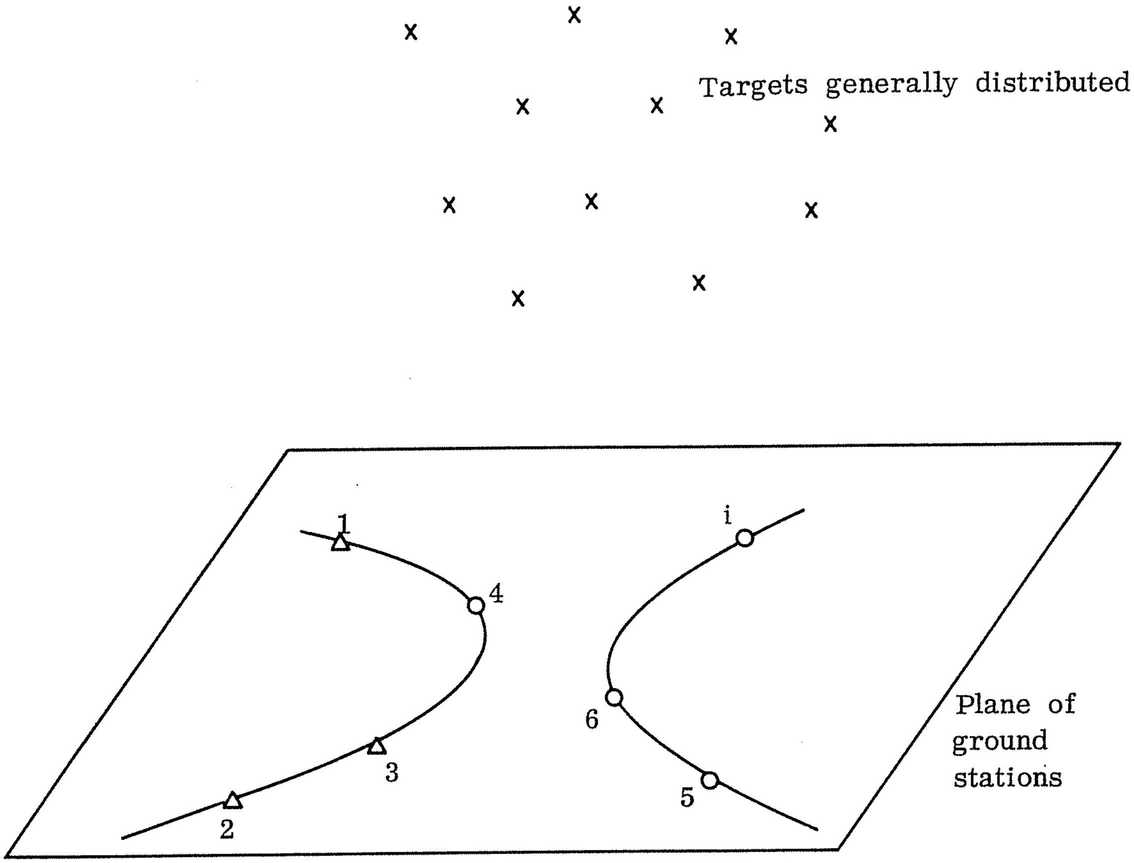


Figure 4

ILLUSTRATION OF SINGULARITY C): All stations observe all targets; all stations are on a second order curve.

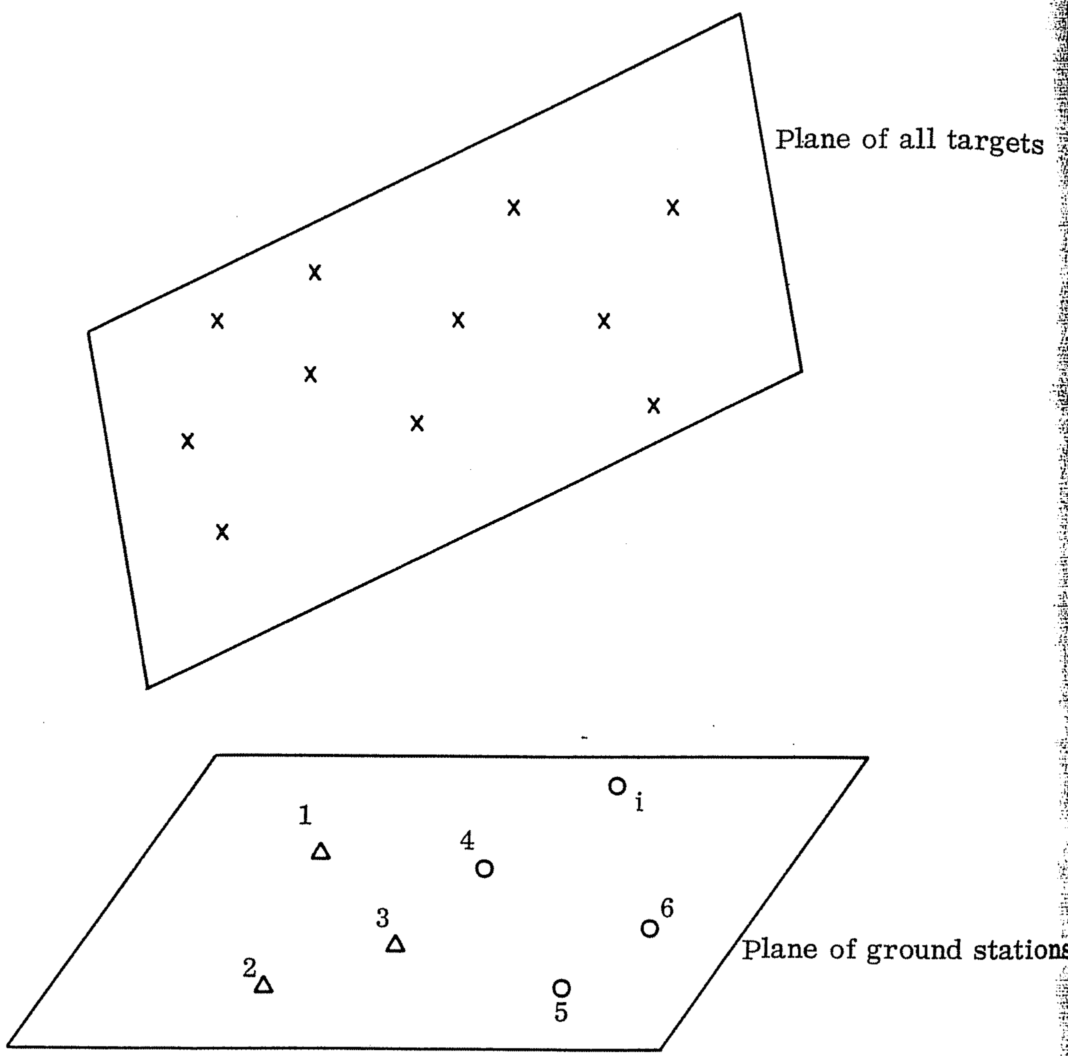


Figure 5

ILLUSTRATION OF SINGULARITY C): All stations observe all targets; all targets are in a plane.

1.4 Principle of Replacing of Stations

In the last sections at least three ground stations were treated as observing distances to all the satellite points. All the derivations presented there are naturally valid when all the ground stations in a network observe all the targets (in this sense "observe" means strictly "observe distances"). This would be an ideal way of observing range networks, based on simultaneous or quasi-simultaneous observations from all or most of the ground stations. The latter mode of quasi-simultaneous ranges would involve precise timing and such an interpolation procedure as to yield simultaneous observations for the range adjustment. When such an observational mode realizes, the analysis treating three ground stations as observing all the targets will then be sufficient and complete. However, the data presently available is not of this nature. A great number of range observations have been made using the SECOR observational mode, when only four ground stations are observing simultaneously. Even in this case it would be possible to have three stations observing all the targets, while the fourth station would be moving. In practice, however, even if networks extend to relatively small areas, all the stations are gradually displaced and occupy new positions in a fashion called "leapfrogging". If this is the case, no three stations observe all the satellite points in general. It is then of interest to analyse the critical configurations for this new procedure in a way similar to that used in the previous sections.

To procede with such an analysis as clearly as possible certain notations for groups of satellite points will be introduced. The ground stations (considered again as lying all in one plane) will be grouped by four, which corresponds to "quads" arising when the SECOR observations are used. Any satellite group will be represented by the letter j subscripted by the number (or letter) of a ground station which does not appear in any other quad or which observes this group for the first time. Thus the quads consisting of ground stations 1, 2, 3, 4, then 1, 2, 3, k, and 1, 2, k, s are said to observe satellite groups j_4, j_k, and j_s respectively.

Each station of one quad is assumed to observe each satellite in the corresponding group. Theoretically, some or all of the satellite groups could coincide, i.e., contain the same targets, which would happen if more than four stations could make simultaneous observations. This would not affect the derivations at all, since the coordinates of all the targets are eliminated and expressed in terms of the observing stations. In the above illustration, station k effectively replaced station 3 in the observations of j_s, which were thus made from stations 1, 2, k, and s. This is the reason why such a procedure is referred to as "replacing of stations". Station k will be considered throughout as performing the first replacement. The stations following station k will be all denoted as "s-stations" even if their number is more than one, in which case they will be distinguished by primes: s', s'', etc. The corresponding satellite groups will be then j_s', j_s'', etc.

Replacing of stations will be carried out on three levels, according to the number of replacements. One replacement will be analyzed in section 1.41, which will be the most detailed of the sections dealing with replacements; two replacements will be treated in section 1.42, and more replacements in section 1.43.

When dealing with one replacement, the observing stations (quads) and the corresponding satellite groups can be conveniently arranged in the following way:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & j_4 \\ & & \vdots & & \\ 1 & 2 & 3 & k & j_k \\ 1 & 2 & k & s' & j_s' \\ 1 & 2 & k & s'' & j_s'' \\ & & \vdots & & \end{array}$$

where the dots express the possibility of more quads present with the first three stations the same as in the preceding quad. Thus, one or more quads "i" could be introduced between the quads of station 4 and station k, namely:

$$\begin{array}{ccccc} 1 & 2 & 3 & i & j_i. \end{array}$$

When two replacements are taking place, then one s-station replaces station 2 in addition to station 3 having been replaced by station k. This s-station will be denoted as s'. A similar pattern will arise in this case:

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & j_4 \\
 & & \vdots & & \\
1 & 2 & 3 & k & j_k \\
1 & 2 & k & s' & j_{s'} \\
 & & \vdots & & \\
1 & k & s' & s'' & j_{s''} \\
 & & \vdots & &
\end{array}
$$

where the dots have the same interpretation as in the previous part. Further replacements can be carried out in an analogous manner.

1.41 <u>One Replacement: Station 3 Replaced by Station k.</u>

Up to and including station k, the elimination of the parameters ∂X, ∂Y, ∂Z, associated with the satellite points can be done in the same manner as presented in section 1.2. With the same definition of the coordinate system, the parameters for each target in j_4 through j_k can be eliminated using observations from stations 1, 2, 3, which lead to the equations (1.2-6a) - (1.2-6c) in section 1.2. These three equations after multiplying each of them by s_{1j} (distance ground-satellite) can be expressed in a matrix form as

$$
\begin{bmatrix} X_j & Y_j & Z_j \\ X_j - x_2 & Y_j & Z_j \\ X_j - x_3 & Y_j - y_3 & Z_j \end{bmatrix} \begin{bmatrix} \partial X_j \\ \partial Y_j \\ \partial Z_j \end{bmatrix} = \begin{bmatrix} 0 \\ (X_j - x_2)\,\partial x_2 \\ (X_j - x_3)\,\partial x_3 + (Y_j - y_3)\,\partial y_3 \end{bmatrix} \quad (1.4\text{-}1)
$$

where j stands for any satellite point of j_4 through j_k. The solution using the matrices is equivalent to that presented in section 1.2 and gives ∂X_j, ∂Y_j, and ∂Z_j such as found in (1.2-7b), (1.2-7c), and (1.2-7a'). The determinant of the (3 x 3) matrix in (1.4-1) can be expressed as $D = Z_j x_2 y_3$, giving raise to the conditions

$$x_2 \neq 0, \quad y_3 \neq 0, \quad Z_j \neq 0, \tag{1.4-2}$$

already discussed in section 1.2. When the results for ∂X_j, ∂Y_j, ∂Z_j are substituted for in (1.2-4a) associated with the observations from "i-stations" (i can be any station between 4 and k inclusively), then the equation (1.2-9) is obtained. Equations of this type were used to form $\widetilde{A}$ matrix, presented in Table (1.2-2).

Elimination of the parameters associated with the satellite group(s) j_s will be done using stations 1, 2, k, co-observing with station(s) s. The observations from stations 1 and 2 lead to the same two equations for j_s as expressed in the first two lines of (1.4-1). For the observations from station k it holds similarly that

$$(X_{j_s}-x_k)(\partial X_{j_s}-\partial x_k) + (Y_{j_s}-y_k)(\partial Y_{j_s}-\partial y_k) + (Z_{j_s}-z_k)(\partial Z_{j_s}-\partial z_k) = 0.$$

In the matrix form the three equations can be written as

$$\begin{bmatrix} X_{j_s} & Y_{j_s} & Z_{j_s} \\ X_{j_s}-x_2 & Y_{j_s} & Z_{j_s} \\ X_{j_s}-x_k & Y_{j_s}-y_k & Z_{j_s}-z_k \end{bmatrix} \begin{bmatrix} \partial X_{j_s} \\ \partial Y_{j_s} \\ \partial Z_{j_s} \end{bmatrix} = \begin{bmatrix} 0 \\ (X_{j_s}-x_2)\partial x_2 \\ (X_{j_s}-x_k)\partial x_k + (Y_{j_s}-y_k)\partial y_k + (Z_{j_s}-z_k)\partial z_k \end{bmatrix}. \tag{1.4-3}$$

The determinant of the (3 x 3) matrix in the above expression is given by

$$D = x_2 (y_k Z_{j_s} - z_k Y_{j_s}), \tag{1.4-4}$$

which is the same as

$$D = - \begin{vmatrix} -x_2 & 0 & 0 \\ x_k - x_2 & y_k & z_k \\ X_{j_s} - x_2 & Y_{j_s} & Z_{j_s} \end{vmatrix} . \tag{1.4-5}$$

This form can be obtained also upon using the equivalence operations on the matrix in (1.4-3). If the determinant in (1.4-5) is equal to zero, then stations 1, 2, k, and the point(s) of j_s, with the coordinates $(X_{j_s}, Y_{j_s}, Z_{j_s})$, all lie in a plane. Consequently, the condition for (1.4-3) to have a unique solution for ∂X_{j_s}, ∂Y_{j_s}, ∂Z_{j_s} with respect to any target in j_s is that none of the points in j_s lies in a plane with stations 1, 2, and k. The solution of (1.4-3) with ground stations in general configuration (not necessarily in one plane) is:

$$\partial X_{J_s} = (x_2 - X_{J_s})\frac{\partial x_2}{x_2} \tag{1.4-6a}$$

$$\partial Y_{J_s} = \frac{1}{y_k Z_{J_s} - z_k Y_{J_s}}\Big[-(x_k Z_{J_s} - z_k X_{J_s})(x_2 - X_{J_s})\frac{\partial x_2}{x_2} + Z_{J_s}(x_k - X_{J_s})\partial x_k + \\ + Z_{J_s}(y_k - Y_{J_s})\partial y_k + Z_{J_s}(z_k - Z_{J_s})\partial z_k\Big] \tag{1.4-6b}$$

$$\partial Z_{J_s} = \frac{1}{y_k Z_{J_s} - z_k Y_{J_s}}\Big[-(y_k X_{J_s} - x_k Y_{J_s})(x_2 - X_{J_s})\frac{\partial x_2}{x_2} - Y_{J_s}(x_k - X_{J_s})\partial x_k - \\ - Y_{J_s}(y_k - Y_{J_s})\partial y_k - Y_{J_s}(z_k - Z_{J_s})\partial z_k\Big]. \tag{1.4-6c}$$

When these values are substituted in the equation for observations from station(s) s, i.e., in

$$(X_{J_s} - x_s)(\partial X_{J_s} - \partial x_s) + (Y_{J_s} - y_s)(\partial Y_{J_s} - \partial y_s) + (Z_{J_s} - \partial z_s)(\partial Z_{J_s} - \partial z_s) = 0,$$

then the following expression is obtained:

$$(y_k Z_{J_s} - z_k Y_{J_s})(X_{J_s} - x_s)\partial x_s + (y_k Z_{J_s} - z_k Y_{J_s})(Y_{J_s} - y_s)\partial y_s + (y_k Z_{J_s} - z_k Y_{J_s})(Z_{J_s} - z_s)\partial z_s - \\ -(y_s Z_{J_s} - z_s Y_{J_s})(X_{J_s} - x_k)\partial x_k - (y_s Z_{J_s} - z_s Y_{J_s})(Y_{J_s} - y_k)\partial y_k - (y_s Z_{J_s} - z_s Y_{J_s})(Z_{J_s} - z_k)\partial z_k - \\ -[X_{J_s}(z_k y_s - y_k z_s) + Y_{J_s}(x_k z_s - z_k x_s) + Z_{J_s}(y_k x_s - x_k y_s)](X_{J_s} - x_2)\frac{\partial x_2}{x_2} = 0. \tag{1.4-7}$$

Using equations of the type (1.2-9) for i-stations and of the type (1.4-7) for s-station(s), $\widetilde{A}$ matrix for one replacement with the ground stations generally distributed can be obtained, such as presented in Table (1.4-1). If there are more than one s-station, the table can be easily expanded, using the same type of terms for any further s-stations; for each such additional station three columns and as many rows as the number of targets observed by it would have to be added. The dots in each row block of Table (1.4-1) and any further table indicate that the same rows figure in the whole row block with the targets' coordinates as the only changing elements in them; if there are any columns in which these coordinates do not figure, then in such columns the elements do not change within the same row block.

Table (1.4-1)

$\widetilde{A}$ Matrix with Station 3 Replaced by Station k
(General Distribution of Ground Stations)

$\vdots$	∂x_4	∂y_4	∂z_4	$\cdots$	∂x_k	∂y_k	∂z_k	∂x_s	∂y_s	∂z_s	$\partial x_3/y_3$	$\partial y_3/y_3$	$\partial x_2/x_2$
From 4	$\vdots$ $Z_{J_4}(X_{J_4}-x_4)$ $\vdots$	$\vdots$ $Z_{J_4}(Y_{J_4}-y_4)$ $\vdots$	$\vdots$ $Z_{J_4}(Z_{J_4}-z_4)$ $\vdots$								$\vdots$ $(z_4 Y_{J_4}-y_4 Z_{J_4})(X_{J_4}-x_3)$ $\vdots$	$\vdots$ $(z_4 Y_{J_4}-y_4 Z_{J_4})(Y_{J_4}-y_3)$ $\vdots$	$\vdots$ $[z_4(X_{J_4}-Y_{J_4}\frac{x_3}{y_3}) -$ $-Z_{J_4}(x_4-y_3\frac{x_3}{y_3})](X_{J_4}-x_2)$ $\vdots$
$\cdots$													
From k					$\vdots$ $Z_{J_k}(X_{J_k}-x_k)$ $\vdots$	$\vdots$ $Z_{J_k}(Y_{J_k}-y_k)$ $\vdots$	$\vdots$ $Z_{J_k}(Z_{J_k}-z_k)$ $\vdots$				$\vdots$ $(z_k Y_{J_k}-y_k Z_{J_k})(X_{J_k}-x_3)$ $\vdots$	$\vdots$ $(z_k Y_{J_k}-y_k Z_{J_k})(Y_{J_k}-y_3)$ $\vdots$	$\vdots$ $[z_k(X_{J_k}-Y_{J_k}\frac{x_3}{y_3}) -$ $-Z_{J_k}(x_k-y_k\frac{x_3}{y_3})](X_{J_k}-x_2)$ $\vdots$
From s					$\vdots$ $-(y_s Z_{J_s}-z_s Y_{J_s}) \times$ $\times (X_{J_s}-x_k)$ $\vdots$	$\vdots$ $-(y_s Z_{J_s}-z_s Y_{J_s}) \times$ $\times (Y_{J_s}-y_k)$ $\vdots$	$\vdots$ $-(y_s Z_{J_s}-z_s Y_{J_s}) \times$ $\times (Z_{J_s}-z_k)$ $\vdots$	$\vdots$ $(y_k Z_{J_s}-z_k Y_{J_s}) \times$ $\times (X_{J_s}-x_s)$ $\vdots$	$\vdots$ $(y_k Z_{J_s}-z_k Y_{J_s}) \times$ $\times (Y_{J_s}-y_s)$ $\vdots$	$\vdots$ $(y_k Z_{J_s}-z_k Y_{J_s}) \times$ $\times (Z_{J_s}-z_s)$ $\vdots$			$\vdots$ $-[X_{J_s}(z_k y_s-y_k z_s) +$ $+Y_{J_s}(x_k z_s-z_k x_s +$ $+Z_{J_s}(y_k x_s-x_k y_s)](X_{J_s}-x_2)$ $\vdots$

When dealing with ground stations which are all lying in one plane, the z-coordinate for any station is set to zero and $\widetilde{A}$ matrix becomes considerably simplified. Its determinant then becomes, upon considering (1.4-3):

$$D = x_2 y_k Z_{J_s},$$

which means that in addition to the conditions in (1.4-2),

$$y_k \neq 0, \quad Z_{J_s} \neq 0 \tag{1.4-8}$$

also has to hold. Since it is the rank of $\widetilde{A}$ which is of interest as it was also the case in the previous sections, certain equivalence operations will be performed to further simplify $\widetilde{A}$ matrix. If there were more than one s-station, they would be treated the same way (e.g., $\partial x_k \to \partial x_k + \frac{y_s}{y_k}\partial x_s$ would become $\partial x_k \to \partial x_k + \frac{y_{s'}}{y_k}\partial x_{s'} + \frac{y_{s''}}{y_k}\partial x_{s''} + \cdots$, etc.). The equivalence operations with respect to $\widetilde{A}$ matrix of Table (1.4-1) are the following:

(1) Divide each row by the corresponding $Z_J \neq 0$.

(2) Multiply each of the last three columns by -1.

(3) Perform on the three column block of station k:

$$\partial x_k \to \partial x_k + \frac{y_s}{y_k}\partial x_s, \quad \partial y_k \to \partial y_k + \frac{y_s}{y_k}\partial y_s, \quad \partial z_k \to \partial z_k + \frac{y_s}{y_k}\partial z_s.$$

(4) Perform on the last three column block:

$$\frac{\partial x_3}{y_3} \to \frac{\partial x_3}{y_3} - y_4\partial x_4 - \dots - y_k\partial x_k, \quad \frac{\partial y_3}{y_3} \to \frac{\partial y_3}{y_3} - y_4\partial y_4 - \dots - y_k\partial y_k,$$

$$\frac{\partial x_2}{x_2} \to \frac{\partial x_2}{x_2} - (x_4 - y_4\frac{x_3}{y_3})\partial x_4 - \dots - (x_k - y_k\frac{x_3}{y_3})\partial x_k - \frac{y_k x_s - x_k y_s}{y_k}\partial x_s.$$

(5) Divide each of the rows "From s" by $y_k \neq 0$.

(6) Perform further on the last column:

$$\frac{\partial x_2}{x_2} \to \frac{\partial x_2}{x_2} + \frac{x_3}{y_3}\,\frac{\partial x_3}{y_3}.$$

The matrix thus obtained is called $\widetilde{\widetilde{A}}$ matrix with one replacement and with ground stations lying in one plane; it is presented in Table (1.4-2). There appear two s-stations in this table, denoted as s′ and s″. At this point some notations

Table (1.4-2)

$\tilde{\tilde{A}}$ Matrix with Station 3 Replaced by Station k
(Ground Stations in Plane)

	∂x_4	∂y_4	∂z_4	...	∂x_i	∂y_i	∂z_i	...	∂x_k	∂y_k	∂z_k	$\partial x_s'$	$\partial y_s'$	$\partial z_s'$	...	$\partial x_s''$	$\partial y_s''$	$\partial z_s''$	...	$\partial x_3/y_3$	$\partial y_3/y_3$	$\partial x_2/x_2$
station 4	$X_{J_4}-x_4$	$Y_{J_4}-y_4$	Z_{J_4}																	$y_4(x_4-x_3)$	$y_4(y_4-y_3)$	$x_4(x_4-x_2)-y_4\frac{x_3}{y_3}(x_3-x_2)$
...																						
station i					$X_{J_i}-x_i$	$Y_{J_i}-y_i$	Z_{J_i}													$y_i(x_i-x_3)$	$y_i(y_i-y_3)$	$x_i(x_i-x_2)-y_i\frac{x_3}{y_3}(x_3-x_2)$
...																						
station k									$X_{J_k}-x_k$	$Y_{J_k}-y_k$	Z_{J_k}									$y_k(x_k-x_3)$	$y_k(y_k-y_3)$	$x_k(x_k-x_2)-y_k\frac{x_3}{y_3}(x_3-x_2)$
station s'									$(x_k-x_s')\frac{y_s'}{y_k}$	$(y_k-y_s')\frac{y_s'}{y_k}$	0	$X_{J_s}'-x_s'$	$Y_{J_s}'-y_s'$	Z_{J_s}'						$y_s'(x_s'-x_k)$	$y_s'(y_s'-y_k)$	$x_s'(x_s'-x_2)-y_s'\frac{x_k}{y_k}(x_k-x_2)$
...																						
station s''									$(x_k-x_s'')\frac{y_s''}{y_k}$	$(y_k-y_s'')\frac{y_s''}{y_k}$	0					$X_{J_s}''-x_s''$	$Y_{J_s}''-y_s''$	Z_{J_s}		$y_s''(x_s-x_k)$	$y_s''(y_s''-y_k)$	$x_s''(x_s-x_2)-y_s'\frac{x_k}{y_k}(x_k-x_2)$

ill be introduced in accordance with the previous sections; they will mainly onsist of the f-terms from the last three column block of $\widetilde{\widetilde{A}}$ matrix and some ew terms (p-terms), arising from the presence of s' and s'' stations:

$$
\begin{aligned}
&f_4^1 = y_4(x_4-x_3), \quad f_4^2 = y_4(y_4-y_3), \quad f_4^3 = x_4(x_4-x_2)-y_4\frac{x_3}{y_3}(x_3-x_2)\\
&\qquad\vdots \qquad\qquad\qquad\quad \vdots \qquad\qquad\qquad\qquad \vdots\\
&f_k^1 = y_k(x_k-x_3), \quad f_k^2 = y_k(y_k-y_3), \quad f_k^3 = x_k(x_k-x_2)-y_k\frac{x_3}{y_3}(x_3-x_2)\\
&f_{s'}^1 = y_{s'}(x_{s'}-x_k), \quad f_{s'}^2 = y_{s'}(y_{s'}-y_k), \quad f_{s'}^3 = x_{s'}(x_{s'}-x_2)-y_{s'}\frac{x_k}{y_k}(x_k-x_2)\\
&f_{s''}^1 = y_{s''}(x_{s''}-x_k), \quad f_{s''}^2 = y_{s''}(y_{s''}-y_k), \quad f_{s''}^3 = x_{s''}(x_{s''}-x_2)-y_{s''}\frac{x_k}{y_k}(x_k-x_2)
\end{aligned}
\tag{1.4-9}
$$

$$
\begin{aligned}
p_{s'}^1 &= -\frac{1}{y_k} f_{s'}^1 & \qquad p_{s'}^2 &= -\frac{1}{y_k} f_{s'}^2\\
p_{s''}^1 &= -\frac{1}{y_k} f_{s''}^1 & \qquad p_{s''}^2 &= -\frac{1}{y_k} f_{s''}^2
\end{aligned}
\tag{1.4-10}
$$

In the analysis of the rank of $\widetilde{\widetilde{A}}$ matrix the necessary conditions for $\widetilde{\widetilde{A}}$ to be on-singular will be presented first. The above notations and earlier notations rom section 1.332 will be used in a similar approach as in that section. This neans that when all the a-coefficients and b-coefficients have to be zero in rder that the systems associated with $\widetilde{\widetilde{A}}$ matrix and denoted as (1.4-11) and 1.4-11a) be consistent, then $\widetilde{\widetilde{A}}$ will be said to be non-singular. These two ystems are presented in Table (1.4-3). The systems (1.4-11) and (1.4-11a) an be also imagined in a matrix form which is helpful for certain rank considerations. When the a-terms and c-terms are arranged in a column vector, then he system (1.4-11) is arrived at by pre-multiplying of this vector by what is aid to be the "matrix of the system (1.4-11)", or the "matrix of (1.4-11)". Similarly, with the b-terms arranged in a column vector, the "matrix of (1.4-11a)" vill have the form

Table (1.4-3)

Systems (1.4-11) and (1.4-11a) Associated with $\tilde{\tilde{A}}$ Matrix

$$
\begin{aligned}
&a_4^x \tilde{V}_4^x + a_4^y \tilde{V}_4^y + a_4^z \tilde{V}_4^z && + c_4 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \\
&\quad\vdots \\
&a_k^x \tilde{V}_k^x + a_k^y \tilde{V}_k^y + a_k^z \tilde{V}_k^z && + c_k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \\
&a_k^x p^1_{s'} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_k^y p^2_{s'} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + 0 + a_s^{x'} \tilde{V}_s^{x'} + a_s^{y'} \tilde{V}_s^{y'} + a_s^{z'} \tilde{V}_s^{z'} && + c_s' \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \\
&a_k^x p^1_{s''} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_k^y p^2_{s''} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + 0 + a_s^{x''} \tilde{V}_s^{x''} + a_s^{y''} \tilde{V}_s^{y''} + a_s^{z''} \tilde{V}_s^{z''} && + c_s'' \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0
\end{aligned}
$$

(1.4-11)

$$
\begin{aligned}
b_1 f_4^1 + b_2 f_4^2 + b_3 f_4^3 &= c_4 \\
&\vdots \\
b_1 f_k^1 + b_2 f_k^2 + b_3 f_k^3 &= c_k \qquad (1.4\text{-}11a) \\
b_1 f_{s'}^1 + b_2 f_{s'}^2 + b_3 f_{s'}^3 &= c_s' \\
b_1 f_{s''}^1 + b_2 f_{s''}^2 + b_3 f_{s''}^3 &= c_s''.
\end{aligned}
$$

$$F = \begin{bmatrix} f_4^1 & f_4^2 & f_4^3 \\ \vdots & \vdots & \vdots \\ f_k^1 & f_k^2 & f_k^3 \\ f_s^{1\prime} & f_s^{2\prime} & f_s^{3\prime} \\ f_s^{1\prime\prime} & f_s^{2\prime\prime} & f_s^{3\prime\prime} \end{bmatrix} . \tag{1.4-12}$$

For stations 4 through k, the corresponding rows in F have the same form as the same rows in the last three column block of $\tilde{\tilde{A}}$ matrix with three stations observing all the targets (presented in Table (1.3-1)), except that to the third column the $\frac{x_3}{y_3}$ -multiple of the first column has been added. Therefore, a row corresponding to any station beyond station 4 up to and including station k would be linearly dependent on the row of station 4 under exactly the same conditions as presented in section 1.321, namely, if that station lay in a straight line with all except one of stations 1, 2, 3, and 4. When the c-terms from (1.4-11a) are substituted into (1.4-11) this system now contains the full set of a-terms and b-terms and wil be called "full system of (1.4-11)", or "full system". Its corresponding matrix is exactly $\tilde{\tilde{A}}$ matrix. However, it is easier to work with $\tilde{\tilde{A}}$ matrix with the notations of (1.4-11) and (1.4-11a), i.e., in terms of the full system, taking advantage of the abbreviated notations. In this context it will be preferable to call $\tilde{\tilde{A}}$ matrix as "full matrix" associated with the full system. The necessary conditions for the full matrix to be non-singular will be divided into two principle groups: one dealing with its rwo blocks and called "row conditions", and the other dealing with its three column blocks and called "column conditions". The latter group will be subdivided into two parts: conditions necessary to prevent signularity of any of the three column blocks except the last one, i.e., to prevent singularity A), and conditions necessary to prevent singularity of the last three column block, i.e., to prevent singularity B). With all the notations and definitions introduced, the above necessary conditions can be systematically investigated.

When the row conditions are to be examined, it has to be taken into consideration that no row block in the matrix of (1.4-11) can have higher rank than four. Assuming that each row block there has the rank equal to at least three, it is clear that at least three row blocks in the matrix of (1.4-11) or in the full matrix have to have the rank equal to four. Otherwise the sum of the row ranks would not even reach the number of columns in the full matrix, since each row block introduces three new columns in the left-hand part of $\widetilde{\widetilde{A}}$ matrix which does not include the last three column block (in (1.4-11) this part and the last three column block are separated by a vertical dotted line). In other words, the row conditions imply that at least three quads must observe their targets off-plane when the terminology of section 1.332 is used and when the expression (1.3-17a) with the text below it in the same section is considered.

When analyzing the column conditions, singularity A) will be examined first. Upon considering $\widetilde{A}$ matrix of Table (1.4-1), it is clear that singularity A) occurs for any distribution of ground stations whenever it holds that

$$\begin{vmatrix} X_1 - x_k & Y_1 - y_k & Z_1 - z_k \\ X_2 - x_k & Y_2 - y_k & Z_2 - z_k \\ X - x_k & Y - y_k & Z - z_k \end{vmatrix} = 0$$

where (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) are taken as denoting the first two targets in j_k; (X, Y, Z) denotes any further target in j_k or any target in j_s. The above equation expresses any point (X, Y, Z) as lying in the plane through the first two targets in j_k and station k. This result indicates that singularity A) occurs whenever all the targets (in one satellite group or more) observed by one station lie in the plane containing this station regardless of the distribution of ground stations. Singularity A) caused by such a distribution of targets will be sometimes referred to as "general singularity A)". Next, ground stations are assumed to be in the plane $z = 0$.

Considering the matrix of (1.4-11), it is evident that singularity A) for any station except station k occurs under the circumstances specified in section 1.31, i.e., when all the satellite points in any of j_i (except j_k) or j_s lie in a plane through corresponding stations i or s. Singularity A) for the three column block of station k can happen only if, in addition to all satellite points of j_k lying in one plane through, in such case called "plane π" (which expresses the condition (1.3-1) in section 1.31), the following relation also holds:

$$\begin{vmatrix} X_1 - x_k & Y_1 - y_k & Z_1 \\ X_2 - x_k & Y_2 - y_k & Z_2 \\ p_s^1 & p_s^2 & 0 \end{vmatrix} = 0. \qquad (1.4\text{-}13)$$

Here (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) denote again the coordinates of the first two targets in j_k, namely j_{k_1} and j_{k_2}. In the present case, (1.4-13) would hold with the third row pertaining to either s' or s''. When the coordinates of s' or s'' are denoted as variables x and y (the z coordinate being zero), then (1.4-13) holds when either

$$y = 0, \qquad (1.4\text{-}13a)$$

or

$$\begin{vmatrix} X_1 - x_k & Y_1 - y_k & Z_1 \\ X_2 - x_k & Y_2 - y_k & Z_2 \\ x - x_k & y - y_k & 0 \end{vmatrix} = 0 \qquad (1.4\text{-}13b)$$

is fulfilled. The equation (1.4-13b) expresses the fact that the variable point (x, y, z) lies in a plane with the points j_{k_1}, j_{k_2}, and k, subject to the condition $z = z_k = 0$. But this implies that the corresponding station s' or s'' lies on a straight line denoted by the letter "ℓ", which is generated by two intersecting planes: plane π and the plane of ground stations (i.e., plane z = 0). It will be of interest to compute the direction of the line ℓ, passing necessarily through station k. From the general equation of a straight line,

$$ax + by + c = 0,$$

its direction, given by the angle α (measured from the positive direction of the x-axis), can be computed from the formula

$$\text{tj}\ \alpha = -\frac{a}{b}.$$

Thus, if

$$a_x = x_k (Z_2 - Z_1) + Z_1 X_2 - X_1 Z_2 \tag{1.4-14a}$$

and

$$a_y = y_k (Z_2 - Z_1) + Z_1 Y_2 - Y_1 Z_2, \tag{1.4-14b}$$

then (1.4-13b) can be written as

$$x\, a_y - y\, a_x - (x_k\, a_y - y_k\, a_x) = 0, \tag{1.4-14c}$$

from which

$$\text{tj}\ \alpha = -\frac{a_y}{a_x}. \tag{1.4-14d}$$

It can be summarized that singularity A) for station k can occur only if all satellite points observed by it (here targets of j_k and j_s) are in π (plane through station k), called also general singularity A), or if all targets of j_k are in π and each of the s-stations is fulfilling either (1.4-13a) or (1.4-13b), i.e., if it is lying either on the x-axis of the local coordinate system (line connecting stations 1 and 2) or on ℓ (line of intersection between π and the plane of ground stations).

Singularity B) arises when any (3×3) submatrix of F, given by (1.4-12), is singular. Then any row vector in that matrix would have to be a linear combination of its two chosen vectors (here the vectors corresponding to stations 4 and k), i.e., it would have to lie in the subspace V_2 spanned by those two row vectors:

$$V_2 \sim \begin{bmatrix} f_4^1 & f_4^2 & f_4^3 \\ f_k^1 & f_k^2 & f_k^3 \end{bmatrix} . \tag{1.4-15}$$

The two rows which span V_2 are assumed to be independent. This is always true unless station k is in a straight line with all except one of stations 1, 2, 3, and 4, as it was evidenced following (1.4-12). Such special configurations are assumed non-existent. Thus, for any station i it would have to hold that

$$\begin{vmatrix} f_4^1 & f_4^2 & f_4^3 \\ f_k^1 & f_k^2 & f_k^3 \\ f_i^1 & f_i^2 & f_i^3 \end{vmatrix} = 0, \tag{1.4-16}$$

which represents a second order curve for station i (i.e., in x_i, y_i). This curve can in general be specified also by finding five points through which it passes. As it is evident from the relations in (1.4-9),

$$f_i^1 = f_i^2 = f_i^3 = 0, \text{ whenever } i \equiv 1, 2, \text{ or } 3; \tag{1.4-16a}$$

further,

$$[f_i^1 \;\; f_i^2 \;\; f_i^3] = [f_4^1 \;\; f_4^2 \;\; f_4^3] \quad \text{if } \; i \equiv 4, \tag{1.4-16b}$$

and

$$[f_i^1 \;\; f_i^2 \;\; f_i^3] = [f_k^1 \;\; f_k^2 \;\; f_k^3] \quad \text{if } \; i \equiv k. \tag{1.4-16c}$$

From here it is evident that the second order curve for station i, expressed by (1.4-16), would have to pass through the stations 1, 2, 3, 4, k. This can also be seen from section 1.32 where station 5 was used rather than station k. Further, for any station s it would have to hold that

$$\begin{vmatrix} f_4^1 & f_4^2 & f_4^3 \\ f_k^1 & f_k^2 & f_k^3 \\ f_s^1 & f_s^2 & f_s^3 \end{vmatrix} = 0, \tag{1.4-17}$$

which now represents a second order curve for station(s) s (in the present case it would have to hold for stations s' and s''). With the simplifications

$$[f_s^1 \;\; f_s^2 \;\; f_s^3] \equiv f_s, \text{ etc.},$$

it again follows from (1.4-9) that

$$f_s = 0, \quad \text{whenever } s \equiv 1, 2, k; \tag{1.4-17}$$

further,

$$f_s = af_k, \quad a = -\frac{y_3}{y_k} \quad \text{if } s \equiv 3, \tag{1.4-17}$$

and

$$f_s = f_4 + bf_k, \quad b = -\frac{y_4}{y_k} \quad \text{if } s \equiv 4. \tag{1.4-17}$$

Thus the second order curve for station(s) s, expressed by (1.4-17), also passes through stations 1, 2, 3, 4, k. Whether considering (1.4-16) or (1.4-17), the coefficients in the equation of second degree never all vanished; this would happen if and only if V_2 of (1.4-15) were of dimension one, similar to section 1.321. However, such cases were assumed non-existent, as mentioned following (1.4-15). Consequently, singularity B) occurs when all the stations lie on one second order curve, which is the same conclusion as the one reached in section 1.32 when three ground stations were considered to be observing all the targets. A similar definition as in section 1.332 can be made now: any stations lying on a second order curve passing through stations 1, 2, 3, 4, k are said to be "on-curve" stations, while otherwise they are called "off-curve" stations.

According to the earlier sections, when $\widetilde{\widetilde{A}}$ matrix is singular in absence of singularity A) and singularity B), then singularity C) is said to have occurred. The necessary conditions to avoid singularity C) are the row conditions described earlier. They correspond to the necessary conditions of section 1.331 with three stations considered observing all the targets. Next, the sufficient conditions to avoid singularity C) (or global singularity) will be examined. Subsequently, the conclusions about singularity and non-singularity of $\widetilde{\widetilde{A}}$ matrix will be reached. Since such a discussion is fairly complex, it will be divided into parts, according to certain properties of the group j_k. First of all, it will be assumed throughout that the group j_4 contains off-plane satellite points. Otherwise this group (with singularity A) discarded) could offer no help in analyzing the singularity of $\tilde{\tilde{A}}$. This group together with station 4 could then be completely disregarded and deleted from $\tilde{\tilde{A}}$ matrix. If this "new" $\tilde{\tilde{A}}$ matrix were non-singular, the observations from 1, 2, 3 would determine the targets in j_4 ($Z \neq 0$) and station 4 could in turn be uniquely determined from j_4, which is possible in absence of singularity A). But stations 1, 2, k in this new $\tilde{\tilde{A}}$ matrix could be considered as observing all the targets and the problem would thus be reduced to the one delt with in the previous sections. Similar argument would hold for two or more replacements, i.e., if j_4 did not contain off-plane targets, the problem with two replacements would be essentially reduced to the problem with one replacement, etc. Consequently, the group j_4 will be considered off-plane not only in this section, but also in all sections dealing with replacements of stations. Next, two basic possibilities can arise with respect to the group j_k:

(1) j_k contains off-plane satellite targets, which is of practical importance and will be analyzed in section 1.411;

(2) j_k contains in-plane satellites only; this problem, rather of academic interest, is presented for the sake of completeness in section 1.412 and summarized in section 1.431. It can be further subdivided into the case when all the targets in j_k lie in a plane in general position and the case when these targets lie in a plane which happens to pass through station k, i.e., in the plane π (see earlier notation). From the general point of view, nothing essential is lost if the part of section 1.412, starting with case (a) and continuing with other rather special cases and lenthy derivations, is skipped; all the results and conclusions of this section are listed in section 1.413.

1.411 Group j_k Considered as "Off-Plane Targets".

Since singularity A) for station k as well as for station 4 is automatically discarded, the remaining necessary conditions are: avoiding singularity A) for the other stations (i. e., no corresponding satellite group should be in plane with such stations), avoiding singularity B) (i. e., the ground stations should not be all on one second order curve), and presence of at least one more quad observing off-plane targets. Since

$$\operatorname{rank}\left[\tilde{v}_k^x, \tilde{v}_k^y, \tilde{v}_k^z, \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\right] = 4,$$

it holds that

$$a_k^x = a_k^y = a_k^z = c_k = 0.$$

Similarly,

$$a_4^x = a_4^y = a_4^z = c_4 = 0.$$

The remaining part of the system (1.4-11) can be written as

$$\begin{array}{l}\vdots\\ a_s^{x\prime}\tilde{v}_s^{x\prime} + a_s^{y\prime}\tilde{v}_s^{y\prime} + a_s^{z\prime}\tilde{v}_s^{z\prime} \qquad\qquad + c_s{}'\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = 0\\[2ex] \qquad\qquad a_s^{x\prime\prime}\tilde{v}_s^{x\prime\prime} + a_s^{y\prime\prime}\tilde{v}_s^{y\prime\prime} + a_s^{z\prime\prime}\tilde{v}_s^{z\prime\prime} \qquad + c_s{}'\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = 0\end{array}$$

where the dots can represent row blocks for any number of i stations. For the remaining a-coefficients to be zero it is necessary that all the remaining c-coefficients be zero. However, not all the stations associated with the above system need observe their targets off-plane. If one of them, such that its corresponding row in F matrix of (1.4-12) is independent of the rows f_4 and f_k observes off-plane targets, the corresponding c-term is brought to zero (besides c_4 and c_k). But then the only solution of (1.4-11a) is

$$b_1 = b_2 = b_3 = 0,$$

and consequently all the c-terms are necessarily zero. Any station having the above property is clearly an off-curve station. Since singularity B) has been discarded, such a station must exist and the discussion is complete. As a conclusion, in this section the necessary and sufficient conditions for $\widetilde{\widetilde{A}}$ matrix being not singular are: elimination of singularity A) for all the other stations (besides stations 4 and k), and the property, that at least one of those stations is off-curve and observes off-plane targets at the same time.

1.412 Group j_k Considered as "In-Plane Targets".

Like in all examined cases, due to the group j_4 containing off-plane targets, it holds that

$$a_4^x = a_4^y = a_4^z = 0. \tag{1.4-18}$$

For the following analysis it will be assumed that

$$f_4^1 \neq 0, \quad \text{i.e.,} \quad y_4 \neq 0, \quad x_4 \neq x_3. \tag{1.4-19}$$

Due to (1.4-18), one b-term can be eliminated from the first row of (1.4-11a); it is chosen to be the b_1 term, namely

$$b_1 = -\frac{f_4^2}{f_4^1} b_2 - \frac{f_4^3}{f_4^1} b_3, \tag{1.4-20}$$

for which the assumption (1.4-19) was needed. Otherwise, a different elimination procedure would have to be used. Next, the full system can be rewritten in the form

$$\begin{array}{lllll|lll}
\vdots & & & & & & & \\
 & a_k^x \widetilde{v}_k^x & + a_k^y \widetilde{v}_k^y & + a_k^z \widetilde{v}_k^z & & + b_2 q_k^2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & + b_3 q_k^3 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0 \\
a_k^x p_s^1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & + a_k^y p_s^2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & + \quad 0 & & a_s^x \widetilde{v}_s^x + a_s^y \widetilde{v}_s^y + a_s^z \widetilde{v}_s^z & + b_2 q_s^2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & + b_3 q_s^3 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0
\end{array} \tag{1.4-21}$$

where the dots make allowance for some i-stations, and where several s-stations can appear (here s′, s″). In the above, the following notations have been made:

$$\vdots$$

$$q_k^2 = f_k^2 - f_k^1 \frac{f_4^2}{f_4^1}, \quad q_k^3 = f_k^3 - f_k^1 \frac{f_4^3}{f_4^1}$$

$$q_s^2 = f_s^2 - f_s^1 \frac{f_4^2}{f_4^1}, \quad q_s^3 = f_s^3 - f_s^1 \frac{f_4^3}{f_4^1}. \tag{1.4-22}$$

The necessary row conditions for the matrix of (1.4-21) to be non-singular are such that at least two row blocks (including those for i stations, if present) have the rank equal to four, while the others have the rank of at least three; otherwise the row ranks would not even add up to reach the number of columns. Consequently, at least two stations besides station 4 have to observe off-plane targets.

The column conditions with respect to singularity A) are the same as those presented in the more general section 1.41. (They are automatically fulfilled for station 4.) In the analysis of singularity B), the last two columns in the matrix of (1.4-21) will be considered. They would have rank less than two if the relations

$$\vdots$$

$$q_k^3 = c\, q_k^2$$

$$q_s^3 = c\, q_s^2 \tag{1.4-23}$$

held, together with

$$q_i^3 = c\, q_i^2 \tag{1.4-23a}$$

if some i stations were also present.

In the case of

$$q_k^2 = 0 \tag{1.4-24a}$$

it would have to hold simultaneously that

$$q_k^2 = q_k^3 = 0, \tag{1.4-24b}$$

according to (1.4-23). The condition (1.4-24a) leads to

$$(x_4 - x_3)(y_k - y_3) = (y_4 - y_3)(x_k - x_3), \qquad (1.4\text{-}25a)$$

implying that stations 3,4 and k are lying on a straight line. The condition $q_k^3 = 0$ leads to the equation of a second order curve in $(x_k, y_k) \equiv x^T$,

$$x^T A x + x^T a = 0. \qquad (1.4\text{-}25b)$$

Denoting a_{ij} as the ij th element of the (2×2) matrix A and a_i as the i th element of the 2-(column) vector a, it holds that

$$a_{11} = 1,$$

$$a_{12} = a_{21} = -\frac{1}{2}\frac{f_4^3}{f_4^1},$$

$$a_{22} = 0,$$

$$a_1 = -x_2,$$

$$a_2 = \frac{x_3 x_4}{f_4^1}\left[x_4 - x_2 - \frac{y_4}{y_3}(x_3 - x_2)\right].$$

The above curve is a hyperbola, passing through stations 1,2,3,4 (if the coordinates of those stations are substituted for the coordinates of station k, the equation (1.4-25b) is fulfilled); it will be called a "special hyperbola". Should $q_k^2 = 0$ and $q_k^3 = 0$ hold simultaneously, station k would have to lie on the intersection of the line given by (1.4-25a) and of the special hyperbola, which would constraint the location of k to at most two isolated points. Throughout in this study, the policy will be accepted to discard all cases when the location of any station or satellite point is restricted to some isolated points. Thus, critical configurations consisting of straight lines, curves, and later surfaces will be examined, with little attention paid to some isolated singularity, even though it may be easy to compute it; in many instances, existence of such isolated points will be only mentioned.

With the case (1.4-24b) discarded, the constant c can be expressed from (1.4-23) as

$$c = \frac{q_k^3}{q_k^2}.$$

In presence of station(s) i, the equation (1.4-23a) yields:

$$(f_4^2 f_k^3 - f_4^3 f_k^2) f_i^1 + (f_4^3 f_k^1 - f_4^1 f_k^3) f_i^2 + (f_4^1 f_k^2 - f_4^2 f_k^1) f_i^3 = 0, \qquad (1.4\text{-}26a)$$

which gives rise to a second order curve for station i. Upon using (1.4-16a)-(1.4-16c), it is seen that this second order curve passes through stations 1, 2, 3, 4, and k. For any station s (here s' and s''), the relations (1.4-23) yield

$$(f_4^2 f_k^3 - f_4^3 f_k^2) f_s^1 + (f_4^3 f_k^1 - f_4^1 f_k^3) f_s^2 + (f_4^1 f_k^2 - f_4^2 f_k^1) f_s^3 = 0, \qquad (1.4\text{-}26b)$$

giving rise to second order curves for stations s' and s''. The form of this equation closely resembles (1.4-26a). Both equations (1.4-26a) and (1.4-26b) would have all the coefficients of f_i or f_s equal to zero under exactly the same conditions which lead to V_2 of (1.4-15) having dimension one. Such cases were assumed non-existent. Upon using (1.4-17a) - (1.4-17c), this equation is satisfied and the second order curve for any s stations is seen to pass through stations 1, 2, 3, 4, and k as well. Thus, the same conclusion is reached with respect to singularity B) as in the more general section 1.41. In other words, singularity B) occurs when there exist no off-curve stations.

As stated in section 1.41, singularity C) has to be examined in order to find the sufficient conditions for non-singular $\widetilde{\widetilde{A}}$ matrix. With station j_4 observing always off-plane targets and station k taken in this section as observing in-plane targets, three basic cases will be examined. They will be called:

Case (a), when no station i beyond station 4 observes off-plane targets; thus at least two s-stations must observe off-plane targets to fulfil the necessary conditions.

Case (b), when one station i observes off-plane targets; thus at least one s-station is required to observe off-plane targets in order to fulfil the necessary conditions.

Case (c), when two stations i observe off-plane targets, i. e., when three off-plane satellite groups are observed from stations preceding station k; under certain assumptions, these three groups can prove to fulfill not only the necessary, but also the sufficient conditions for non-singular $\tilde{\tilde{A}}$.

Furthermore, each of these three cases can be further subdivided into two parts, one, in which the targets in j_k lie in a plane in general position and the other, in which this plane passes through station k, i. e., it is the plane π. This subdivision was already mentioned at the end of section 1. 41. The first part will be given number "1" and the second number "2", so that at certain point the notations case (a1), case (a2), etc., will appear. As stated earlier, the rest of this section can be skipped without loss of generality; it is summarized in the following secion 1. 413.

Case (a). Due to the necessary conditions,

$$\operatorname{rank}\left[\tilde{v}_s^x, \tilde{v}_s^y, \tilde{v}_s^z, \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right] = 4 \tag{1.4-27}$$

has to hold for at least two s-stations. It will now be examined under what circumstances this condition is also sufficient for non-singular $\tilde{\tilde{A}}$ matrix with exactly two such s stations, denoted as s' and s''. If

$$a_k^x p_{s'}^1 + a_k^y p_{s'}^2 + b_2 q_{s'}^2 + b_3 q_{s'}^3 = \bar{c}_{s'}$$

and (1.4-28)

$$a_k^x p_{s''}^1 + a_k^y p_{s''}^2 + b_2 q_{s''}^2 + b_3 q_{s''}^3 = \bar{c}_{s''} ,$$

then with $j_{s'}$ and $j_{s''}$ containing off-plane targets, it follows from (1.4-27) and (1.4-21) that

$$a_{s'}^x = a_{s'}^y = a_{s'}^z = \bar{c}_{s'} = 0$$

and

$$a^x_{s''} = a^y_{s''} = a^z_{s''} = \bar{c}_{s''} = 0 .$$

When these two $\bar{c}$-terms are substituted in (1.4-28), it is obtained:

$$\begin{bmatrix} a^x_k \\ a^y_k \end{bmatrix} = - \begin{bmatrix} p^1_{s'} & p^2_{s'} \\ p^1_{s''} & p^2_{s''} \end{bmatrix}^{-1} \begin{bmatrix} q^2_{s'} & q^3_{s'} \\ q^2_{s''} & q^3_{s''} \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} . \qquad (1.4\text{-}29)$$

For the unique solution to exist, it has to hold for the determinant of the matrix to be inverted:

$$D \neq 0$$

where

$$D = p^1_{s'} \, p^2_{s''} - p^2_{s'} \, p^1_{s''} . \qquad (1.4\text{-}29a)$$

(The symbol D should not be confused with the one used in section 1.41 to denote other types of determinants.) If it held that $D = 0$, then (1.4-29a) would result in

$$y_{s'} y_{s''} (x_{s'} - x_k)(y_{s''} - y_k) = y_{s'} y_{s''} (y_{s'} - y_k)(x_{s''} - x_k) , \qquad (1.4\text{-}29b)$$

which would be fulfilled if:

1) $y_{s'} = 0$, station s' lying on a straight line through stations 1, 2.

2) $y_{s''} = 0$, station s'' lying on a straight line through stations 1, 2. (1.4-29b')

3) Stations k, s', s'' lying on a straight line (of any direction).

The analysis in this fashion can proceed only if the above three conditions are eliminated, which is hereby assumed. The solution of (1.4-29) can be then written as

$$\begin{aligned} a_k^x &= u_1 b_2 + u_2 b_3 \\ a_k^y &= v_1 b_2 + v_2 b_3 \ , \end{aligned} \tag{1.4-30}$$

where

$$\begin{aligned} u_1 &= \frac{1}{D}\,(p_{s'}^3 q_{s''}^2 - q_{s'}^3 p_{s''}^2)\ , \\ u_2 &= \frac{1}{D}\,(p_{s'}^2 q_{s''}^3 - q_{s'}^3 p_{s''}^2)\ , \\ v_1 &= \frac{1}{D}\,(q_{s'}^2 p_{s''}^1 - p_{s'}^1 q_{s''}^2)\ , \\ v_2 &= \frac{1}{D}\,(q_{s'}^3 p_{s''}^1 - p_{s'}^1 q_{s''}^3)\ . \end{aligned} \tag{1.4-31}$$

These terms can be further developed and the following identities made:

$$\begin{aligned} (D y_k f_4^1)\, u_1 &= f_4^2 t\ , \\ (D y_k f_4^1)\, v_1 &= f_4^1 t \end{aligned} \tag{1.4-32}$$

where

$$t = f_{s'}^2 f_{s''}^1 - f_{s'}^1 f_{s''}^2 \ ;$$

thus

$$u_1 = -\frac{f_4^2}{f_4^1}\, v_1 \ .$$

Further,

$$\begin{aligned} (D\, y_k f_4^1) u_2 &= f_4^3 f_{s'}^2 f_{s'}^1 + (f_4^1 f_{s'}^3 - f_4^3 f_{s'}^1)\, f_{s''}^2 - f_4^1 f_{s'}^2 f_{s''}^3\ , \\ (D\, y_k f_4^1) v_2 &= -f_4^1 (f_{s'}^3 f_{s''}^1 - f_{s'}^1 f_{s''}^3)\ . \end{aligned} \tag{1.4-33}$$

When the relations (1.4-30) are used in the row block for station k in the system (1.4-21), it is obtained that

$$M \begin{bmatrix} a_k^z \\ b_2 \\ b_3 \end{bmatrix} = 0 \tag{1.4-34a}$$

where

$$M = [\widetilde{v}_k^z \,,\; u_1\widetilde{v}_k^x + v_1\widetilde{v}_k^y + q_k^2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \,,\; u_2\widetilde{v}_k^x + v_2\widetilde{v}_k^y + q_k^3 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}]. \tag{1.4-34b}$$

It is easy to show that when

$$\text{rank } M = 3, \tag{1.4-35}$$

then $\widetilde{\widetilde{A}}$ matrix is non-singluar. Clearly, when (1.4-35) holds, then the only solution of (1.4-34a) is

$$a_k^z = b_2 = b_3 = 0 .$$

But then

$$a_k^x = a_k^y = 0 ,$$

which follows from (1.4-29) and

$$b_1 = 0 ,$$

from (1.4-20). Thus all the b-terms and consequently all the c-terms are equal to zero. Since singularity A) is assumed to be eliminated as a necessary condition, the a-terms for the remaining i-stations and s-stations must be equal to zero as well, which completes the proof for non-singular $\widetilde{\widetilde{A}}$.

It remains to examine the rank of matrix M. If the conditions for

$$\text{rank } M < 3 \tag{1.4-36}$$

re found and eliminated, then the sufficient conditions for non-singular have been specified. If (1.4-36) holds, then

$$| M | = 0 , \tag{1.4-37}$$

hich is

$$\begin{vmatrix} Z_1 & u_1(X_1 - x_k) + v_1(Y_1 - y_k) + q_k^2 & u_2(X_1 - x_k) + v_2(Y_1 - y_k) + q_k^3 \\ Z_2 & u_1(X_2 - x_k) + v_1(Y_2 - y_k) + q_k^2 & u_2(X_2 - x_k) + v_2(Y_2 - y_k) + q_k^3 \\ Z & u_1(X - x_k) + v_1(Y - y_k) + q_k^2 & u_2(X - x_k) + v_2(Y - y_k) + q_k^3 \end{vmatrix} = 0. \tag{1.4-38a}$$

ere (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) denote the first two targets in j_k (j_{k_1} and $_2$), while the variable point (X, Y, Z)stands for any further target in j_k. he above relation represents the equation of a plane in (X, Y, Z), which isses through j_{k_1} and j_{k_2} (the equation is fulfilled when the coordinates of or j_{k_2} are substituted for variable point). Upon performing the obvious uivalence operations on (1.4-38a), it becomes:

$$\begin{vmatrix} Z_1 & u_1(X_1 - x_k) + v_1(Y_1 + y_k) + q_k^2 & u_2(X_1 - x_k) + v_2(Y_1 - y_k) + q_k^3 \\ Z_2 - Z_1 & u_1(X_2 - X_1) + v_1(Y_2 - Y_1) & u_2(X_2 - X_1) + v_2(Y_2 - Y_1) \\ Z - Z_1 & u_1(X - X_1) + v_1(Y - Y_1) & u_2(X - X_1) + v_2(Y - Y_1) \end{vmatrix} = 0. \tag{1.4-38b}$$

ie second and third rows in this determinant are dependent if it holds that

$$\frac{X - X_1}{X_2 - X_1} = \frac{Y - Y_1}{Y_2 - Y_1} = \frac{Z - Z_1}{Z_2 - Z_1} ,$$

e., if any further target in j_k lies on the straight line connecting j_{k_1} and $_2$. Consequently, (1.4-36) holds and $\widetilde{A}$ matrix is singular whenever all targets in j_k lie on a straight line.

From the form of (1.4-38a) it is also clear that $|M| = 0$ whenever

$$u_2 = cu_1, \quad v_2 = cv_1, \text{ and } q_k^3 = cq_k^2 . \tag{1.4-39}$$

Then the third column of M is the c-multiple of the second column identically for any X, Y, Z. The conditions leading to (1.4-39) will be now examined in detail. First, the case with $q_k^2 = q_k^3 = 0$ is discarded in accordance with (1.4-24a), (1.4-24b), and the discussion which followed. With $c = q_k^3/q_k^2$, the relation (1.4-39) yields two conditions:

$$u_2 q_k^2 = u_1 q_k^3$$

and

$$v_2 q_k^2 = v_1 q_k^3 .$$

The first condition leads to

$$(f_4^2 f_k^3 f_{s'}^2 - f_4^3 f_k^2 f_{s'}^2)\, f_{s''}^1 + (f_4^2 f_k^1 f_{s'}^3 + f_4^3 f_k^2 f_{s'}^1 - f_4^1 f_k^2 f_{s'}^3 - f_4^2 f_k^3 f_{s'}^1)\, f_{s''}^2 +$$

$$+ (f_4^1 f_k^2 f_{s'}^2 - f_4^2 f_k^1 f_{s'}^2)\, f_{s''}^3 = 0 \tag{1.4-40a}$$

and the second condition leads to

$$(f_4^1 f_k^2 f_{s'}^3 - f_4^2 f_k^1 f_{s'}^3 - f_4^1 f_k^3 f_{s'}^2 + f_4^3 f_k^1 f_{s'}^2)\, f_{s''}^1 + (f_4^1 f_k^3 f_{s'}^1 - f_4^3 f_k^1 f_{s'}^1)\, f_{s''}^2 +$$

$$+ (f_4^2 f_k^1 f_{s'}^1 - f_4^1 f_k^2 f_{s'}^1)\, f_{s''}^3 = 0. \tag{1.4-40b}$$

Both (1.4-40a) and (1.4-40b) represent a second degree curve (in general different from each other) for s'', passing through stations 1, 2, k, and s'; this follows from (1.4-17a) and from the fact that $f_{s''} = f_{s'}$ whenever $s'' \equiv s'$. Unless (1.4-40a) and (1.4-40b) represent the same curve, a common solution for s'' would be restricted to some isolated points. Accordingly, such cases will be discarded. Should the above two equations represent the same curve, it would have to hold identically (for any s''):

$$(f_4^2 f_k^3 - f_4^3 f_k^2) f_s^{2\prime} = c\,[(f_4^3 f_k^1 - f_4^1 f_k^3)\, f_s^{2\prime} + (f_4^1 f_k^2 - f_4^2 f_k^1)\, f_s^{3\prime}]\,, \tag{1.4-41a}$$

$$(f_4^3 f_k^2 - f_4^2 f_k^3)\, f_s^{1\prime} + (f_4^2 f_k^1 - f_4^1 f_k^2)\, f_s^{3\prime} = c\,(f_4^1 f_k^3 - f_4^3 f_k^1)\, f_s^{1\prime}\,, \tag{1.4-41b}$$

$$(f_4^1 f_k^2 - f_4^2 f_k^1)\, f_s^{2\prime} = c\,(f_4^2 f_k^1 - f_4^1 f_k^2)\, f_s^{1\prime}\,. \tag{1.4-41c}$$

Since

$$f_4^1 f_k^2 - f_4^2 f_k^1 \neq 0\,,$$

(due to the earlier specification that $q_k^2 \neq 0$) it follows from (1.4-41c) that

$$f_s^{2\prime} = -c\, f_s^{1\prime}\,. \tag{1.4-42}$$

Apart from the cases when $y_s' = 0$ which was eliminated in (1.4-29b¹) due to $D \neq 0$, or $x_s' = x_k$, it follows, when (1.4-42) is substituted to either of (1.4-41a) or (1.4-41b), that

$$(f_4^2 f_k^3 - f_4^3 f_k^2)\, f_s^{1\prime} + (f_4^3 f_k^1 - f_4^1 f_k^3)\, f_s^{2\prime} + (f_4^1 f_k^2 - f_4^2 f_k^1)\, f_s^{3\prime} = 0\,. \tag{1.4-43}$$

This is the equation of a second order curve for s', passing through stations 1, 2, 3, 4, k (if the substitutions are made for f_s' according to (1.4-17a) - (1.4-17c), the equation (1.4-43) is fulfilled). With this specification regarding station s' made, the conditions for station s'' can be further examined. If $s'' \equiv 3$ is considered in (1.4-40a), then (1.4-43) is obtained (multiplied by $-f_k^2$)

and the curve (1.4-40a) is seen to pass through station 3. If $s'' \equiv 4$ is considered in the same equation, then two terms are obtained: one which was already shown to be zero in the substitution $s \equiv 3$, and the other which yields again (1.4-43), multiplied by $-f_4^2$. Similar conclusions hold if the equation (1.4-40b) is used with the same substitutions for s'' (leading to (1.4-43) multiplied by f_k^1 and f_4^1). Thus it has been shown that for (1.4-39) to hold, both stations s' and s'' would have to lie on the same second order curve passing through stations 1, 2, 3, 4, k (except for some special cases which were discarded). In other words, $|M| = 0$ would hold identically for any X, Y, Z if s' and s'' were on-curve stations.

Next, an investigation will be made in order to determine whether there are some more general conditions than those expressed above, under which $|M| = 0$ would hold identically for any X, Y, Z. Such conditions would imply that

$$a = b = c = d = 0 \tag{1.4-44}$$

in the equation of a plane for X, Y, Z, given as

$$aX + bY + cZ + d = 0 .$$

First, the following notations will be made:

$$\bar{a}_{11} = u_1(X_1 - x_k) + v_1(Y_1 - y_k) + q_k^2 , \tag{1.4-45a}$$

$$\bar{a}_{12} = u_1(X_2 - x_k) + v_1(Y_2 - y_k) + q_k^2 , \tag{1.4-45b}$$

$$\bar{a}_{21} = u_2(X_1 - x_k) + v_2(Y_1 - y_k) + q_k^3 , \tag{1.4-45c}$$

$$\bar{a}_{22} = u_2(X_2 - x_k) + v_2(Y_2 - y_k) + q_k^3 . \tag{1.4-45d}$$

The conditions (1.4-44) yield respectively, upon considering the equation (1.4-38a):

$$u_1(\bar{a}_{21}Z_2 - \bar{a}_{22}Z_1) = u_2(\bar{a}_{11}Z_2 - \bar{a}_{12}Z_1)\ ,$$

$$v_1(\bar{a}_{21}Z_2 - \bar{a}_{22}Z_1) = v_2(\bar{a}_{11}Z_2 - \bar{a}_{12}Z_1)\ ,$$

$$\bar{a}_{11}\bar{a}_{22} = \bar{a}_{12}\bar{a}_{21}\ ,$$

$$q_k^2(\bar{a}_{21}Z_2 - \bar{a}_{22}Z_1) = q_k^3(\bar{a}_{11}Z_2 - \bar{a}_{12}Z_1)\ .$$

When these four equations are further developed, the first combined with the hird, and the relations (1.4-45a) - (1.4-45d) considered, the following results re obtained respectively:

$$[(Y_1-y_k)Z_2 - (Y_2-y_k)Z_1](u_1v_2-v_1u_2)+(Z_2-Z_1)(u_1q_k^3 - q_k^3u_2) = 0\ , \qquad (1.4\text{-}46a)$$

$$[(X_1-x_k)Z_2 - (X_2-x_k)Z_1](u_1v_2-v_1u_2)+(Z_1-Z_2)(v_1q_k^3 - q_k^2v_2) = 0\ , \qquad (1.4\text{-}46b)$$

$$[(X_1-x_k)(Y_2-y_k)-(Y_1-y_k)(X_2-x_k)](u_1v_2-v_1u_2)+(X_1-X_2)(u_1q_k^3-q_k^3u_2)+$$
$$+(Y_1-Y_2)(v_1q_k^3 - q_k^2v_2) = 0\ , \qquad (1.4\text{-}46c)$$

$$[(X_1-x_k)Z_2-(X_2-x_k)Z_1](u_1q_k^3-q_k^2u_2)+[(Y_1-y_k)Z_2-(Y_2-y_k)Z_1]\times$$
$$\times(v_1q_k^3-q_k^2v_2) = 0\ . \qquad (1.4\text{-}46d)$$

$Z_1 = Z_2$, the discussion is relatively simple. From (1.4-46a) and (1.4-46b) is obtained:

$$(Y_1 - Y_2)(u_1v_2 - v_1u_2) = 0$$

d

$$(X_1 - X_2)(u_1v_2 - v_1u_2) = 0\ ;$$

m these, it holds in general that

$$\frac{u_2}{u_1} = \frac{v_2}{v_1}\ .$$

her of the equations (1.4-46c) or (1.4-46d) yields in general:

$$\frac{q_k^3}{q_k^2} = \frac{v_2}{v_1} \quad .$$

Considering the last two relations, (1.4-39) is again found to be the condition which makes $|M| = 0$ hold identically for any X, Y, Z. However, in general $Z_1 \neq Z_2$. From the equations (1.4-46a) and (1.4-46b), it is obtained:

$$u_1 q_k^3 - q_k^2 u_2 = -c_y (u_1 v_2 - v_1 u_2) , \tag{1.4-47a}$$

$$u_1 q_k^3 - q_k^2 v_2 = c_x (u_1 v_2 - v_1 u_2) , \tag{1.4-47b}$$

where

$$c_x = -\frac{a_x}{Z_2 - Z_1} \quad , \quad c_y = -\frac{a_y}{Z_2 - Z_1} \quad , \tag{1.4-47c}$$

and where a_x and a_y were defined by (1.4-14a) and (1.4-14b). The equations (1.4-46c) and (1.4-46d) are fulfilled automatically when (1.4-47a) and (1.4-47b) are used, and therefore, do not have to be considered anymore. Finally, (1.4-47a) and (1.4-47b) can be rewritten as

$$u_1 (q_k^3 + c_x v_2) = u_2 (q_k^2 + c_y v_1) \tag{1.4-48a}$$

and

$$v_1 (q_k^3 + c_x u_2) = v_2 (q_k^2 + c_x u_1) \quad . \tag{1.4-48b}$$

Clearly, these two equations hold identically, whenever

$$u_2 = c u_1 , \quad v_2 = c v_1 , \quad \text{and} \quad q_k^3 = c q_k^2 ,$$

which is the familiar expression (1.4-39). Otherwise, each of them can be shown to represent a fourth order curve for station s''; in general, they are both fulfilled for station s'' located at their intersections. Since these are isolated points, they will be discarded from further discussion. However, before dismissing the considerations connected with the fourth order curves of (1.4-48a) and (1.4-48b), it will have to be found out when these curves may coincide. They can be both written in terms of $f_s^{1\prime\prime}$, $f_s^{2\prime\prime}$, $f_s^{3\prime\prime}$, $(f_s^{1\prime\prime})^2$, $f_s^{1\prime\prime} f_s^{2\prime\prime}$, etc.; in order that they coincide the coefficients of the same powers in the variables, and therefore also in the terms f_s'', $f_s'' \times f_s''$, would have to be constant multiples of each other. When only the terms $f_s'' \times f_s''$ are considered with c denoting the multiplication constant for the above coefficients, relations of the following type would have to hold:

$$c_y f_4^1 [-f_4^2 f_s^{2\prime} f_s^{3\prime} + f_4^3 (f_s^{2\prime})^2] = c\, c_x [f_4^3 (f_s^{2\prime})^2 - f_4^2 f_s^{2\prime} f_s^{3\prime}] \tag{1.4-48c}$$
$$\vdots$$

Altogether, there are five equations of this type, pertaining respectively to the coefficients of $(f_s^{1\prime\prime})^2$, $f_s^{1\prime\prime} f_s^{2\prime\prime}$, $f_s^{1\prime\prime} f_s^{3\prime\prime}$, $(f_s^{2\prime\prime})^2$, and $f_s^{2\prime\prime} f_s^{3\prime\prime}$, since there are no terms $(f_s^{3\prime\prime})^2$ in either of (1.4-48a), (1.4-48b). If

$$c_x \neq 0 , \tag{1.4-48d}$$

then

$$c = \frac{c_y}{c_x} f_4^1$$

satisfies all five relations (1.4-48c). Considering (1.4-47c), the constant c can be written as

$$c = \frac{a_y}{a_x} f_4^1 . \tag{1.4-48e}$$

Under certain circumstances (namely, when x_k has a specific value of $(Z_1 X_2 - X_1 Z_2)/(Z_2 - Z_1)$), it could happen that

$$c_x = a_x = 0 . \tag{1.4-48f}$$

But then the relations (1.4-48c) would finally lead to the following conditions for s':

$$f_4^3 f_{s'}^1 + 0 - f_4^1 f_{s'}^3 = 0 ,$$

$$f_4^2 f_{s'}^1 - f_4^1 f_{s'}^2 + 0 = 0 .$$

These represent two second order curves for s'. Since the curves cannot coincide, s' would be restricted to isolated points (intersections of the two curves); such cases are discarded according to the earlier statements. Consequently, only the expression (1.4-48e) will be considered in further investigations. Next, the coefficients of $f_{s''}^1$, $f_{s''}^2$, and $f_{s''}^3$ of (1.4-48a) and (1.4-48b) will be compared. Using the same c as in (1.4-48c), it has to hold:

$$q_k^3 f_4^2 f_{s'}^2 - q_k^2 f_4^3 f_{s'}^2 = c\,(q_k^3 f_{s'}^2 - q_k^2 f_{s'}^3) , \tag{1.4-48g}$$

$$q_k^3 f_4^2 f_{s'}^1 + q_k^2 f_4^1 f_{s'}^3 - q_k^2 f_4^3 f_{s'}^1 = c\, q_k^3 f_{s'}^1 , \tag{1.4-48h}$$

and

$$q_k^2 f_4^1 f_{s'}^2 = c\, q_k^2 f_{s'}^1 . \tag{1.4-48i}$$

For q_k^2 such, that

$$q_k^2 \neq 0 , \tag{1.4-48j}$$

the equation (1.4-48i) gives

$$c = \frac{f_4^1 f_{s'}^2}{f_{s'}^1} . \tag{1.4-48k}$$

In this approach it is assumed that $x_{s'} \neq x_k$; then $f_{s'}^1 \neq 0$, since $y_{s'} \neq 0$ is assumed throughout. With this c substituted in (1.4-48g) and (1.4-48h), the following relation is obtained for station s':

$$(q_k^3 f_4^2 - q_k^2 f_4^3)\, f_{s'}^1 + (-q_k^3 f_4^1)\, f_{s'}^2 + (q_k^2 f_4^1)\, f_{s'}^3 = 0 . \tag{1.4-48l}$$

n the other hand, the constant c from either (1.4-48e) or (1.4-48k) must e the same. Consequently,

$$a_y f^1_{s'} - a_x f^2_{s'} = 0 \qquad (1.4\text{-}48m)$$

ust hold. Since $y_{s'} \neq 0$, this expresses an equation of a straight line for ', passing through k and having the direction given by

$$\operatorname{tg} \alpha = \frac{a_y}{a_x}$$

ee earlier notation); as a matter of fact, this line is exactly the line ℓ, enerated by intersection of the plane π and the plane of ground stations. nce (1.4-48l) and (1.4-48m) both give the conditions for station s', it ould restrict s' to isolated points unless the two loci coincide. But for at to occur, it would be necessary that

$$q^3_k f^2_4 - q^2_k f^3_4 = \bar{c}\, a_y ,$$

$$q^3_k f^1_4 = \bar{c}\, a_x ,$$

ıd

$$q^2_k f^1_4 = 0$$

satisfied (for some constant $\bar{c}$), giving immediately

$$q^2_k = 0 \qquad (1.4\text{-}48n)$$

ıd

$$a_x f^2_4 - a_y f^1_4 = 0 . \qquad (1.4\text{-}48o)$$

'he case $q^2_k = q^3_k = 0$ had been discarded.) However, with $q^2_k = 0$ the uations (1.4-48g) - (1.4-48i) hold identically for any s' if

$$c = f^2_4 ,$$

which then replaces (1.4-48k). Equating this constant c with the one given by (1.4-48e), one gets again the expression (1.4-48o). Since $y_4 \neq 0$, (1.4-48o) can be finally written as

$$a_y x_4 - a_x y_4 - (a_y x_3 - a_x y_3) = 0 ,$$

which is an equation of a straight line for station 4, passing through station 3. Its direction is given as

$$\operatorname{tg}\alpha = \frac{a_y}{a_x} ,$$

which means that the line is parallel to the line ℓ. On the other hand, $q_k^2 = 0$ implies that stations 3, 4, and k are lying on a straight line. Consequently, stations 3, 4, and k would have to lie on the line ℓ. Under these conditions the 4th order curve for s'' would be the same, whether given by (1.4-48a), or (1.4-48b). However, this would be a very special case. An important conclusion can now be made: except for some special cases, $|M| = 0$ would hold identically for any X, Y, Z only if the relation (1.4-39) were satisfied, i. e., only if both s' and s'' were on-curve stations.

Case (a1). According to the above conclusion, it is clear that $|M| = 0$ does not hold identically for any arbitrary plane. It holds only for a special plane, called critical plane, which can be computed from (1.4-38a) and avoided. Matrices M and $\tilde{\tilde{A}}$ will then be non-singular. The equation of the critical plane is given as

$$aX + bY + cZ + d = 0 \tag{1.4-49}$$

where the coefficients are found from (1.4-38a); they are:

$$a = (\bar{a}_{21} u_1 - \bar{a}_{11} u_2) Z_2 + (\bar{a}_{12} u_2 - \bar{a}_{22} u_1) Z_1 ,$$

$$b = (\bar{a}_{21} v_1 - \bar{a}_{11} v_2) Z_2 + (\bar{a}_{12} v_2 - \bar{a}_{22} v_1) Z_1 ,$$

$$c = \bar{a}_{11} \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21} , \tag{1.4-50}$$

$$d = [\bar{a}_{11} (x_k u_2 + y_k v_2 - q_k^3) - \bar{a}_{21} (x_k u_1 + y_k v_1 - q_k^2)] Z_2 + [\bar{a}_{22} (x_k u_1 + y_k v_1 - q_k^2) -$$

$$- \bar{a}_{12} (x_k u_2 + y_k v_2 - q_k^3)] Z_1 .$$

'ase (a2). Suppose that the plane of j_k passes through station k, i.e., ; is the plane π. It will be examined under what circumstances the ritical plane generated by $|M| = 0$ and the plane π can coincide assum- ng now that j_k does not contain all the targets in a straight line. When ney are avoided, matrices M and $\tilde{\tilde{A}}$ will again be non-singular. With $x_k, y_k, 0)$ substituted for (X, Y, Z) in the equation (1.4 - 38a), it would ave to hold that

$$\left| \begin{array}{c|c|c} Z_1 & u_1 (X_1 - x_k) + v_1 (Y_1 - y_k) + q_k^2 & u_2 (X_1 - x_k) + v_2 (Y_1 - y_k) + q_k^3 \\ Z_2 & u_1 (X_2 - x_k) + v_1 (Y_2 - y_k) + q_k^2 & u_2 (X_2 - x_k) + v_2 (Y_2 - y_k) + q_k^3 \\ 0 & q_k^2 & q_k^3 \end{array} \right| = 0 ,$$

hould the plane $|M| = 0$ and the plane π coincide. (q_k^2 and q_k^3 can be liminated from the first two rows by equivalence operations.) This is quivalent to the equation

$$a_x a_u + a_y a_v = 0 \tag{1.4-51}$$

here

$$a_u = u_1 q_k^3 - u_2 q_k^2 , \tag{1.4-51a}$$

$$a_v = v_1 q_k^3 - v_2 q_k^2 , \tag{1.4-51b}$$

nd where a_x and a_y were given by (1.4-14a) and (1.4-14b).

Before considering (1.4-51) in general, some special cases, the most important being

$$a_u = a_v = 0 ,$$

will be investigated first. If

$$a_u = 0 ,$$

it is obtained upon considering (1.4-32) and (1.4-33):

$$(q_k^3 f_4^2 f_{s'}^2 - q_k^2 f_4^3 f_{s'}^2) f_{s''}^1 + [-q_k^3 f_4^2 f_{s'}^1 - q_k^2 (f_4^1 f_{s'}^3 - f_4^3 f_{s'}^1)] f_{s''}^2 + q_k^2 f_4^1 f_{s'}^2 f_{s''}^3 = 0 \qquad (1.4\text{-}52)$$

which represents a second order curve for s''. Using the now standard approach of examining through which points a second order curve passes, it is seen that the curve given by the above equation passes through station 1, 2, k, and s'. (The same curve is obtained in terms of s', i.e., when all the elements associated with s' and s'' are interchanged; the curve would then pass through 1, 2, k, and s''.) Using certain previous stipulations (i.e., $f_4^1 \neq 0$ and eliminating of the conditions ($1.4\text{-}29b^1$) which cause $D = 0$) and discarding cases when some of the stations would be restricted to isolated points, only one special case fulfilling (1.4-52) may arise; it is the case when stations 3, 4, and k lie in a straight line parallel to the line connecting stations 1, 2; then $q_k^2 = f_4^2 = 0$, and the equation is satisfied identically for any s', s''. Otherwise, (1.4-52) represents a general second order curve. When

$$a_v = 0$$

is considered, it is similarly obtained:

$$(q_k^2 f_{s'}^3 - q_k^3 f_{s'}^2) f_{s''}^1 + q_k^3 f_{s'}^1 f_{s''}^2 - q_k^2 f_{s'}^1 f_{s''}^3 = 0 \quad . \qquad (1.4\text{-}53)$$

The same comments can be made with respect to this equation as those made when (1.4-52) was considered, except that now (1.4-53) is not satisfied identically when $q_k^2 = f_4^2 = 0$. Should the condition

$$a_u = a_v = 0$$

be fulfilled, station s'' would be in general restricted to some isolated points, generated by simultaneous solutions of (1.4-52) and (1.4-53); they are intersections of two second order curves represented by these equations. Again, such cases are discarded unless the two curves coincide; if that is the case, the corresponding coefficients in (1.4-52) and (1.4-53) are compared the same way, which lead to equations (1.4-48g) - (1.4-48i) and exactly these same equations are obtained now. In this manner one arrives again at the equation (1.4-48l). Since s' and s'' were mutually interchangeable, the same curve as that represented by (1.4-48l) would be obtained for either s' or s''; to simplify the derivations, any of these two stations will be denoted as s and the corresponding f-terms as $\bar{f}_s$. When q_k^2 and q_k^3 are substituted from (1.4-22) into (1.4-48l), the second order equation for s is obtained in the simpliest form:

$$(f_4^3 f_k^2 - f_4^2 f_k^3)\, f_s{}' + (f_4^1 f_k^3 - f_4^3 f_k^1)\, f_s{}^2 + (f_4^2 f_k^1 - f_4^1 f_k^2)\, f_s^3 = 0 .$$

The second order curve for s represented by this equation is seen to pass not only through stations 1, 2, k, but also through stations 3 and 4. Consequently, it can be said that when both stations s' and s'' are on-curve stations, then $a_u = a_v = 0$ holds.

Other special cases of (1.4-51) are easy to formulate and will be mentioned only briefly. If

$$a_x = a_v = 0,$$

then a constraint for the x-coordinate of station k is obtained, namely

$$x_k = \frac{Z_1 X_2 - X_1 Z_2}{Z_2 - Z_1} \; ; \tag{1.4-54}$$

in addition, station s'' has to lie on a second order curve given by (1.4-53). When $Z_1 = Z_2$, the condition (1.4-54) would be replaced by $X_1 = X_2$.

If

$$a_y = a_u = 0,$$

it holds similarly that

$$y_k = \frac{Z_1 Y_2 - Y_1 Z_2}{Z_2 - Z_1} \; ; \tag{1.4-55}$$

in addition, (1.4-52) would have to be fulfilled. If $Z_2 = Z_1$, (1.4-55) would be replaced by $Y_1 = Y_2$. Finally, if

$$a_x = a_y = 0,$$

then, k would be constrained to a prescribed point with the coordinates such as given by (1.4-54) and (1.4-55); as usual, this case is disregarded (if it also held that $Z_1 = Z_2$, this would mean that $j_{k_1} \equiv j_{k_2}$, which obviously is not true).

The general consideration of (1.4-51) again leads to a second order curve for station s''. It has the form:

$$[a_x (q_k^3 f_4^2 f_{s'}^2 - q_k^2 f_4^3 f_{s'}^2) + a_y (q_k^2 f_4^1 f_{s'}^3 - q_k^3 f_4^1 f_{s'}^2) f_{s''}^1 +$$

$$+ \{ a_x [-q_k^3 f_4^2 f_{s'}^1 - q_k^2 (f_4^1 f_{s'}^3 - f_4^3 f_{s'}^1)] + a_y q_k^3 f_4^1 f_{s'}^1 \} f_{s''}^1 +$$

$$+ (a_x q_k^2 f_4^1 f_{s'}^2 - a_y q_k^2 f_4^1 f_{s'}^1) f_{s''}^3 = 0 .$$

This curve passes through stations 1, 2, k, and s'. (Station s' and s'' could be again interchanged). Considering the earlier assumptions and disregarding certain special cases according to previously mentioned specifications, only e further special case will arise:

$$q_k^2 = 0$$

gether with

$$a_x f_4^2 - a_y f_4^1 = 0 \ ,$$

plying that stations 3, 4, k, lie in a straight line of the prescribed

rection

$$\mathrm{tg}\,\alpha = \frac{a_y}{a_x} \ ,$$

gardless of s' and s''. Obviously, it is the line ℓ. Consequently, the lation (1.4-51) is fulfilled identically for any s', s'', if stations 3, 4, k on the line ℓ. Otherwise, (1.4-51) leads to a second order curve for station s'' pressed as

$$x^T A x + x^T a = 0 \tag{1.4-56a}$$

th the elements:

$$
\begin{aligned}
a_{11} &= 1 \ , \\
a_{12} = a_{21} &= \frac{1}{2}\,\frac{G_1}{G} \ , \\
a_{22} &= \frac{G_2}{G} \ , \\
a_1 &= -x_2 \ , \\
a_2 &= \frac{G_3}{G} \ ,
\end{aligned}
\tag{1.4-56b}
$$

d

ere

$$G = q_k^2 f_4^1 (a_x f_s^{2\prime} - a_y f_s^{1\prime}) \ ,$$

$$G_1 = a_x (q_k^3 f_4^3 f_s^{2\prime} - q_k^2 f_4^3 f_s^{3\prime}) + a_y (q_k^2 f_4^1 f_s^{3\prime} - q_k^3 f_4^1 f_s^{2\prime}) \ ,$$

$$G_2 = a_x[-q_k^3 f_4^2 f_{s'}^1 - q_k^2(f_4^1 f_{s'}^3 - f_4^3 f_{s'}^1)] + a_y q_k^3 f_4^1 f_{s'}^1 ,$$

and

$$G_3 = a_x\{(-q_k^3 f_4^2 f_{s'}^2 + q_k^2 f_4^3 f_{s'}^2)x_k + [q_k^3 f_4^2 f_{s'}^1 + q_k^2(f_4^1 f_{s'}^3 - f_4^3 f_{s'}^1)]y_k - \quad (1.4\text{-}56c)$$

$$- q_k^2 f_4^1 f_{s'}^2 \frac{x_k}{y_k}(x_k - x_2)\} + a_y[(-q_k^2 f_4^1 f_{s'}^3 + q_k^3 f_4^1 f_{s'}^2)x_k - q_k^3 f_4^1 f_{s'}^1 y_k +$$

$$+ q_k^2 f_4^1 f_{s'}^1 \frac{x_k}{y_k}(x_k - x_2)] .$$

In conclusion, it can be said that the plane corresponding to $|M| = 0$ and the plane π coincide if station s'' lies on a second degree curve expressed by (1.4-56a) - (1.4-56c). Of the special cases, the most significant was the one characterized by stations s', s'' as being on-curve stations. But for such configuration $|M| = 0$ was fulfilled identically for any X, Y, Z and, therefore, it had to hold for plane π as well; this case thus provided a useful verification of the earlier derivations. A quite important special case also arised when stations 3, 4, k lie on the line ℓ. Finally, when station s'' does not lie on the above curve and none of the special cases occurs, $|M| = 0$ does not hold and M and $\tilde{\tilde{A}}$ matrices are non-singular.

<u>Case (b)</u>. Due to necessary conditions, the relation (1.4-27) will have to hold for at least one s station, i.e., the corresponding j_s will have to contain off-plane targets (in addition to j_4 and j_5). It will be examined under what circumstances this condition is also sufficient for non-singular $\tilde{\tilde{A}}$ matrix when exactly one such s station is observing off-plane targets. With this station denoted as s', it means that

$$\text{rank}\left[\tilde{v}_{s'}^x, \tilde{v}_{s'}^y, \tilde{v}_{s'}^z, \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right] = 4. \quad (1.4\text{-}57)$$

Due to the j_4 and j_5 (containing off-plane targets), it holds that

$$a_4^x = a_4^y = a_4^z = c_4 = 0\,, \tag{1 4-58a}$$

which are already expressed in (1.4-18), and

$$a_5^x = a_5^y = a_5^z = c_5 = 0\ . \tag{1.4-58b}$$

Using the first two rows in the system (1.4-11a), b_1 and b_2 can be expressed as follows:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = -\begin{bmatrix} f_4^1 & f_4^2 \\ f_5^1 & f_5^2 \end{bmatrix}^{-1} \begin{bmatrix} f_4^3 \\ f_5^3 \end{bmatrix} b_3\ . \tag{1.4-59}$$

For the unique solution to exist, it has to hold for the determinant (denoted by D as in previous sections) for the matrix to be inverted:

$$D \neq 0$$

where

$$D = f_4^1 f_5^2 - f_4^2 f_5^1\ . \tag{1.4-60a}$$

For $D = 0$ it would hold that

$$y_4 y_5 (x_4 - x_3)(y_5 - y_3) = y_4 y_5 (y_4 - y_3)(x_5 - x_3), \tag{1.4-60b}$$

which would be fulfilled if any of the following occurred:

1) $y_4 = 0$, station 4 lying on a straight line through stations 1, 2.

2) $y_5 = 0$, station 5 lying on a straight line through stations 1, 2. (1.4-60b′)

3) Stations 3, 4, 5 lying on a straight line (any direction).

These three conditions will be assumed eliminated. The solution of (1.4-59) is then given as

$$\begin{aligned} b_1 &= g b_3 , \\ b_2 &= h b_3 , \end{aligned} \tag{1.4-61}$$

where

$$\begin{aligned} g &= -\frac{1}{D}\left(f_4^3 f_5^2 - f_4^2 f_5^3\right) , \\ h &= -\frac{1}{D}\left(f_4^1 f_5^3 - f_4^3 f_5^1\right) . \end{aligned} \tag{1.4-62}$$

Expressing c_k and c_s' from the system (1.4-11a) and using (1.4-61), it is found that

$$\begin{aligned} c_k &= t_k b_3 , \\ c_s' &= t_s' b_3 \end{aligned}$$

where

$$t_k = g f_k^1 + h f_k^2 + f_k^3 , \tag{1.4-63a}$$

$$t_s' = g f_s^{1\prime} + h f_s^{2\prime} + f_s^{3\prime} . \tag{1.4-63b}$$

With these notations, the system (1.4-11) can be written as

$$\begin{aligned} & a_k^x \tilde{v}_k^x + a_k^y \tilde{v}_k^z + a_k^z \tilde{v}_k^z && + b_3 t_k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \\ & a_k^x p_s^{1\prime} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_k^y p_s^{2\prime} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + 0 + a_s^{x\prime} v_s^{x\prime} + a_s^{y\prime} v_s^{y\prime} + a_s^{z\prime} v_s^{z\prime} && + b_3 t_s' \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \end{aligned} \tag{1.4-64}$$

where the row blocks for some stations i and further stations s could be added in an obvious fashion.

At this point, it will be examined for later use under what conditions

$$t_k = 0 \tag{1.4-66}$$

and

$$t_s{}' = 0 \tag{1.4-66}$$

can hold. The condition (1.4-65) leads to a second degree equation in x_k, y_k, namely

$$(f_4^3 f_5^2 - f_4^2 f_5^3) f_k^1 + (f_4^1 f_5^3 - f_4^3 f_5^1) f_k^2 + (f_4^2 f_5^1 - f_4^1 f_5^2) f_k^3 = 0\ . \tag{1.4-67}$$

Since

$$\begin{aligned} f_k &= 0 \quad \text{whenever} \quad k \equiv 1, 2, 3, \\ f_k &= f_4 \quad \text{if } k \equiv 4, \end{aligned} \tag{1.4-68}$$

and

$$f_k = f_5 \quad \text{if} \quad k \equiv 5,$$

the second degree curve for station k corresponding to the above equation is seen to pass through stations 1, 2, 3, 4, 5. The condition (1.4-66) leads to

$$(f_4^3 f_5^2 - f_4^2 f_5^3) f_s^1{}' + (f_4^1 f_5^3 - f_4^3 f_5^1) f_s^2{}' + (f_4^2 f_5^1 - f_4^1 f_5^2) f_s^3{}' = 0. \tag{1.4-69}$$

It is clear that the second order curve for s' represented by this equation passes through stations 1, 2, k. If $f_s{}'$ were replaced by f_4 or f_5, the equation (1.4-69) would hold. Since (1.4-17c) would have similar form with respect to station 5, i.e.,

$$f_s = f_5 + b f_k\,, \quad b = -\frac{y_5}{y_k} \quad \text{if} \quad s \equiv 5\,,$$

it follows that (1.4-69) holds for $s' \equiv 4$ and $s' \equiv 5$ whenever it holds for $s' \equiv 3$; but this happens only if (1.4-67) is satisfied. Consequently, the second order curve for $t_s' = 0$ passes through stations 1, 2, 3, 4, 5 only if $t_k = 0$ holds. In this case the second order curves for s′ and k given by (1.4-6?) and (1.4-65) coincide. Otherwise the curve for s′, passing through 1, 2, k, is given by

$$x^T A x + x^T a = 0, \qquad (1.4\text{-}70a)$$

where

$$\begin{aligned} a_{11} &= 1 \quad , \\ a_{12} = a_{21} &= \frac{1}{2}\frac{D_1}{D} , \\ a_{22} &= \frac{D_2}{D} \quad , \\ a_1 &= -x_2 \quad , \\ a_2 &= \frac{D_1}{D} x_k - \frac{D_2}{D} y_k - \frac{x_k}{y_k}(x_k - x_2) \ , \end{aligned} \qquad (1.4\text{-}70b)$$

and where

$$\begin{aligned} D_1 &= f_4^2 f_5^3 - f_4^3 f_5^2 \quad , \\ D_2 &= f_4^3 f_5^1 - f_4^1 f_5^3 \quad . \end{aligned} \qquad (1.4\text{-}70e)$$

After these considerations a more general discussion of case (b) can resume. Due to (1.4-57), the solution of

$$\left[\ \tilde{v}_s^{x\prime} , \ \tilde{v}_s^{y\prime} , \ \tilde{v}_s^{z\prime} , \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \ \right] \begin{bmatrix} a_s^{x\prime} \\ a_s^{y\prime} \\ a_s^{z\prime} \\ c \end{bmatrix} = 0$$

is given as

$$a_s^{x\prime} = a_s^{y\prime} = a_s^{z\prime} = c = 0 \qquad (1.4\text{-}71)$$

where

$$c = a_k^x p_s^{1\prime} + a_k^y p_s^{2\prime} + b_3 t_s'$$

s seen from (1.4-64). From here, the coefficient b_3 will be substituted in he first row block of (1.4-64); namely,

$$b_3 = -\frac{1}{t_s'} \left(a_k^x p_s^1{}' + a_k^y p_s^2{}'\right) \tag{1.4-72}$$

nd

$$\overline{M} \begin{bmatrix} a_k^z \\ a_k^x \\ a_k^y \end{bmatrix} = 0 \tag{1.4-73a}$$

here

$$\overline{M} = [\widetilde{v}_k^z, \ \widetilde{v}_k^x - p_s^1{}' \frac{t_k}{t_s'} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \widetilde{v}_y - p_s^2{}' \frac{t_k}{t_s'}] . \tag{1.4-732b}$$

ecessarily, it has to hold that

$$t_s' \neq 0 \tag{1.4-74}$$

that s' cannot lie on the second order curve described in (1.4-69) through .4-70c).

Again, it is easy to show that when

$$\text{rank } \overline{M} = 3 \tag{1.4-75}$$

lds, $\tilde{\tilde{A}}$ matrix is non-singular. The only solution of (1.4-73a) is then

$$a_k^x = a_k^y = a_k^z = 0 ; \tag{1.4-75a}$$

en

$$b_1 = b_2 = b_3 = 0$$

ıst hold due to (1.4-72) and (1.4-61). Thus, all the c-terms are equal to ro, and since by necessary assumptions singularity A) is supposed to be eliminated, the a-terms associated with any i and s stations added to the system (1.4-64) would now have to be zero (for stations 4, 5, k, and s′ the trivial solution for the a-coefficients was already demonstrated in (1.4-58a), (1.4-58b), (1.4-75a), and (1.4-71), respectively). This completes the proof for the non-singularity of $\tilde{\tilde{A}}$ matrix.

It remains to examine the rank of matrix $\bar{M}$. If the conditions for

$$\text{rank } \bar{M} < 3 \tag{1.4-76}$$

are found and eliminated, then the sufficient conditions for non-singular $\tilde{\tilde{A}}$ have been specified. If (1.4-76) holds, then

$$|\bar{M}| = 0 , \tag{1.4-77}$$

which is

$$\begin{vmatrix} Z_1 & X_1 - x_k - p_s^{1\prime} \dfrac{t_k}{t_s{}'} & Y_1 - y_k - p_s^{2\prime} \dfrac{t_k}{t_s{}'} \\ Z_2 & X_2 - x_k - p_s^{1\prime} \dfrac{t_k}{t_s{}'} & Y_2 - y_k - p_s^{2\prime} \dfrac{t_k}{t_s{}'} \\ Z & X - x_k - p_s^{1\prime} \dfrac{t_k}{t_s{}'} & Y - y_k - p_s^{2\prime} \dfrac{t_k}{t_s{}'} \end{vmatrix} = 0 , \tag{1.4-78a}$$

with the same description as that made with respect to (1.4-38a); namely, the above relation is the equation of a plane in (X, Y, Z), which passes through j_{k_1} and j_{k_2}. With the same equivalence operations which lead to (1.4-38b), (1.4-78a) now becomes:

$$\begin{vmatrix} Z_1 & X_1 - x_k - p_s^{1\prime} \dfrac{t_k}{t_s{}'} & Y_1 - y_k - p_s^{2\prime} \dfrac{t_k}{t_s{}'} \\ Z_2 - Z_1 & X_2 - X_1 & Y_2 - Y_1 \\ Z - Z_1 & X - X_1 & Y - Y_1 \end{vmatrix} = 0 . \tag{1.4-78b}$$

Again, the second and third rows are dependent if any further target in j_k lies on a straight line connecting j_{k_1} and j_{k_2}. Consequently, $\widetilde{A}$ matrix is singular whenever all the targets in j_k lie on a straight line.

Next, the conditions will be determined under which $|M| = 0$ holds identically for any X,Y,Z. As seen earlier in (1.4-44), such conditions would imply that

$$a = b = c = d = 0 \tag{1.4-79}$$

in the equation (1.4-78a) of a plane for X,Y,Z, written as

$$aX + bY + cZ + d = 0.$$

It cannot happen when $Z_1 = Z_2$, since (1.4-79) would then lead to $j_{k_1} \equiv j_{k_2}$, which is not true. When $Z_1 \neq Z_2$, then the conditions in (1.4-79) imply that

$$x_k + p_s^{1\prime} \frac{t_k}{t_s{}'} = k_x$$

and (1.4-79a)

$$y_k + p_s^{2\prime} \frac{t_k}{t_s{}'} = k_y ,$$

where $k_x = \dfrac{X_1 Z_2 - Z_1 X_2}{Z_2 - Z_1}$ and $k_y = \dfrac{Y_1 Z_2 - Z_1 Y_2}{Z_2 - Z_1}$. (1.4-79b)

The relations in (1.4-79a) can be expressed as

$$[y_k(y_k - k_y)g]\, f_s^{1\prime} + [y_k(y_k - k_y)h - t_k]\, f_s^{2\prime} + [y_k(y_k - k_y)]\, f_s^{3\prime} = 0 \tag{1.4-80a}$$

and

$$[y_k(x_k - k_x)g - t_k]\, f_s^{1\prime} + [y_k(x_k - k_x)h]\, f_s^{2\prime} + [y_k(x_k - k_x)]\, f_s^{3\prime} = 0, \tag{1.4-80b}$$

which represent two second order curves for s'; in general, s' is restricted to some isolated locations, unless the two curves coincide. In the latter case it would have to hold according to the now standard procedure:

$$y_k(y_k - k_y)g = c[y_k(x_k - k_x)g - t_k],$$

$$y(y_k - k_y)h - t_k = c\, y_k(x_k - k_x)h,$$

$$y(y_k - k_y) = c\, y_k(x_k - k_x).$$

Whether

$$x_k = k_x \text{ and } y_k = k_y \tag{1.4-81}$$

holds or not, the above equations are fulfilled only when $t_k = 0$, i.e., when station k lies on a second order curve through stations 1,2,3,4,5. But, then (1.4-81) would have to hold due to (1.4-79a). These two conditions for **k** are either impossible or restricting station k to an isolated point and will be therefore discarded. Consequently, $|M| = 0$ cannot hold identically for arbitrary points X, Y, Z.

<u>Case (b1)</u>. According to the above conclusion, $|M| = 0$ holds only for a special plane, called critical plane, which can be computed from (1.4-78a) and avoided. Matrices $\overline{M}$ and $\tilde{\tilde{A}}$ will then be non-singular. The equation of the critical plane is given as

$$aX + bY + cZ + d = 0 \tag{1.4-82}$$

where the coefficients are found from (1.4-78a); they are:

$$a = Y_1Z_2 - Z_1Y_2 - (y_k + p_s^{2\prime}\frac{t_k}{t_s{}'})(Z_2 - Z_1),$$

$$b = Z_1X_2 - X_1Z_2 + (x_k + p_s^{1\prime}\frac{t_k}{t_s{}'})(Z_2 - Z_1),$$

$$c = X_1Y_2 - Y_1X_2 + (y_k + p_s^{2\prime}\frac{t_k}{t_s{}'})(X_2 - X_1) - (x_k + p_s^{1\prime}\frac{t_k}{t_s{}'})(Y_2 - Y_1),$$

$$d = (y_k + p_s^{2\prime}\frac{t_k}{t_s{}'})(X_1 Z_2 - Z_1 X_2) - (x_k + p_s{}'\frac{t_k}{t_s{}'})(Y_1 Z_2 - Z_1 Y_2). \qquad (1.4\text{-}82a)$$

ase(b 2). Suppose that the plane π is now the plane containing all the targets 1 j_k and that all the targets in j_k do not lie in a straight line. Should the plane $\bar{M}| = 0$ and the plane π coincide, it would have to hold that

$$\begin{vmatrix} Z_1 & X_1 - x_k & Y_1 - y_k \\ Z_2 & X_2 - x_k & Y_2 - y_k \\ 0 & -p_s^{1\prime}\frac{t_k}{t_s{}'} & -p_s^{2\prime}\frac{t_k}{t_s{}'} \end{vmatrix} = 0 \quad .$$

henever $y_s{}' = 0$ or $t_k = 0$, this relation holds, since the third row inside e determinant contains only zeros. In general ($y_s{}' \neq 0$, $t_k \neq 0$), the above quation holds if

$$a_x f_s^{2\prime} = a_y f_s^{1\prime}$$

satisfied, which yields:

$$x_s{}'a_y - y_s{}'a_x - (x_k a_y - y_k a_x) = 0 .$$

ut, this is the equation of a straight line for station s'; this line passes rough station k and its direction is given as

$$t g\alpha = \frac{a_y}{a_x} \quad ,$$

hich means that it is the line ℓ. Consequently, $|\overline{M}| = 0$ holds in any of e following three cases:

1) $y_s' = 0$, station s' lying on a straight line through stations 1,2.

2) Station s' lying on the line ℓ. (1.4-83)

3) Station k lying on a second order curve through stations 1,2,3,4,5 (i.e., $t_k = 0$).

Otherwise $\overline{M}$ and $\tilde{\tilde{A}}$ matrices are non-singular.

Case (c). The satellite groups j_4, j_5, j_6, are now all considered to contain off-plane targets. Therefore,

$$c_4 = c_5 = c_6 = 0 \tag{1.4-84}$$

holds, which is also true for the corresponding a-terms. If station 6 lies on a second order curve through stations 1,2,3,4,5, then (1.4-84) alone would not imply that all three b-terms in the system (1.4-11a) must be zero. It would then be possible to express b_1 and b_2 in terms of b_3 and the analysis would be carried out the same way as it was done for case (b). Consequently, it will be assumed that station 6 is not lying on a second order curve through stations 1,2,3,4,5. This gives for the b-coefficients:

$$b_1 = b_2 = b_3 = 0$$

as the only possibility, making all the c-terms equal to zero. With singularity A) eliminated as a necessary condition, all the a-terms for stations i preceding station k are zeros. The system (1.4-11) is thus reduced to

$$\begin{aligned} a_k^x \tilde{v}_k^x + a_k^y \tilde{v}_k^y + a_k^z \tilde{v}_k^z &= 0 \\ a_k^x p_s^1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_k^y p_s^2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + 0 + a_s^x \tilde{v}_s^x + a_s^y \tilde{v}_s^y + a_s^z \tilde{v}_s^z &= 0 \end{aligned} \tag{1.4-85}$$

where any number of s-stations can be used.

Case (c 1). When the targets in j_k are not in plane with station k, then it holds that

$$\operatorname{rank} [\tilde{v}_k^x , \tilde{v}_k^y , \tilde{v}_k^z] = 3 , \qquad (1.4\text{-}86)$$

which forces the three a-terms for station k to be zero. Due to singularity A) eliminated, also a-terms for all s-stations must be zero which thus brings all the a-terms and b-terms to zero. Accordingly, $\tilde{\tilde{A}}$ matrix is always non-singular in this case.

Case (c 2). When the targets in j_k lie in the plane π (as a special case they can also lie in a straight line), the relation(1.4-86) does not hold and a-terms for station k are not automatically equal to zero. If the rank of the second group in (1.4-85) were less than four, the system (1.4-85) would not have to have the trivial solution and $\tilde{\tilde{A}}$ would be singular. Thus, at least one of the groups j_s will have to contain off-plane targets. If the corresponding s-station is such that it lies in a straight line with stations 1 and 2, namely

$$y_s = 0 ,$$

then it holds that

$$p_s^1 = p_s^2 = 0 \qquad (1.4\text{-}87)$$

and the a-coefficients for this station must be zeros. But that leaves the a-coefficients for station k unchanged (not necessarily zeros). Therefore, it will be assumed that

$$y_s \neq 0 ,$$

which means that (1.4-87) cannot hold (unless $s \equiv k$ which is not true). Since s is assumed off-plane, it holds that

$$a_s^x = a_s^y = a_s^z = c = 0 \qquad (1.4\text{-}88a)$$

where

$$c = a_k^x p_s^1 + a_k^y p_s^2 \; ; \qquad (1.4\text{-}88b)$$

it is further assumed that

$$p_s^1 \neq 0 , \quad p_s^2 \neq 0 \; ;$$

otherwise the following proof would be slightly different and somewhat shorter. From (1.4-88a) and (1.4-88b) it follows that

$$a_k^y = -a_k^x \frac{x_s - x_k}{y_s - y_k} \; ; \qquad (1.4\text{-}89)$$

when substituted in the first row block of (1.4-85), this yields

$$\overline{\overline{M}} \begin{bmatrix} a_k^x \\ a_k^z \end{bmatrix} = 0 \qquad (1.4\text{-}90a)$$

where

$$\overline{\overline{M}} = [\tilde{v}_k^x - \frac{x_s - x_k}{y_s - y_k} \tilde{v}_k^y , \quad \tilde{v}_k^z \;] , \qquad (1.4\text{-}90b)$$

or

$$\overline{\overline{M}} = \left[\begin{array}{c|c} X_1 - x_k - \frac{x_s - x_k}{y_s - y_k}(Y_1 - y_k) & Z_1 \\ X_2 - x_k - \frac{x_s - x_k}{y_s - y_k}(Y_2 - y_k) & Z_2 \\ \vdots & \\ X - x_k - \frac{x_s - x_k}{y_s - y_k}(Y - y_k) & Z \end{array}\right] ; \qquad (1.4\text{-}90c)$$

here, the coordinates (X, Y, Z) can denote any point in π.

Clearly, when it holds that

$$\text{rank } \overline{\overline{M}} = 2 , \qquad (1.4\text{-}91)$$

then matrix $\tilde{\tilde{A}}$ is non singular. It follows from (1.4-90a) and (1.4-89) which bring the a-terms for station k to zero, from (1.4-88a) for station s, and from the fact that singularity A) was eliminated (for any further station s).

It is then sufficient to specify when (1.4-91) does not hold and to avoid such cases; namely, it has to be found when the determinant of an (2 x 2) submatrix of $\overline{\overline{M}}$ is equal to zero. For the first such submatrix, one would obtain that

$$[X_2 - x_k - \frac{x_s - x_k}{y_s - y_k}(Y_2 - y_k)]\ Z_1 = [X_1 - x_k - \frac{x_s - x_k}{y_s - y_k}(Y_1 - y_k)]\ Z_2\ ,$$

or, after some algebraic manipulations:

$$\begin{vmatrix} X_1 - x_k & Y_1 - y_k & Z_1 \\ X_2 - x_k & Y_2 - y_k & Z_2 \\ x_s - x_k & y_s - y_k & 0 \end{vmatrix} = 0\ ,$$

which stipulates that station s lies on the line ℓ. The same results would be obtained for any other point in π replacing j_{k_1} or j_{k_2}; in other words, with s on the line ℓ the above determinant is equal to zero with any two targets in π.

As seen above, if s is such that it lies in line with stations 1 and 2, or on the line ℓ, then the fact that j_s contains off-plane targets is not sufficient to make $\tilde{\tilde{A}}$ matrix non-singular. In order to achieve it, still another satellite group would have to contain off-plane targets. As demonstrated earlier, any such group would be of no help if $y_s = 0$ held for the corresponding station. Therefore, only $y_s \neq 0$ and the corresponding j_s will be considered. Suppose that two such satellite groups are denoted as $j_{s'}$ and $j_{s''}$. If any one of the stations s' or s'' is such that it lies on the line ℓ, the corresponding group j_s alone can be of no help to make $\tilde{\tilde{A}}$ non-singular, as seen from the above derivation. If both stations lie on ℓ, then, necessarily,

$$\frac{x_s'' - x_k}{y_s'' - y_k} = \frac{x_s' - x_k}{y_s' - x_k}$$

holds and the relation between a_k^x and a_k^y is unchanged; it remains such as given by (1.4-89). The solution for the a-coefficients for station k is then the same, whether one or both stations s are used. This means that no matter how many s-stations have their satellite groups off-plane, as long as they all lie on the line ℓ their contribution cannot make $\tilde{\tilde{A}}$ matrix non-singular.

Consequently, $\bar{\bar{M}}$ and $\tilde{\tilde{A}}$ matrices are non-singular if there exists such s station observing off-plane satellites, that it is not lying on the line connecting stations 1 and 2, or on the line ℓ. (Using the earlier terminology, one could also say that singularity A) for station k should be eliminated with respect to that particular station s.)

1.413 Summary for Group j_k Containing "In-Plane Targets".

In this section, several most important conclusions will be repeated in order to summarize the whole of section 1.412 which deals with the group j_k containing only in-plane targets. The necessary conditions for non-singular $\tilde{\tilde{A}}$ matrix were such that at least two satellite groups besides j_4 had to contain off-plane targets, that singularity B) did not occur (all stations lying on one second order curve), and that singularity A) was eliminated (it would occur if for stations other than k the corresponding satellite groups lay in one plane through that station, and for station k, besides the case of all targets in j_k and j_s lying in one plane through k called plane π, if in addition to the targets in j_k lying in π all stations s would lie on the line ℓ generated as intersection of the plane π and the plane of ground stations). To stipulate sufficient conditions in order that $\tilde{\tilde{A}}$ matrix be non-singular, the distinction was made whether the group j_k lay in a general plane, or in the plane π (passing through station k and including the configuration when all the targets in j_k lay in a straight line). When a general plane was considered the cases were examined with the following satellite groups off-plane: j_4, j_s', and j_s'' called case (a1), j_4, j_5, and j_s' called case (b1), and j_4, j_5, and j_6 (station 6 not lying on the second order curve through stations 1,2,3,4, and 5) called case (c1). Assuming that singularity A) does not exist, the necessary and sufficient

;onditions for $\widetilde{\widetilde{A}}$ matrix to be non-singular were found to be:

In case (a1) to avoid such plane for j_k as given by (1.4-49) and (1.4-50), and of special cases to avoid s' and s'' being both on-curve stations (i.e., lying on the second order curve through stations 1,2,3,4, and k).

In case (b1) to avoid such plane for j_k as given by (1.4-82) and (1.4-82a); further, to avoid station s' lying on the second order curve through stations 1,2,3,4, and 5, in case that station k lies on it, otherwise the curve to be avoided for s' is given by (1.4-70a) - (1.4-70c).

In case (c1) no further specifications were necessary.

When the targets of j_k lay in the plane π, the same satellite groups as above 'ere examined and the analysis correspondingly divided into case (a2), case (b2), nd case (c2) with the same stipulation for station 6. Assuming again that singu- ırity A) does not exist, the corresponding necessary and sufficient conditions ere found to be:

In case (a2) to avoid station s'' lying on a second order curve expressed by (1.4-56a) - (1.4-56c) and the special case with stations 3,4 and k lying on the line ℓ; a special configuration which has to be avoided for any plane (not only π) was already mentioned in case (a1) as s' and s'' being both on-curve stations. Another special configuration to be avoided is the one with all the targets in j_k lying on a straight line.

In case (b2) to avoid station s' lying on a straight line through stations 1 and 2, or lying on the line ℓ, and to avoid station k lying on a second order curve through stations 1,2,3,4, or 5; a special configuration to be avoided is when all the targets in j_k lie in a straight line.

In case (c2) to avoid station s - one further station which is required to observe off-plane targets due to j_k lying in the plane π whether all the targets in j_k form a straight line or not — lying on a straight line through stations 1 and 2, or lying on the line ℓ. If only the groups j_4, j_5, and j_6 contain off-plane targets the problem is singular in this case.

1.42 Two Replacements: Stations 3 and 2 Replaced by Stations k and s'.

The difference between this section and section 1.41 consists in further replacements of stations at the level of s-stations. For some station s denoted as s' (co-observing with stations 1,2, and k), any further s-stations will be assumed to co-observe with stations 1, k, and that station s', which thus effectively replaced station 2. The following derivations will make use of some results obtained in section 1.41; building of $\tilde{\tilde{A}}$ matrix will be basically the same up to and including station s'. Of further s-stations, one denoted as s'' will be considered in the following derivations. Naturally, it will differ from station s'' dealt with in previous sections in that s'' is now co-observing with stations 1, k, and s' rather than with stations 1, 2, and k. The approach in this section will again consist of eliminating the parameters associated with the satellite group $j_{s''}$ using observations from stations 1, k and s'. The now standard procedure for obtaining matrix $\tilde{\tilde{A}}$ and examining its rank will be used: first, the necessary conditions for non-singular $\tilde{\tilde{A}}$ matrix will be analyzed with no specifications for the satellite groups j_k and $j_{s'}$; then in addition to the usual assumption about j_4 containing off-plane targets, only the practical cases will be examined, namely when the targets in j_k and $j_{s'}$ are also lying off-plane; finally, the sufficient conditions for non-singular $\tilde{\tilde{A}}$ matrix with the above assumptions will be formulated.

The parameters associated with $j_{s''}$ will be eliminated using the relations resulting from stations 1, k, and s' observing the targets in $j_{s''}$. When considering the first two stations (1 and k), two equations similar to those given in section

1.41 are obtained. Due to the observations from station s' (all stations are lying in a plane), it holds that

$$(X_{j_s''} - x_s')(\partial X_{j_s''} - \partial x_s') + (Y_{j_s''} - y_s')(\partial Y_{j_s''} - y_s') + Z_{j_s''}(\partial Z_{j_s''} - \partial z_s') = 0 \quad (1.4\text{-}92)$$

where j_s'' stands for any target in the satellite group j_s''. All three equations can be arranged in a matrix form as follows:

$$\begin{bmatrix} X_{j_s''} & Y_{j_s''} & Z_{j_s''} \\ X_{j_s''}-x_k & Y_{j_s''}-y_k & Z_{j_s''} \\ X_{j_s''}-x_s' & Y_{j_s''}-y_s' & Z_{j_s''} \end{bmatrix} \begin{bmatrix} \partial X_{j_s''} \\ \partial Y_{j_s''} \\ \partial Z_{j_s''} \end{bmatrix} = \begin{bmatrix} 0 \\ (X_{j_s''}-x_k)\,\partial x_k + (Y_{j_s''}-y_k)\,\partial y_k + Z_{j_s''}\partial z_k \\ (X_{j_s''}-x_s')\partial x_s' + (Y_{j_s''}-y_s')\partial y_s' + Z_{j_s''}\partial z_s' \end{bmatrix} .$$

The determinant of the (3×3) matrix in this expression is given as

$$D = Z_{j_s''}(x_k y_s' - y_k x_s').$$

In order that $D \neq 0$ holds, the cases with $Z_{j_s''} = 0$ and with stations 1, k, and s' lying in a straight line have to be eliminated (this in addition to the earlier stipulations: $x_2 \neq 0$, $y_3 \neq 0$, $y_k \neq 0$, and $Z \neq 0$ for all previous satellite groups). The satellite parameters are then obtained as

$$\partial X_{j_s''} = \frac{1}{x_k y_s' - y_k x_s'} \, (-y_s' T_k + y_k T_s'),$$

$$\partial Y_{j_s''} = \frac{1}{x_k y_s' - y_k x_s'} \, (x_s' T_k - x_k T_s'),$$

and

$$\partial Z_{j_s''} = \frac{1}{Z_{j_s''}(x_k y_s' - y_k x_s')} \, [(X_{j_s''} y_s' - Y_{j_s''} x_s') T_k + (Y_{j_s''} x_k - X_{j_s''} y_k) \, T_s'],$$

where

$$T_k = (X_{j_s''} - x_k)\partial x_k + (Y_{j_s''} - y_k)\partial y_k + Z_{j_s''}\partial z_k$$

and

$$T_s' = (X_{j_s''} - x_s')\partial x_s' + (Y_{j_s''} - y_s')\partial y_s' + Z_{j_s''}\partial z_s'.$$

Using these expressions in (1.4-92), the following equation is obtained after some algebraic manipulations:

$$- (x_s'y_s''-y_s'x_s'')\left[(X_{J_s}''-x_k)\partial x_k + (Y_{J_s}''-y_k)\partial y_k + Z_{J_s}''\partial z_k\right] +$$
$$+ (x_k y_s''-y_k x_s'')\left[(X_{J_s}''-x_s')\partial x_s' + (Y_{J_s}''-y_s')\partial y_s' + Z_{J_s}''\partial z_s'\right] - \qquad (1.4\text{-}92a)$$
$$- (x_k y_s'-y_k x_s')\left[(X_{J_s}''-x_s'')\partial x_s'' + (Y_{J_s}''-y_s'')\partial y_s'' + Z_{J_s}''\partial z_s''\right] = 0.$$

Now it is possible to form $\widetilde{A}$ matrix: all the row blocks up to and including the one for station s' are the same as in Table (1.4-1), where the z-coordinates for all stations are set to zero and the notation s' replaces s; the row block for station s'' is represented by (1.4-92a). $\widetilde{A}$ matrix for two replacements with ground stations lying in one plane is presented in Table (1.4-4).

To further simplify $\widetilde{A}$ matrix the following equivalence operations will be performed:

(1) Divide each row pertaining to stations up to and including station k by the corresponding $Z_J \neq 0$.

(2) Multiply each of the last three columns by -1.

(3) Perform on the three column block of station s':

$$\partial x_s' \to \partial x_s' + P\partial x_s'', \quad \partial y_s' \to \partial y_s' + P\partial y_s'', \quad \partial z_s' \to \partial z_s' + P\partial z_s'',$$

where

$$P = \frac{x_k y_s'' - y_k x_s''}{x_k y_s' - y_k x_s'} .$$

(4) Perform on the three column block of station k:

$$\partial x_k \to \partial x_k - Q\partial x_s'', \quad \partial y_k \to \partial y_k - Q\partial y_s'', \quad \partial z_k \to \partial z_k - Q\partial z_s'',$$

where

$$Q = \frac{x_s'y_s'' - y_s'x_s''}{x_k y_s' - y_k x_s'} .$$

(5) Divide each of the rows "From s''" by $-(x_k y_s' - y_k x_s') \neq 0$.

(6) Perform further on the three column block of station k:

$$\partial x_k \to \partial x_k + \frac{y_s'}{y_k}\partial x_s', \quad \partial y_k \to \partial y_k + \frac{y_s'}{y_k}\partial y_s', \quad \partial z_k \to \partial z_k + \frac{y_s'}{y_k}\partial z_s'.$$

(7) Perform on the last three column block:

Table (1.4-4)

$\tilde{A}$ Matrix with Stations 3 and 2 Replaced by Stations k and s′, Respectively
(Ground Stations in Plane)

...	∂x_4	∂y_4	∂z_4	...	∂x_k	∂y_k	∂z_k	$\partial x_s'$	$\partial y_s'$	$\partial z_s'$	$\partial x_s''$	$\partial y_s''$	$\partial z_s''$	$\partial x_3/y_3$	$\partial y_3/y_3$	$\partial x_2/x_2$
From 4	$Z_{j_4}(X_{j_4}-x_4)$	$Z_{j_4}(Y_{j_4}-y_4)$	$Z^2_{j_4}$											$-y_4 Z_{j_4}(X_{j_4}-x_3)$	$-y_4 Z_{j_4}(Y_{j_4}-y_3)$	$-Z_{j_4}(x_4-y_4\frac{x_3}{y_3})(X_{j_4}-x_2)$
...																
From k					$Z_{j_k}(X_{j_k}-x_k)$	$Z_{j_k}(Y_{j_k}-y_k)$	$Z^2_{j_k}$							$-y_k Z_{j_k}(X_{j_k}-x_3)$	$-y_k Z_{j_k}(Y_{j_k}-y_3)$	$-Z_{j_k}(x_k-y_k\frac{x_3}{y_3})(X_{j_k}-x_2)$
From s′					$-y_s' Z_{j_s}'(X_{j_s}'-x_k)$	$-y_s' Z_{j_s}'(Y_{j_s}'-y_k)$	$-y_s' Z^2_{j_s}{}'$	$y_k Z_{j_s}'(X_{j_s}'-x_s')$	$y_k Z_{j_s}'(Y_{j_s}'-y_s')$	$y_k Z^2_{j_s}{}'$						$-Z_{j_s}'(y_k x_s'-x_k y_s')(X_{j_s}'-x_2)$
From s″					$-(x_s' y_s''-y_s' x_s'')\times(X_{j_s}''-x_k)$	$-(x_s' y_s''-y_s' x_s'')\times(Y_{j_s}''-y_k)$	$-(x_s' y_s''-y_s' x_s'')\times Z_{j_s}''$	$(x_k y_s''-y_k x_s'')\times(X_{j_s}''-x_s')$	$(x_k y_s''-y_k x_s'')\times(Y_{j_s}''-y_s')$	$(x_k y_s''-y_k x_s'')\times Z_{j_s}''$	$-(x_k y_s'-y_k x_s')\times(X_{j_s}''-x_s'')$	$-(x_k y_s'-y_k x_s')\times(Y_{j_s}''-y_s'')$	$-(x_k y_s'-y_k x_s')\times Z_{j_s}''$			

$$\frac{\partial x_3}{y_3} \rightarrow \frac{\partial x_3}{y_3} - y_4 \partial x_4 - \cdots - y_k \partial x_k, \quad \frac{\partial y_3}{y_3} \rightarrow \frac{\partial y_3}{y_3} - y_4 \partial y_4 - \cdots - y_k \partial y_k,$$

$$\frac{\partial x_2}{x_2} \rightarrow \frac{\partial x_2}{x_2} - (x_4 - y_4 \frac{x_3}{y_3}) \partial x_4 - \cdots - (x_k - y_k \frac{x_3}{y_3}) \partial x_k - \frac{y_k x_s' - x_k y_s'}{y_k} \partial x_s'.$$

(8) Perform further on the last column:

$$\frac{\partial x_2}{x_2} \rightarrow \frac{\partial x_2}{x_2} + \frac{x_3}{y_3} \frac{\partial x_3}{y_3}.$$

(9) Divide each of the rows "From s′" by $y_k \neq 0$.

The matrix thus obtained is called $\tilde{\tilde{A}}$ matrix for two replacements with ground stations lying in one plane; it is presented in Table (1.4-5). Up to and including the rows for station s' it is identical with Table (1.4-2) for one replacement; the notations for f-terms and p-terms would also be the same as introduced in (1.4-9) and (1.4-10) with respect to these stations. However, new notations will be needed in connection with station s'':

$$\begin{aligned} f^1_{s''} &= y_k y_s' x_s''(x_k - x_s') + y_k x_s' y_s''(x_s'' - x_k) + x_k y_s' y_s''(x_s' - x_s''), \\ f^2_{s''} &= y_k y_s' x_s''(y_k - y_s') + y_k x_s' y_s''(y_s'' - y_k) + x_k y_s' y_s''(y_s' - y_s''), \\ f^3_{s''} &= x_k y_s' x_s''(x_k - x_s'') + x_k x_s' y_s''(x_s' - x_k) + y_k x_s' x_s''(x_s'' - x_s'), \end{aligned} \quad (1.4\text{-}93a)$$

$$\begin{aligned} p^1_{s''} &= -\frac{1}{y_k} f^1_{s''}, \quad p^2_{s''} = -\frac{1}{y_k} f^2_{s''}, \\ r^1_{s''} &= (x_k y_s'' - y_k x_s'')(x_s'' - x_s'), \quad r^2_{s''} = (x_k y_s'' - y_k x_s'')(y_s'' - y_s'). \end{aligned} \quad (1.4\text{-}93b)$$

With the same simplified notations as in (1.4-17a)-(1.4-17c), it holds that

$$\begin{aligned} f_{s''} &= 0, \quad \text{whenever } s'' \equiv 1,\ k,\ s' \\ f_{s''} &= a f_s' \text{ where } a = -x_2 y_3 \text{ if } s'' \equiv 2, \\ f_{s''} &= b f_s' + c f_k \text{ where } b = y_3 x_k - x_3 y_k \text{ and } c = \frac{y_3}{y_k}(x_k y_s' - y_k x_s') \text{ if } s'' \equiv 3, \end{aligned} \quad (1.4\text{-}94)$$

and

$$f_{s''} = d f_s' + e f_k + f f_4 \quad \text{if} \quad s'' \equiv 4,$$

where

Table (1.4-5)

$\tilde{\tilde{A}}$ Matrix with Stations 3 and 2 Replaced by Stations k and s$'$, Respectively
(Ground Stations in Plane)

$\vdots$	∂x_4	∂y_4	∂z_4	...	∂x_k	∂y_k	∂z_k	$\partial x_s'$	$\partial y_s'$	$\partial z_s'$	$\partial x_s''$	$\partial y_s''$	$\partial z_s''$	$\partial x_3/y_3$	$\partial y_3/y_3$	$\partial x_2/x_2$
From 4	$X_{J_4}-x_4$	$Y_{J_4}-y_4$	Z_{J_4}											$y_4(x_4-x_3)$	$y_4(y_4-y_3)$	$x_4(x_4-x_2)-y_4\frac{x_3}{y_3}(x_3-x_2)$
...																
From k					$X_{J_k}-x_k$	$Y_{J_k}-y_k$	Z_{J_k}							$y_k(x_k-x_3)$	$y_k(y_k-y_3)$	$x_k(x_k-x_2)-y_k\frac{x_3}{y_3}(x_3-x_2)$
From s$'$					$(x_k-x_s')\frac{y_s'}{y_k}$	$(y_k-y_s')\frac{y_s'}{y_k}$	0	$X_{J_s}'-x_s'$	$Y_{J_s}'-y_s'$	Z_{J_s}'				$y_s'(x_s'-x_k)$	$y_s'(y_s'-y_k)$	$x_s'(x_s'-x_2)-y_s'\frac{x_k}{y_k}(x_k-x_2)$
From s$''$					$y_s'x_s''(x_s'-x_k)+$ $+x_s'y_s''(x_k-x_s'')+$ $+\frac{x_k}{y_k}y_s'y_s''(x_s''-x_s')$	$y_s'x_s''(y_s'-y_k)+$ $+x_s'y_s''(y_k-y_s'')+$ $+\frac{x_k}{y_k}y_s'y_s''(y_s''-y_s')$	0	$(x_ky_s''-y_kx_s'')\times$ $\times(x_s''-x_s')$	$(x_ky_s''-y_kx_s'')\times$ $\times(y_s''-y_s')$	0	$X_{J_s}''-x_s''$	$Y_{J_s}''-y_s''$	Z_{J_s}''	$y_ky_s'x_s''(x_k-x_s')+$ $+y_kx_s'y_s''(x_s''-x_k)+$ $+x_ky_s'y_s''(x_s'-x_s'')$	$y_ky_s'x_s''(x_k-y_s')+$ $+y_kx_s'y_s''(y_s''-y_k)+$ $+x_ky_s'y_s''(y_s'-y_s'')$	$x_ky_s'x_s''(x_k-x_s'')+$ $+x_kx_s'y_s''(x_s'-x_k)+$ $+y_kx_s'x_s''(x_s''-x_s')$

$$d = y_4 x_k - x_4 y_k, \quad e = \frac{y_4}{y_k}(x_k y_s' - y_k x_s'), \quad f = y_k x_s' - x_k y_s'.$$

The basic analysis of the necessary conditions for $\tilde{\tilde{A}}$ matrix to be non-singular will be made in a general sense when no assumptions are made concerning the satellite groups j_k and j_s' as yet. The systems associated with $\tilde{\tilde{A}}$ matrix are presented in Table (1.4-6); they are denoted as systems (1.4-95) and (1.4-95a), and they closely resemble the systems (1.4-11) and (1.4-11a) of Table (1.4-3), section 1.41. Matrices of these systems will be called similarly to those which represented (1.4-11) and (1.4-11a). This hold also for the full matrix (i.e., $\tilde{\tilde{A}}$ matrix) corresponding to the full system. The matrix of (1.4-95a) has the same form as the matrix F given by (1.4-12). The necessary conditions for the full matrix to be non-singular will be again divided into the row conditions and column conditions; the latter, pertaining to singularity A) and singularity B), will be treated separately.

The row conditions are exactly the same as those investigated in section 1.41 with respect to the system (1.4-11). They require that at least three quads observe their targets off-plane.

For the analysis of column conditions, singularity A) will be examined first. Upon considering $\tilde{A}$ matrix of Table (1.4-4), it is observed that singularity A) occurs for station k whenever all the targets observed by it (here targets in j_k and in both groups j_s) lie in the plane with it (called plane π) with a similar conclusion for station s$'$. Upon retracing the steps leading to the above $\tilde{A}$ matrix, it is seen that singularity A) occurs under the same conditions for any distribution of ground stations. The method for analyzing these cases was presented in section 1.41 where the name "general singularity A)" was also introduced. Next, ground stations are assumed to be in the plane $z = 0$.

Singularity A) with respect to any station except k and s$'$ would occur if the corresponding satellite group contained only in-plane targets, the plane of which would pass through that station. Singularity A) with respect to station

Table (1.4-6)

Systems (1.4-95) and (1.4-95a) Associated with $\tilde{\tilde{A}}$ Matrix

$$
\begin{array}{ll|ll}
a_4^x \tilde{v}_4^x + a_4^y \tilde{v}_4^y + a_4^z \tilde{v}_4^z & & + c_4 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0 \\
\vdots & & & \\
a_k^x \tilde{v}_k^x + a_k^y \tilde{v}_k^y + a_k^z \tilde{v}_k^z & & + c_k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0 \\
a_k^x p^1_{s'} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_k^y p^2_{s'} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + 0 & + a_s^{x'} \tilde{v}_s^{x'} + a_s^{y'} \tilde{v}_s^{y'} + a_s^{z'} \tilde{v}_s^{z'} & + c_s' \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0 \\
a_k^x p^1_{s''} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_k^y p^2_{s''} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + 0 & + a_s^{x'} r^1_{s''} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_s^{y'} r^2_{s''} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + 0 + a_s^{x''} \tilde{v}_s^{x''} + a_s^{y''} \tilde{v}_s^{y''} + a_s^{z''} \tilde{v}_s^{z''} & + c_s'' \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} & = 0
\end{array}
$$

(1.4-95)

$$
\begin{aligned}
b_1 f_4^1 + b_2 f_4^2 + b_3 f_4^3 &= c_4 \\
&\vdots \\
b_1 f_k^1 + b_2 f_k^2 + b_3 f_k^3 &= c_k \\
b_1 f_{s'}^1 + b_2 f_{s'}^2 + b_3 f_{s'}^3 &= c_s' \\
b_1 f_{s''}^1 + b_2 f_{s''}^2 + b_3 f_{s''}^3 &= c_s''.
\end{aligned}
\qquad (1.4\text{-}95a)
$$

k would be the same as in section 1.41 if there were no station s''; it would occur if in addition to j_k lying in the plane π, station s' laid in a straight line with stations 1 and 2, or if it laid on the line ℓ. However, with station s'' present, singularity A) for station k would occur only if in addition to the above conditions also the following equation held:

$$\begin{vmatrix} X_1 - x_k & Y_1 - y_k & Z_1 \\ X_2 - x_k & Y_2 - y_k & Z_2 \\ p_{s''}^1 & p_{s''}^2 & 0 \end{vmatrix} = 0, \qquad (1.4\text{-}96)$$

where the p-terms are those of (1.4-93b); otherwise, this equation in terms of s'' has the same form as (1.4-13) in terms of s. It would be exactly the same equation in terms of s'' (upon considering (1.4-93a)) if s' were lying on a straight line given by

$$y_s' = 0.$$

In this case, (1.4-96) would hold if s'' were lying on a straight line with station 1 and 2 (i.e., $y_s'' = 0$), or on the line ℓ. When s' is such that

$$y_s' \neq 0,$$

the discussion is somewhat longer. It will be convenient to replace the second row of (1.4-96) with the row pertianing to station s', since this row is assumed to lie in the same row space according to the above stipulations. If the $y_k x_s''$-multiple of this new second row is added to the third row in (1.4-96), this equation can be rewritten as

$$\begin{vmatrix} X_1 - x_k & Y_1 - y_k & Z_1 \\ (x_k - x_s')\dfrac{y_s'}{y_k} & (y_k - y_s')\dfrac{y_s'}{y_k} & 0 \\ x_s' y_s''(x_k - x_s'') + \dfrac{x_k}{y_k} y_s' y_s''(x_s'' - x_s') & x_s' y_s''(y_k - y_s'') + \dfrac{x_k}{y_k} y_s' y_s''(y_s'' - y_s') & 0 \end{vmatrix} = 0 \qquad (1.4\text{-}96a)$$

Clearly, (1.4-96a) holds whenever

$$y_s'' = 0.$$

Otherwise its third row can be divided by $y_s' y_s'' \neq 0$, and the determinant expanded. After some algebraic manipulations, it is obtained that

$$x_s''(y_k - y_s') - y_s''(x_k - x_s') + x_k y_s' - y_k x_s' = 0,$$

vhich is the equation of a straight line for station s''. This line passes through ;tations k and s' (as seen upon substitution for s''); therefore it is exactly the line , to which s' had been restricted. Thus it is seen that singularity A) for station : occurs under the same circumstances as those derived in section 1.41; namely, f in addition to the targets in j_k lying in π each of the stations s', s'' lie on the traight line with stations 1 and 2, or on the line ℓ. To complete the discussion, ingularity A) for station s' will be also analyzed. First, a few notations will be ıtroduced: (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) will now be the coordinates of the (first) vo targets in j_s'. If all the targets in j_s' are lying in a plane through station s', ıch a plane will be denoted as π'; the line of intersection between the plane π' ıd the plane of the ground stations will be denoted as ℓ'. Singularity A) for ;ation s' will occur if in addition to the targets in j_s' lying in π', also the follow- ıg relation holds:

$$\begin{vmatrix} X_1 - x_s' & Y_1 - y_s' & Z_1 \\ X_2 - x_s' & Y_2 - y_s' & Z_2 \\ (x_k y_s'' - y_k x_s'')(x_s'' - x_s') & (x_k y_s'' - y_k x_s'')(y_s'' - y_s') & 0 \end{vmatrix} = 0.$$

learly, this can be true if

$$x_k y_s'' - y_k x_s'' = 0$$

fulfilled, i.e., if station s'' lies on the straight line with stations 1 and k. herwise, the third row in the above determinant will be divided by $(x_k y_s'' - y_k x_s'') \neq 0$, giving

$$\begin{vmatrix} X_1 - x_s' & Y_1 - y_s' & Z_1 \\ X_2 - x_s' & Y_2 - y_s' & Z_2 \\ x_s'' - x_s' & y_s'' - y_s' & 0 \end{vmatrix} = 0,$$

which is the equation of the straight line for station s''. This line is exactly ℓ and constitutes another case when singularity A) for station s' could occur.

Conditions leading to singularity B) will also be treated in a way similar to that in section 1.41. Namely, any row vector of the matrix F given by (1.4-12) would have to lie in subspace V_2, given by (1.4-15), should singularity B) occur. The rows of F pertaining to all stations except station s'' were already treated in section 1.41. They are in V_2 only if the corresponding stations lie on the second order curve through stations 1, 2, 3, 4, k, in other words, if they are on-curve stations. Still to be found are conditions for station s'' in this respect. Should its corresponding row in F lie in V_2, the following would have to hold:

$$\begin{vmatrix} f_4^1 & f_4^2 & f_4^3 \\ f_k^1 & f_k^2 & f_k^3 \\ f_{s''}^1 & f_{s''}^2 & f_{s''}^3 \end{vmatrix} = 0;$$

this represents a second order curve for station s''. The coefficients of $f_{s''}$ in this equation would all vanish under exactly the same conditions which lead to V_2 of (1.4-15) having dimension one. Such cases were assumed non-existent. If s'' is replaced by any of 1, 2, 3, 4, or k, $f_{s''}$ is either a zero row or a linear combination of the rows f_4, f_k, or $f_{s'}$ according to (1.4-94). However, if station s' is itself an on-curve station, then its row is a linear combination of the rows f_4 and f_k; under these circumstances the above equation would hold and the corresponding second order curve or $f_{s'}$ according to (1.4-94). However, if station s' is itself an on-curve station, then its row is a linear combination of the rows f_4 and f_k; under these circumstances the above equation would hold and the corresponding second order curve for station s'' would also pass through stations 1, 2, 3, 4, and k. This leads to the same conclusion as was made in section 1.41: singularity B) occurs when all the ground stations lie on the same second order curve.

For the practical discussion in this section, it will be assumed that the satellite groups j_4, j_k, and j_s' contain off-plane targets. Consequently, singularity A) for stations k and s′ is automatically eliminated; the necessary conditions of this kind for non-singular $\widetilde{\widetilde{A}}$ matrix are limited to those, specifiying that no further satellite group is lying in the plane through the corresponding ground station. Necessary conditions for singularity B) remain unchanged, while the row conditions are automatically fulfilled.

Finally, it will be examined when the assumptions about j_4, $\hat{j}_k$, and j_s' constitute also the sufficient conditions for non-singular $\widetilde{\widetilde{A}}$ matrix. It can be seen from the system (1.4-95) that

$$a_4^x = a_4^y = a_4^z = c_4 = 0$$

and

$$a_k^x = a_k^y = a_k^z = c_k = 0,$$

since j_4 and j_k contain off-plane targets. Since the a-coefficients for station k are now zeroes and since j_s' also contains off-plane targets, it must further hold that

$$a_s^{x\prime} = a_s^{y\prime} = a_s^{z\prime} = c_s' = 0.$$

With these a-coefficients for station s′, it holds for station s″ and similarly for any additional station i (if present):

$$a_s^{x\prime\prime} \widetilde{v}_s^{x\prime\prime} + a_s^{y\prime\prime} \widetilde{v}_s^{y\prime\prime} + a_s^{z\prime\prime} \widetilde{v}_s^{z\prime\prime} + c_s'' \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0.$$

Now the a-coefficients for all the remaining stations must be zero (due to singularity A) eliminated) only if all the remaining c-coefficients are equal to zero as the only possibility. But this happens only if those rows of F whose c-coefficients are zero can form a submatrix with full rank. When no further satellite group contains off-plane satellites, this would imply that

$$\text{rank}\begin{bmatrix} f_4^1 & f_4^2 & f_4^3 \\ f_k^1 & f_k^2 & f_k^3 \\ f_{s'}^1 & f_{s'}^2 & f_{s'}^3 \end{bmatrix} = 3.$$

Then all three b-coefficients and all the c-coefficients would have to be equal to zero. The above relation stipulates that station s' must be an off-curve station. With more stations observing off-plane satellites, at least one of them would have to be an off-curve station.

This section can be concluded by summarizing the necessary and sufficient conditions for a non-singular $\tilde{\tilde{A}}$ matrix when stations 4, k, and s' observe off-plane targets: $\tilde{\tilde{A}}$ is non-singular if in the absence of singularity A), station s' (or further stations - besides 4, k - observing off-plane targets) is an off-curve station.

1.43 <u>More than Two Replacements.</u>

If six ground stations form a non-singular network which is to be expanded, then the new network is in general also non-singular. The sufficient conditions that it be so is that singularity A) is eliminated also for any additional station, and that no satellite point is lying in the plane of ground stations. The fact that the network remains non-singular no matter how many stations are added and no matter how many replacements are carried out can be visualized in a simple manner. Let the ground stations observe their respective satellite groups in the following fashion:

$$
\begin{array}{ccccccccl}
4 & 3 & 2 & 1 & & & & \ldots\, j_4 & \\
 & \vdots & & & & & & \vdots & \\
 & 3 & 2 & 1 & k & & & \ldots\, j_k & \text{Non-singular} \\
 & & 2 & 1 & k & s' & & \ldots\, j_{s'} & \\
 & & & \vdots & & & & \vdots & \qquad\qquad\qquad (1.4\text{-}97) \\
\hline
 & & & 1 & k & s' & s'' & \ldots\, j_{s''} & \\
 & & & & \vdots & & & \vdots & \\
 & & & & k & s' & s'' \ \bar{s} & \ldots\, j_{\bar{s}} & \text{Singularity A) avoided} \\
 & & & & & \vdots & & \vdots &
\end{array}
$$

There can be any number of stations and replacements in the above illustration. The network consisting of six stations, e.g., of stations 1, 2, 3, 4, k, and s′ (stations 4, k, and s′ observe off-plane targets), is the smallest possible for a non-singular solution (to avoid singularity B) more than 5 stations are needed, and to avoid singularity C) at least three stations - understood as each representing a quad - should observe off-plane targets). Assuming the block containing stations 1, 2, 3, 4, k, and s′ to be non-singular, each of these stations can be uniquely determined; any three of them, co-observing with some new station, are able to determine the coordinates of any target (provided it does not lie in a plane with them). But since the new station does not lie in a plane with these targets, it can be uniquely determined from them. Thus, the number of known ground stations has been increased and the same argument may be repeated.

In the illustration (1.4-97), the original non-singular part of the network was obtained using one replacement (namely, station k replaced station 2). Naturally, a similar network of six ground stations could be obtained without replacements (i.e., three ground stations could observe all the targets as discussed in the early sections). However, when the first replacement is carried out, the network necessarily consists of at least six ground stations. Therefore, one replacement is of fundamental importance and deserves much attention. That was the reason behind an extensive treatment of all its aspects in section 1.41. It may be of interest to mention a type of problem when the network is singular with three stations (assumed to be 1,2,3) observing all the targets, while with replacements of stations it becomes non-singular. Suppose that singularity A) for station k occurred in the former case (all the targets in j_k laid in π), and suppose that station k replaced station 2. Then, however, singularity A) for station k would occur only if in addition to the above property of j_k station s' laid on the line ℓ or j_s' laid in π. Otherwise singularity A) is removed.

Although the sufficient conditions for non-singular networks for two or more replacements are simple to state and plausible without derivations, somewhat detailed procedure for two replacements was presented in section 1.42. Considering the practical cases (such as j_4, j_k, and j_s' containing off-plane targets), the results supported the theory and illustrations in this section. Concretely, the conditions stipulating that singularity A) must be eliminated and that station s' must be an off-curve station, summarized at the end of section 1.42, guarantee that the part of the network presented above the horizontal line in (1.4-97) and corresponding to one replacement is indeed non-singular; consequently, these conditions make further expansion of the network possible. From the theoretical point of view, the necessary conditions for two replacements in section 1.43 were given regardless whether one replacement was possible or not, i.e., with no assumptions concerning the groups j_4, j_k, and j_s'. In this context it would be possible to have a singular network with one replacement, namely, if the targets in j_s' lay in π', while with two replacements singularity A) for station s' would be removed if station s'' did not lie on ℓ' and j_s'' did not lie in π'.

From the purely practical point of view, the derivations in the part of section 1.41 following section 1.411 and those in section 1.42 were not absolutely needed, since in general a large number of satellite points can be observed from any quad and they most often do not all lie in, or nearly in one plane, much less in a plane through a specific ground station. The configurations of ground stations - as their number is limited - appears to be much more important. Many of the derivations were mainly of academic interest and they were made for the sake of completeness; occasionally, however, they could be put in use, especially when the number of targets is limited.

In conclusion, this section can be summarized by saying that any non-singular network can be successfully expanded, the sufficient conditions being: no target should be in the plane of ground stations and singularity A) should be eliminated also for each further station. The smallest non-singular range network can consist of no less than six ground stations. In case of replacement of stations (leading to "leapfrogging"), the first replacement can result in a non-singular network and is therefore of fundamental importance.

1.5 Numerical Examples and Verifications of Theory.

Most of the numerical solutions were carried out during the first part of this study, dealing with ground stations in a plane and three of them observing all the targets. With exception of Example 2, all the points (stations and targets) were generated in an arbitrary coordinate system; the data was also generated (mostly with attached random errors), whenever it was judged appropriate. In many instances, however, only the trace of the weight coefficient matrix (inverse matrix of normal equations, N^{-1}) was needed as an indicator of critical configurations (singular problems). In all the examples, the coordinate system was chosen the "best" possible way in order to eliminate undue numerical difficulties and large uncertainties in the adjusted parameters (reflected in the large trace of N^{-1}), due to poor definitions of the coordinate system. These aspects are treated in [16] where the constraints which materialize the "best" coordinate system are given explicitly with respect to all the points of a cluster to be adjusted or a chosen subset of these (e.g., the ground stations). If Q represents the whole matrix N^{-1} or its part corresponding to the chosen subset of points, then these constraints, called inner adjustment constraints, have the property that they render

$$\mathrm{Tr}(Q) = \text{minimum}, \tag{1.5-1}$$

as compared to any other set of constraints defining the coordinate system differently. Consequently, if the matrix N^{-1} indicates that a network to be adjusted is singular or nearly singular, it may be inferred that this is due to the critical distribution of points rather than to a weak definition of the coordinates system. The adjustment method used in all the problems was the method of variation of parameters in the least squares adjustment, called also "A method" (see [5]). In many generated problems only the necessary number of observations were used, so that the solution for the parameters could be obtained without forming the normal equations. But since $\mathrm{Tr}(N^{-1})$ was of the main importance in the type of analysis presented in this section, the normal equations were formed, bordered with the inner adjustment constraints and inverted. Unless otherwise specified, these constraints are given with respect to all the points of a network to be adjusted. If the notation $\mathrm{Tr}(N^{-1})gr$ is used when the results are presented, the trace of only that portion of N^{-1} is given which corresponds to the ground stations alone.

For the numerical solution of a number of problems, certain points were used repeatedly and are presented in Table (1.5-1). The ground stations are denoted by the numbers between 1 and 7, while the targets are designated with numbers higher than 150. As it can be observed from the table, the ground stations are lying in one plane and the targets, grouped by three, are lying on specific straight lines, with exception of 191, 192, 193 which are in general configuration. All the coordinates are given in meters.

The observation equations for adjustment of different range networks were generated in a program called Auxiliary range program. The input for this program had to contain the coordinates of all points in the network as well as the parameters specifying up to which point the observations are generated on a one-to-one basis, the nature of observations (errorless, etc.), number of

Table (1.5-1)

Cartesian Coordinates of Some Generated Points

Point	x	y	z
1	-100,000.	-150,000.	0.
2	1,000,000.	-200,000.	0.
3	300,000.	1,400,000.	0.
4	1,200,000.	1,700,000.	0.
5	1,500,000.	1,300,000.	0.
$\overline{5}$	700,000.	1,300,000.	0.
7	600,000.	350,000.	0.
151	500,000.	-200,000.	1,000,000.
152	500,000.	500,000.	1,000.000.
153	500,000.	1,200,000.	1,000,000.
161	1,200,000.	500,000.	1,600,000.
162	500,000.	500,000.	1,600,000.
163	-200,000.	500,000.	1,600,000.
171	200,000.	200,000.	2,000,000.
172	800,000.	800,000.	2,000,000.
173	1,400,000.	1,400,000.	2,000,000.
181	100,000.	900,000.	1,500,000.
182	600,000.	900,000.	1,500,000.
183	1,000,000.	900,000.	1,500,000.
191	200,000.	-500,000.	1,050,000.
192	-650,000.	600,000.	1,400,000.
193	1,200,000.	1,050,000.	800,000.

ground stations and satellite points, the nature of the inner adjustment constraints, if any (for all the points or the ground stations only), and some additional characteristics. The punched output consisted of the coefficients for the matrix of observation equations and constraints, and of the constant vector of observation equations. This represented the input to an adjustment program, called A method program. Input parameters for this program specified among other things the actual geometry of the network which was to be adjusted, since all the possible range observations were made available by the Auxiliary range program. Thus, when deleting different number of different observations, many problems could be solved using the same input deck. The weight matrix for the observations was stipulated to be a unit matrix in all cases. The judgment on singularity or non-singularity of a particular problem was based on different numerical checks indicating the validity of the solution (in many cases no redundant observations were used and therefore, the residuals had to be theoretically equal to zero). Correlation coefficient matrix was also considered in all investigated cases as well as N^{-1} matrix and its trace. Of these, $Tr(N^{-1})$ and sometimes $Tr(N^{-1})gr$ will be presented in the following examples, as they are expressed by a single number.

Example 1. A special network of four ground stations and eight targets was generated in order to examine the behavior of the adjustment when the ground stations gradually depart from a plane. The targets used in this example are: 151, 152, 153; 161, 162, 163; 171, and 172; their coordinates are presented in Table (1.5-1). The x and y coordinates of the four ground stations, denoted as I, II, III, IV, are given as

I:	$x = 0$	$y = 0$
II:	$x = 0$	$y = 1,000,000$
III:	$x = 1,000,000$	$y = 0$
IV:	$x = 1,000,000$	$y = 1,000,000.$

The z-coordinate of I, II, III is equal to zero, while for IV it is varying between zero and 500 km. When it is zero, the problem is necessarily singular. Altogether, seven cases are presented in Table (1.5-2). The column with the heading "d" gives the average distance between the four ground stations and the best fitting plane; the heading "ℓ" represents the average length in the quad I, II, III, IV (on a one-to-one basis). The last column gives the ratio d/ℓ, which can serve as a certain plausible measure of expected "goodness" of an adjustment.

Table (1.5-2)

Different Configurations of Stations I, II, III, IV.

Case	z (km)	d (km)	ℓ (km)	d/ℓ
1	0.	0.00	1,138.	0.0000
2	10.	2.50	1,138.	0.0022
3	25.	6.25	1,138.	0.0055
4	50.	12.49	1,138.	0.0110
5	100.	24.94	1,140.	0.0219
6	200.	49.50	1,147.	0.0432
7	500.	117.07	1,192.	0.0982

The results of the adjustment are given in Table (1.5-3). The first two columns (beyond case numbers) give $Tr(N^{-1})$ and $Tr(N^{-1})gr$ when the inner adjustment constraints are used with respect to all the points of the network (i.e., eleven points), while the following two columns give the same values when the inner adjustment constraints are used with respect to the ground stations only (four points). The number of degrees of freedom (d.f.) for all the cases treated this way is one. In the last two columns, the results are given when all the possible observations (on a one-to-one basis) are used in the otherwise unchanged network; the inner adjustment constraints used in this part are those with respect to all the points of the network. The a-posteriori standard deviation for this part, denoted as $\hat{\sigma}$, is also given; it appears in the last column. Since the random errors generated by the library subroutine and attached to the errorless observations came from the normal distribution with zero mean and unit variance, the a-posteriori standard deviation is not expected to depart from unity by a great amount. The problem denoted as case 1 is singular when only ground-satellite observations are considered. Numbers on the diagonal of N^{-1} are extremely large and negative. As a matter of fact, case 2 is nearly singular with the results invalidated by

Table (1.5-3)

Results from the Adjustment of the Generated Network

Case	Constraints: All Points		Constraints: Stations Only			
	$Tr(N^{-1})$	$Tr(N^{-1})gr$	$Tr(N^{-1})$	$Tr(N^{-1})gr$	$Tr(N^{-1})$	$\hat{\sigma}$
1					10.995	1.01
2	3,879,000	1,667,000	3,766,000	1,558,000	10.989	1.08
3	579,000	249,000	592,000	244,000	10.981	1.05
4	141,700	60,880	145,500	59,830	10.968	1.23
5	34,320	14,720	35,650	14,420	10.944	.85
6	8,085	3,447	8,760	3,358	10.903	.94
7	1,130	462	1,404	437	10.830	.98

numerical problems. This is also apparent from the above table, where $Tr(N^{-1})$ for the inner adjustment constraints (first column) for all the points is actually larger than its counterpart (third column) where these constraints apply only to the ground stations, which is obviously wrong. For cases 3-7, this relation is correct, while the relation for $Tr(N^{-1})gr$ is opposite (i.e., the numbers in the second column are larger than their counterparts in the fourth column). When all the possible observations are used, the adjustment in all cases appears to be very strong. As illustrated in Appendix 1, an improvement in the solution was to be expected due to the new observations which were added to an existing network. In this adjustment the results for $Tr(N^{-1})$ improve very little when station IV moves away from the plane of stations I, II, III.

Example 2. In this example, several quads of the Pacific network, presented in [2], are adjusted separately. These quads are denoted by small letters, while the stations are represented by the same numbers under which they appeared in [2]. The correspondence between the quads and the stations is listed below:

a ... 1, 2, 3, 4

b ... 4, 5, 6, 7

c ... 3, 7, 8, 10

d ... 3, 6, 7, 8

f ... 2, 3, 5, 6

g ... 2, 3, 4, 5.

Due to the A method program limitations, only ten satellite points were chosen for each quad to be adjusted. They were selected to be as well distributed and representative as possible. In order to visualize the relations between $Tr(N^{-1})$ and the relative distance of one quad's points from the best fitting plane, the results were arranged in the Table (1.5-4) in a fashion similar to Example 1. From the table it appears that larger distances from the plane that 4% of ground distances would be necessary to obtain fairly strong solutions for individual quads. This would imply larger quads and/or different distribution of the ground stations. In order to evidence the improvement in $Tr(N^{-1})gr$ when the inner adjustment constraints are used with respect to the ground stations only, quad f was also adjusted this way. Both cases are presented in Table (1.5-5) for comparison. Naturally, all the residuals must be theoretically the same no

Table (1.5-4)

Quads of the Pacific Network

Quad	Ave. Distance From Plane	Ave. Distance Between Points	Ratio of the Previous 2 Columns	n^{ii}_{max} for * Ground Stations	$Tr(N^{-1})$
a	36.40 km	1,672 km	.0218	2,376	39,391
b	73.91	2,327	.0318	650	11,833
c	44.66	2,746	.0163	27,587	133,215
d	92.26	2,884	.0320	305	3,828
f	65.03	1,835	.0354	156	4,887
g	11.97	1,530	.0078	19,598	678,172

* n^{ii} represents ith diagonal element from N^{-1} matrix

matter which definition of the coordinate system is adopted.

Table (1.5-5)

Quad f Using Different Sets of Inner Adjustment Constraints

Inner Adjustment Constraints for:	n^{ii}_{max} for Ground Stations	$Tr(N^{-1})gr$	$Tr(N^{-1})$
all points	156.0	740	4,887
ground stations	111.6	502	5,847

To visualize possible increase in strength of an adjustment of one quad by changing its shape (and, thereby, the average distance from the best fitting plane) without enlarging its size, further experiments were performed with quads c and g, which previously had the weakest solution. In quad c, station 10 was displaced to occupy the middle area of stations 3, 7, and 8. It was chosen to have the coordinates $\varphi = 10^\circ$ and $\lambda = 180^\circ$ (this corresponds approximately to

$$j_1 \; . \; . \; . \; 151,\ 152,\ 171,\ 183$$

$$j_2 \; . \; . \; . \; 161,\ 162,\ 173,\ 181 \qquad (1.5\text{-}3)$$

$$j_3 \; . \; . \; . \; 153,\ 163,\ 172,\ 182.$$

The observations are organized as given below:

1	2	3	4			...	j_1
	2	3	4	5		...	j_2
	2	3	4		6	...	j_3

(1.5-4)

In addition to the above eleven cases, category 1 contains also two additional cases, denoted as case 12 and case 13; they were designed to illustrate singularity A) and singularity C), respectively. In both cases the coordinates of station 6 are the same as in case 1 and the coordinates of satellite points 152, 171, and 183 were changed so that in case 12 j_1 contains targets lying on a straight line and in case 13 j_1 contains targets lying in one plane. The new targets for case 12 denoted by primes, have the coordinates:

152′	...	0	300,000.	1,350,000.
171′	...	-380,234.5	680,234.5	1,616,164.2
183′	...	-633,724.2	933,724.2	1,793,607.0

The new targets for case 13', denoted by double primes, have the coordinates:

152″	...	0	300,000	1,350,000.
171″	...	800,000	150,000	1,420,000.
183″	...	600,000	700,000	1,899,230.8

Category 2 requires that only two satellite groups be formed. In this category only eight targets are used (151-172), which makes the d.f. = 4. The observations are arranged as follows:

1	2	3	4	5		...	151, 152, 161, 172
1	2	3	4		6	...	153, 162, 163, 171.

If the observations were arranged in quads with the two satellite groups denoted respectively as j_1 and j_2, this category could be represented with respect to $\tilde{\tilde{A}}$ matrix as

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & & & \ldots j_3 \\ 1 & 2 & 3 & & 5 & & \ldots j_1 \\ 1 & 2 & 3 & & & 6 & \ldots j_2 \end{array}$$

where j_3 is obtained by merging j_1 and j_2. This representation corresponding to Table (1.3-1) serves only for theoretical analysis, since the stations denoted as 1,2, and 3 observe each target only once.

<u>Category 3</u> requires only one satellite group, which in this case contains the same eight targets (151-172) as presented in category 2; this makes d.f. = 12 (there are $(6+8)3 = 42$ unknowns and $6 \times 8 = 48$ observations with 6 constraints). With respect to $\tilde{\tilde{A}}$ matrix, the observations in this category can be thought of as being arranged in the following way:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & & & \ldots j_3 \\ 1 & 2 & 3 & & 5 & & \ldots j_3 \\ 1 & 2 & 3 & & & 6 & \ldots j_3 . \end{array}$$

Clearly, all the satellite groups here coincide, since all stations are co-observing. Again, since any of stations 1,2, and 3 observe each target only once (rather than three times) this arrangement corresponding to $\tilde{\tilde{A}}$ matrix serves only for the theoretical analysis (unless proper weighting is applied).

The results for all three categories in all eleven cases (thirteen cases for category 1) are presented in Table (1.5-7). This table illustrates when singularity occurs and how $Tr(N^{-1})$ behaves in singular cases. The best results in all categories are obtained in case 2, when station 6 coincides with x_o, the center of the ellipse. The second best result is reached in case 1, when station 6 coincides with the point $+\frac{b}{2}$. As expected, the results are getting worse in the vicinity of the critical curve, as seen in case 8 and in case 10. Case 12 and case 13 in category 1 are singular as expected. Singularity C) also occurs whenever one of the three satellite groups contains only three targets (necessarily in plane), independently of the number of targets in the other two groups. It is also clear that significant improvement occurs when five stations rather than four observe simultaneously, as evidenced in category 2. Further drastic improvement is to be expected when six ground stations co-observe, judging by the results in category 3.

Table (1.5-7)

Results of Adjustment in Three Categories with Ground Stations in Plane, Critical Curve Being Ellipse

Case	Station 6	Coordinates: Station 6		$Tr(N^{-1})$ categories:			Singularity
		x	y	1	2	3	
1	$+\frac{b}{2}$	340,000	790,000	60,530	22,100	387	
2	x_o	680,606	630,724	51,260	15,300	275	
3	$-\frac{b}{2}$	1,020,000	470,000	124,300	42,000	437	
4	$-b$	1,361,060	301,806	$-.6\times10^8$	$-.5\times10^8$	$-.4\times10^8$	B)
5	$-\frac{3b}{2}$	1,710,000	130,000	214,600	57,900	681	
6	$-2b$	2,060,000	-30,000	95,980	21,900	400	
7	$-\frac{a}{3}$	500,000	250,000	89,010	28,800	278	
8	$-\frac{2a}{3}$	330,000	-90,000	369,200	115,100	474	
9	$-a$	163,256	-439,546	$-.3\times10^8$	$-.4\times10^8$	$-.3\times10^8$	B)
10	$-\frac{4a}{3}$	0	-780,000	860,000	195,000	1,602	
11	$-\frac{5a}{3}$	-170,000	-1,120,000	323,200	65,300	855	
12	$+\frac{b}{2}$	same as case (1)		$-.9\times10^8$			A)
13	$+\frac{b}{2}$	same as case (1)		$-.5\times10^8$			C)

Example 4. Similar experiments to those in Example 3 are performed with the critical curve being now a hyperbola. The difference between these two examples consists in the fact that station $\overline{5}$, given in Table (1.5-1), rather than station 5 is used to define the critical curve; The letters a and b now pertain to the real and imaginary axes of a hyperbola as opposed to (1.5-2), where they were associated with the major and minor axes of an ellipse. All the other aspects are the same as in the previous example, except that category 1 is missing. The remaining results are presented in Table (1.5-8). The results in this table bear certain resemblances to those listed in Table (1.5-7); in general, however, they appear to be somewhat inferior. When station 6 coincides with x_o in this example, the

Table (1.5-8)

Results of Adjustment in Two Categories with Ground Stations in Plane, Critical Curve Being Hyperbola

Case	Station 6	Coordinates: Station 6		$Tr(N^{-1})$ categories:		Singularity
		x	y	2	3	
1	$+2a$	590,000	1,790,000	9,300	674	
2	$+\frac{3a}{2}$	570,000	1,530,000	51,800	1,186	
3	$+a$	553,768	1,273,890	$-.2 \times 10^9$	$-.2 \times 10^9$	B)
4	$+\frac{a}{2}$	535,000	1,010,000	275,800	4,199	
5	x_o	519,344	744,236	206,700	3,302	
6	$-\frac{a}{2}$	500,000	480,000	479,200	6,104	
7	$-a$	484,921	214,584	$-.6 \times 10^8$	$-.2 \times 10^9$	B)
8	$-\frac{3a}{2}$	460,000	-50,000	321,600	1,549	
9	$-2a$	445,000	-310,000	85,100	425	
10	$+b$	132,471	769,380	57,100	719	
11	$-b$	906,218	719,092	66,200	1,141	

solution is quite weak; the best results now correspond to the points $+2a$, $-2a$, and $+\frac{3a}{2}$. The assymetry of the results in cases 1-9 is natrually due to the distribution of the targets, which are not symmetrical with respect to the hyperbola.

Example 5. Two cases from Example 3, category 1 are further modified. They are case 1 and case 9. First, in case 1 station 4 is moved from the plane of ground stations upwards by 100 km, i.e.,

$$z_4 = 100,000.$$

With everything else unchanged, this results in

$$\mathrm{Tr}(N^{-1}) = 34,180.$$

If station 6 moves the same way instead of station 4, i.e., if

$$z_6 = 100,000$$

then it is obtained that

$$\mathrm{Tr}(N^{-1}) = 77,570.$$

These two modifications do not cause any remarkable changes in the quality of the solution. If, however, the same two modifications are applied to case 9 which was singular, the following results are obtained:

$$\mathrm{Tr}(N^{-1}) = 1,406,000$$

and

$$\mathrm{Tr}(N^{-1}) = 1,727,000.$$

The solution is quite weak, but the singularity has been removed. If the first modification is made by 200 km rather than by 100 km, i.e., if

$$z_4 = 200,000$$

then the solution further improves; namely,

$$\mathrm{Tr}(N^{-1}) = 400,100.$$

This solution is nearly as strong as the one given in case (8). In this particular example, moving station 4 by 200 km in the vertical direction from the plane helps to strengthen the solution approximately as much as moving station 6 by 400 km in the plane of ground stations toward the center of the ellipse.

Example 6. Having the same satellite groups as in category 1 of example 3, replacing of stations is illustrated with the same position of station 6 as in case 1. The three satellite groups were presented in (1.5-3). The observations are now arranged in the following way:

$$\begin{array}{cccccccl} 1 & 2 & 3 & 4 & & & \ldots & j_1 \\ & 2 & 3 & 4 & 5 & & \ldots & j_2 \\ & 2 & 3 & & 5 & 6 & \ldots & j_3. \end{array} \qquad (1.5\text{-}5a)$$

Comparing this with the original observations given in (1.5-4), it is seen that station 5 replaced station 4 (formerly observing all the targets) with respect to the satellite group j_3. An adjustment gives:

$$\mathrm{Tr}(N^{-1}) = 70{,}320;$$

this result is only slightly worse than the corresponding result (category 1, case 1) from Table (1.5-7), which gives the trace as being equal to 60,530. If the observations of the same case are arranged another way, namely

$$\begin{array}{cccccccl} 1 & 2 & 3 & 4 & & & \ldots & j_1 \\ & 2 & 3 & 4 & 5 & & \ldots & j_2 \\ & & 3 & 4 & 5 & 6 & \ldots & j_3, \end{array} \qquad (1.5\text{-}5b)$$

then it is obtained that

$$\mathrm{Tr}(N^{-1}) = 149{,}000.$$

In this arrangement station 5 replaced station 2.

Similar computations may be made using seven ground stations (four of them co-observing) and fifteen targets. This gives again only the necessary number of observations. All the stations and satellite points used in the previous two arrangements remain the same and one ground station with one satellite group are added. Station 7 with the coordinates listed in Table (1.5-1) is the added station and

$$j_4 \; . \; . \; . \; 191, \; 192, \; 193$$

is the added satellite group; the coordinates of its targets are listed in Table (1.5-1) as well. The first arrangement in this part is a continuation of the network represented in (1.5-5a), namely

$$\begin{array}{cccccccl}
1 & 2 & 3 & 4 & & & \ldots & j_1 \\
 & 2 & 3 & 4 & 5 & & \ldots & j_2 \\
 & 2 & 3 & & 5 & 6 & \ldots & j_3 \\
 & 2 & 3 & 4 & & 7 & \ldots & j_4 .
\end{array} \qquad (1.5\text{-}6a)$$

The adjustment of this network yields

$$\mathrm{Tr}(N^{-1}) = 116{,}700.$$

Similarly, the second arrangement results from expanding the network of (1.5-5b), namely

$$\begin{array}{cccccccccl}
1 & 2 & 3 & 4 & & & & \ldots & j_1 \\
 & 2 & 3 & 4 & 5 & & & \ldots & j_2 \\
 & & 3 & 4 & 5 & 6 & & \ldots & j_3 \\
 & & & 4 & 5 & 6 & 7 & \ldots & j_4 .
\end{array} \qquad (1.5\text{-}6b)$$

From the adjustment of this network it is obtained:

$$\mathrm{Tr}(N^{-1}) = 277{,}800.$$

The arrangement of observations in (1.5-6b) is a typical case of "leapfrogging". Since the first replacements in (1.5-5a) and (1.5-5b) resulted in non-singular solutions, the sufficient conditions for both (1.5-6a) and (1.5-6b) to form non-singular networks are given by stipulating that the satellite gourp j_4 does not have any targets in the plane of ground stations (z = 0), and that those targets do not lie in the plane through station 7. Since both these conditions are fulfilled, the adjustment of (1.5-6a) and (1.5-6b) yields non-singular results (larger traces were caused mainly by larger N matrices due to twelve additional parameters).

Example 7. This last example illustrates that singularity A) can be removed when replacing of stations is applied. Since singularity A) is not likely to happen in practice, this example is mainly of theoretical interest. Originally, singularity A) was achieved by modifying the satellite group j_4 and station 7 so that they form a plane. The points which were modified, denoted by primes, are given as

7′ . . .	300,000.	0	0
193′ . . .	-247,462.4	350,000	800,000

The observations resulting in singularity A) are arranged in the following way:

$$\begin{array}{cccccccl} 1 & 2 & 3 & 4 & & & & \ldots j_1 \\ & 2 & 3 & 4 & 5 & & & \ldots j_2 \\ & 2 & 3 & 4 & & 6 & & \ldots j_3 \\ & 2 & 3 & 4 & & & 7' & \ldots j_{4'} \end{array} ;$$

the adjustment gives

$$Tr(N^{-1}) = .7 \times 10^9.$$

Next, station 7′ replaces station 4 (previously observing all the targets) for observations of the group j_3. This new arrangement is presented below:

$$\begin{array}{cccccccl} 1 & 2 & 3 & 4 & & & & \ldots j_1 \\ & 2 & 3 & 4 & 5 & & & \ldots j_2 \\ & 2 & 3 & 4 & & & 7' & \ldots j_{4'} \\ & 2 & 3 & 7' & & 6 & & \ldots j_3 . \end{array}$$

The adjustment gives in this case:

$$\mathrm{Tr}(N^{-1}) = 675,300.$$

Even though $j_{4'}$ (i.e., j_k in terms of section 1.41) lies in the plane through 7′ (i.e., k), station 6 (i.e., s) does not lie on the intersection of this plane and the plane of ground stations (i.e., the line ℓ). Accordingly, singularity A) was removed.

As a matter of fact, further replacements can be applied and the final arrangement may be such, that three different stations observe all the targets. If station 7′ is one of them, singularity A) is again removed: there is no condition which stipulates that no satellite group can be in plane with one of the three stations observing all the targets. Consequently, another adjustment was made with the observations arranged as follows:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 7' & & & \ldots j_1 \\ & 2 & 3 & 7' & 4 & & \ldots j_2 \\ & 2 & 3 & 7' & & 5 & \ldots j_3 \\ & 2 & 3 & 7' & & & 6 \ \ldots j_{4'}. \end{array}$$

The results in this case are such that

$$\mathrm{Tr}(N^{-1}) = 216,100.$$

In other words, when stations 2, 3, 7′ rather than stations 2, 3, 4 observe all the targets, singularity A) due to $j_{4'}$ and 7′ lying in one plane is removed.

1.6 Conclusions

In the past sections a rather detailed analysis has been carried out for the range observations. The ground stations have been assumed to be lying in one plane. They were denoted by numbers and letters in the sequence 1, 2, 3, 4, ... i ..., k, s', s'', ..., while the satellite groups observed by these stations were denoted as j_4, ... j_i ..., j_k, $j_{s'}$, $j_{s''}$ ..., respectively. A satellite group consists of those satellite points (targets) which are observed by a given quadrant (quad) of stations. The convention used for the subscript of a certain satellite group is such that the index indicates the number or letter of that station in the quad observing this satellite group which has not observed any other satellite group and/or which is listed as the last station in the quad; for example, the quad consisting of stations 1, 2, 3, and 4 observes the satellite group j_4. The division of a network into quads is convenient from the practical point of view. Considering more than four co-observing stations does not affect the derivations made with the above concept.

The discussion is divided into two basic parts, according to whether the number of ground stations observing all the satellite points is three or more, or less than three. When the number of stations observing all the targets is less than three the principle of replacing of stations (station replacement) is introduced, which leads directly to the concept of "leapfrogging". Both concepts, the first, dealing with at least three stations observing all the targets, and the second, dealing with replacing of stations lead to similar conclusions. The most important conclusion is that except for certain critical configurations of points (stations or targets or both) an adjustment of range networks gives non-singular results, in spite of the fact that all stations are in one plane. The network which can be non-singular with the smallest number of ground stations possible is said to constitute a fundamental unit. When at least three stations observe all the targets a fundamental unit consists of six stations. When the principle of station replacement is utilized a fundamental unit is also six stations, except for one specific observing pattern when the number of required stations is seven.

When three stations denoted as 1,2,3, are observing all the targets, the necessary and sufficient conditions for a network to be non-singular are easy to specify. One of the configurations which makes an adjustment singular is the case when all the targets in one satellite group needed for the determination of a fundamental unit are in a straight line. This is only a special case of a general pattern when all satellite points within a group (e.g., j_1) are in the plane containing the corresponding ground station (i). This case, called singularity A), is illustrated in Figure 1. In a more general sense, singularity A) is said to occur when all targets observed by a certain station - and such targets may be contained in more than one satellite group - are in the plane with this station. When exactly three stations (1,2,3) observe all targets, the targets observed by any particular station besides 1,2,3, are all contained in one satellite group. Under the assumption that singularity A) does not exist the necessary and sufficient conditions for a network to be non-singular are such that at least three stations in addition to those three (1,2,3) observing all the targets must observe targets which are not all in one (general) plane (off-plane targets) and that these three stations must not lie on one second order curve with stations 1,2,3. If these conditions are not fulfilled it is said that singularity C) has occurred; such configuration of points is illustrated in Figure 3. A special case of singualrity C) is singularity B) when all the ground stations are on one second order curve (Figure 2). From the above conditions it is seen that a fundamental unit consists of six ground stations. If such a fundamental unit exists, it is always possible to expand a network by adding further stations and satellite groups, the necessary and sufficient conditions being that no target should lie in the plane of the ground stations and that no station should lie in a plane with its observed targets.

If all ground stations are co-observing, then singularity in a network could occur only if all the stations are on one second order curve, or if all the targets (in this case all the satellite groups coincide) are lying in one plane. These two cases are illustrated in Figures 4 and 5, respectively. Otherwise, the solution is non-singular. Numerical results indicated that when all the stations observed simultaneously the solution was strengthened very significantly.

When dealing with the concept of station replacement, it is concluded that one replacement (leapfrogging) can be sufficient to build a fundamental unit, from which further expansion is possible under certain conditions. Therefore, a great deal of time was devoted analyzing the problem of one replacement where the fundamental unit is assumed to comprise of stations 1,2,3,4, and the satellite group j_4 to contain off-plane targets. After two quads (formed by stations 1,2,3, 4, and stations 1,2,3,k) have completed their observtions, the first replacement will take place. It consists of station k replacing station 3 for the next observations. The satellite group j_s' is then observed by the quad of stations 1,2,k,s', etc. At this point, the discussion is divided into two parts: in the first part the satellite group j_k contains off-plane targets; in the second part, which is rather special and mainly of theoretical interest, the targets in j_k are in one plane. It is true for both parts that a network is singular if the targets in any of the satellite groups (including j_k in the second part) needed for the determination of a fundamental unit are in a straight line. This conclusion is similar to what was mentioned for three stations observing all the targets. It is again assumed that no satellite group lies in a plane passing through the corresponding station. Thus, singularity A) cannot exist. The cases denoted as (a2), (b2), and (c2) in section 1.412 or 1.413 are not considered in the second part, since in these three cases the satellite group j_k was assumed to contain targets lying in one plane with station k.

With the above assumption, the necessary and sufficient conditions for a non-singular solution in the first part (j_k containing off-plane satellites) are similar to those given for three stations observing all the targets. Namely, the network is non-singular if there is at least one more satellite group (in addition to j_4 and j_k) containing off-plane targets and if the corresponding station does not lie on a second order curve with stations 1,2,3,4, and k. In other words, at least three stations not lying on a second order curve with stations 1,2,3 must observe off-plane targets. Therefore, a fundamental unit in this part consists also of six ground stations.

The second part, rather artificial, deals with such cases when the satellite group j_k is composed of targets lying all in one plane (assumed not to pass through station k). The necessary conditions for a non-singular network stipulate that there must be at least two additional satellite groups (besides j_4) which contain off-plane targets. Consequently, a fundamental unit in this part includes seven ground stations (i.e., two stations in addition to stations 1,2,3,4, and k). The two satellite groups of the required property can be chosen in three different ways which in section 1.412 or 1.413 were presented as cases (a1), (b1), and (c1). In case (a1), these two satellite groups correspond to stations s' and s'' (both following station k); a network is singular if the plane of j_k has a specific position, given by (1.4-49) and (1.4-50), or if both s' and s'' are lying on a second order curve through stations 1,2,3,4 and k. In case (b1), one of these groups corresponds to some station i and the other to some station s (in sections 1.412 and 1.413 they were numbered as station 5 and station s'); a network is singular if the plane of j_k has a specific position, given by (1.4-82) and (1.4-82a), or if both stations i and s are lying on a second order curve through stations 1,2,3,4,k, or (in case station i does not have this property), if station s is lying on a specific second order curve with stations 1,2,3, and 4 given by (1.4-70a) - (1.4-70c). In case (c1) both satellite groups correspond to some stations i; a network is singular if these two stations are lying on a second order curve with stations 1,2,3,4. If the circumstances leading to singularity in the cases (a1), (b1), and (c1) are avoided, then the above necessary conditions are also sufficient for a non-singular network.

If the first replacement is successfully carried out, then the resulting fundamental unit can be expanded to become a larger, non-singular network. When new stations and satellite groups are added to it, the necessary and sufficient conditions for the new network to be non-singular are the same as those for similar enlargement when three stations observed all the targets; namely, no target should be in the plane of the ground stations and no station should be in a plane with its observed targets.

The main results of this section are summarized in Table (1.6-1).

Since the number of ground stations is always relatively small compared to the number of targets, the most important conclusion is that ground stations should not be distributed on or near a second order curve.

Table (1.6-1)

Necessary and Sufficient Conditions to Avoid Singular Solutions

When All Ground Stations Are in a Plane

<table>
<tr><th>Type of Singularity</th><th colspan="2">Arrangement of Observations</th><th>Necessary Conditions to Prevent Singularity</th><th>Sufficient Conditions to Prevent Singularity</th><th>Note</th></tr>
<tr><td>Singularity A) (or closely related singularity)</td><td colspan="2">Any</td><td>No station should be in a plane with all its observed targets (distributed over one or more satellite groups)</td><td>No station should be in a plane with the corresponding satellite group</td><td>This singularity is assumed non-existent in analysis of singularity C)</td></tr>
<tr><td rowspan="4">Singularity C) (global type of singularity)</td><td colspan="2">Stations 1, 2, 3 observe all targets</td><td>Three stations in addition to 1, 2, 3 not lying on a second order curve with them should observe off-plane targets</td><td>The same as the necessary conditions</td><td rowspan="3">Special case of singularity C) is singularity B); it occurs when all stations are on a second order curve</td></tr>
<tr><td rowspan="2">Station k replaces station 3 (satellite group j_4 contains off-plane targets)</td><td>Group j_k contains off-plane targets</td><td>One station in addition to 4 and k not lying on a second order curve with 1, 2, 3, 4, k should observe off-plane targets</td><td>The same as the necessary conditions</td></tr>
<tr><td>Group j_k contains in-plane targets</td><td>Two stations in addition to 4 should observe off-plane targets. Always: Avoid all stations lying on a second order curve</td><td>More complex requirements (according to stations which observe off-plane targets)</td></tr>
<tr><td colspan="2">All stations observe all targets (all stations co-observe)</td><td>Avoid all targets lying in a plane (any plane) and all stations lying on a second order curve</td><td>The same as the necessary conditions</td><td></td></tr>
</table>

2. TREATMENT OF RANGE OBSERVATIONS WITH GROUND STATIONS GENERALLY DISTRIBUTED

2.1 Introduction

In this chapter, the ground stations in fundamental range networks are considered to be generally distributed in space. This discussion covers range observations made over a large territory, when ground stations are on the physical surface of the earth, departing significantly from a plane. Since the ground stations in this instance are all approximately on a sphere, their distribution in space is not completely general. However, whenever they depart from a plane, the nature of the problem is the same regardless of further specifications.

The observations are again divided into quads with similar notations as those used previously. Whether four or more ground stations observe simultaneously has again no effect on the derivations. Most of the investigations and derivations will be carried out for such networks where at least three stations observe all the targets. A solution will be shown to be singular when, for each quad and corresponding satellite group, all points involved (four ground stations and all the targets in that satellite group) lie on a specific second order critical surface. This applies regardless whether the ground network consists of one quad or more. A solution could be also singular due to singularity A) discussed earlier. If singularity A) does not exist (this necessarily implies that a satellite group needed for a network should not have all its targets on a straight line) then the critical surfaces can be computed (and thus avoided). There are no specific conditions holding for ground stations only which would lead to singular solutions.

Consequently, with singularity A) non-existent, a solution will be singular if certain (or all) stations together with certain (or all) satellite points lie on specific second order surface(s). However, such cases are not likely to happen in practice for the following reasons:

(a) Distribution of ground stations alone does not induce any type of singularity. Since the number of ground stations is always limited, their distribution presented a cause for concern in the first chapter; it is irrelevant in this chapter, however.

(b) If a network is singular, it is caused by all the satellite points lying on certain second order surfaces (together with some ground stations). This could seldom happen in practice as the number of targets may be very high; thus the probability of all the targets lying on specific second order surfaces ~~would be~~ very small.

The investigations in this chapter can be certainly useful when only a small number of targets is observed because then it could happen that they all lie near one or more specific second order surfaces.[1]

For the reasons cited above, the range investigations for ground stations in general configuration are principally of theoretical interest. They are presented here to make the study related to range observations complete.

[1]It could happen that the satellite passes observed from the middle of a ground network (extending over an area much smaller than a hemisphere) have the lowest altitude, while the passes observed near the edges of the network have increasingly much higher altitudes. If in appropriate scale, such configuration could be approximated by a hyperboloid of two sheets, provided there was not even one target at higher altitude observed from the middle of the network and not one target of lower altitude observed from stations located towards the edges. This case, illustrated in Figure 7, is clearly quite artificial.

2.2 Range Observations from Four Ground Stations in General Configuration

The basic steps needed for defining the coordinate system (local coordinate system) and for obtaining $\tilde{A}$ matrix with the ground stations in general configuration and the satellite parameters eliminated are the same as those used in section 1.2. The explicit form of $\tilde{A}$ matrix is given in Table (1.2-2). Similarly to what was said there, a network is singular or non-singular if the corresponding $\tilde{A}$ matrix is singular or non-singular (the word "singular" used in the same context).

With four ground stations denoted as 1, 2, 3, 4, forming the ground network, the problem can be non-singular if at least six targets are being co-observed. Let the first five targets be denoted by their coordinates as (X_i, Y_i, Z_i), $i = 1, 2, \ldots 5$, and let the sixth and any further target be represented as (X_j, Y_j, Z_j). Matrix $\tilde{A}$ is then identical with the matrix represented by Table (1.2-2), with only the first row block and non-zero column blocks present. It has six columns and as many rows as there are targets observed from the four stations.

2.21 Critical Surface for Four Ground Stations Using Determinant Approach.

Matrix $\tilde{A}$ will be singular if any determinant of its (6 x 6) submatrices is equal to zero. In such case its row space (or column space) is of dimension five in general. It is assumed that five independent rows in $\tilde{A}$ correspond to satellite points 1 through 5. If each further row is in the row space spanned by the above five rows, then all the (6 x 6) submatrices of $\tilde{A}$ are singular and so is $\tilde{A}$. Therefore, $\tilde{A}$ is singular if the determinant of its submatrix $\tilde{A}_4$, corresponding to targets 1 through 5 and every target j, is equal to zero. From Table (1.2-2) it is seen that

$$|\tilde{A}_4| = 0 \tag{2.2-1}$$

represents a second order surface in (X_j, Y_j, Z_j). Since a second order surface is in general defined by nine points, it will be examined what nine points satisfy (2.2-1).

When any of (X_i, Y_i, Z_i), $i = 1, 2, \ldots 5$ is substituted for (X_j, Y_j, Z_j), then (2.2-1) holds, since in the determinant two rows are equal. If the coordinates of ground stations 1, 2, 3, 4, namely $(0, 0, 0)$, $(x_2, 0, 0)$, $(x_3, y_3, 0)$, (x_4, y_4, z_4) are gradually substituted for (X_j, Y_j, Z_j), then (2.2-1) also holds since the last row in the determinant contains only zeroes. Consequently, the second order surface for any target j can be determined as passing through all four ground stations and the first five targets and the problem with four ground stations is singular whenever all the points (stations and targets) are lying on one second order surface. This property was demonstrated also in Appendix 8, where the determinant in (2.2-1) was developed in terms of station 4 rather than in terms of target j. However, this procedure was extremely long and tedious compared to the approach used in this section. It demonstrated, among other things, that it is preferable to work in terms of the targets' coordinates. The numerical computations of a second order surface can be made more easily using the technique of fitting such surface to nine points according to the description given in Appendix 6, rather than to use the approach of Appendix 8; the numerical results in both cases have been found to agree very well, within round-off errors.

From Table (1.2-2), it appears that singularity A) should be also taken into consideration, using the same approach as in the first chapter. One can see immediately that the same conclusions expanded by taking into consideration $Z = 0$, can be drawn now: singularity A) with respect to station 4 occurs if every one of the (corresponding) targets is lying either in one plane through station 4, or in the plane of stations 1, 2, and 3. However, in the case of four ground stations, this represents a special case of the global singularity which occurs when all the points are lying on a second order surface; namely, it represents an intersection of two planes: one, which is the plane of all the targets and station 4, and the other which is the plane of stations 1, 2, and 3. Similarly, what had been defined as singularity B) is only a special case of a second order surface which would arise under certain conditions for the distribution of points; it would be again included in global singularity. This can be said for singularity B) even when more than four ground stations are involved. With general distribution of ground stations (i.e., not lying in one plane) the effect of ground stations cannot be separated from the effect of satellite points; for this reason, singularity B) is completely irrelevant in this chapter. On the other hand, singularity A) will have to be considered and eliminated separately when more than four ground stations are involved; however, no further derivations will be needed in this respect, since singularity A) occurs under the same conditions as presented in the first chapter expanded by taking into consideration $Z = 0$ as it was done for station 4. When considering stations 1, 2, and 3 as observing all the targets, it can be summarized as follows: singularity A) occurs if every one of the targets in some satellite group is lying either in one plane through the corresponding station, or in the plane of stations 1, 2, and 3.

Finally, one very peculiar type of singularity, which could be called "reverse singularity B)" will be mentioned. It is mainly of theoretical interest, however. Since all the four stations observe all the targets, the two sets of points are equivalent in that each point in one set "observes" each point in the other set. Thus, all the targets are "co-observing" all the stations. If the targets were all lying in one plane, singularity B) would occur if they were also lying on a second order curve. Singularity A) or singularity C) could not occur, since the "observed" points (i.e., ground stations) do not lie in a plane. Consequently, the problem could be singular if all the targets were lying on one second order (plane) curve. This could approximately occur in practice if the four ground stations were observing the satellite points on two short passes of approximately the same altitude. Exactly the same conclusions can be drawn for networks with more than four ground stations, whether all the stations are co-observing or not.

2.22 Critical Surface for Four Ground Stations Using Canonical Approach.

The principle of this approach is the following: in case of singularity A) eliminated for station 4, i.e., in case of the non-singular three column block for station 4, it is always possible to bring to zero all except three rows in this three column block by row equivalence operations; these three rows can be assumed to correspond to the first three targets; then, using column equivalence operations, the elements of these three rows in all the (three) remaining columns can be brought to zero. Thus $\widetilde{A}$ matrix has been modified (without having the rank changed) in such a way, that it has a non-singular (3×3) submatrix in its upper left corner, with zeroes everywhere else in the first three rows and columns. Had singularity A) occurred, none of these operations and no further analysis would have been necessary. Thus, using the canonical approach, singularity A) will be assumed eliminated. The following derivation for four ground stations, as well as later for more stations, will be based on this assumption.

The practical way of bringing zeroes to the three column block of station 4 (and any other station in later derivations) is based on the fact that the fourth, fifth, and any further non-zero row in this three column block must lie in the row space of the first three rows which are assumed independent. Consequently, any such row, now denoted as row j, can be brought to zero by adding to it the proper linear combination of the first three rows. The corresponding coefficients of these rows will be denoted as k_1, k_2, k_3. They can be computed as follows:

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = - \begin{bmatrix} Z_1(X_1-x_4) & Z_2(X_2-x_4) & Z_3(X_3-x_4) \\ Z_1(Y_1-y_4) & Z_2(Y_2-y_4) & Z_3(Y_3-y_4) \\ Z_1(Z_1-z_4) & Z_2(Z_2-z_4) & Z_3(Z_3-z_4) \end{bmatrix}^{-1} \begin{bmatrix} Z_j(X_j-x_4) \\ Z_j(Y_j-y_4) \\ Z_j(Z_j-z_4) \end{bmatrix} . \tag{2.2-2}$$

This is seen directly from the corresponding three column block of $\widetilde{A}$ matrix in Table (1.2-2). The inverse in (2.2-2) exists due to the earlier assumptions. The same row operations have to be performed on all the columns of $\widetilde{A}$ matrix (i.e., on the entire row j). This will change the j th row in the last three column block (by adding to it the same linear combination of the first three rows). When this is accomplished for all the rows j (i. e., for the rows corresponding to targets 4, 5, 6, and any further targets), the first three column block has the desired form. Bringing to zeroes the first three rows in the last column block is accomplished at once. If there were o rows in the original $\widetilde{A}_4$ matrix, its form after the above operations would be:

$$\widetilde{A}_4 \sim \left[\begin{array}{c|c} \underset{(3 \times 3)}{P} & \underset{(3 \times 3)}{0} \\ \hline \underset{(o-3) \times 3}{0} & \underset{(o-3) \times 3}{\overline{A}_4} \end{array}\right] . \tag{2.2-3}$$

From (2. 2-3) it is evident that $\widetilde{A}_4$ is singular if and only if $\overline{A}_4$ is singular (i. e., has rank smaller than three). Thus the problem has been reduced to analyzing $\overline{A}_4$ matrix, which has only three columns. Analogous reductions will be made for more than four ground stations.

Next, the rows of $\overline{A}_4$ will be obtained, using the coefficients k. In order to solve (2. 2-2), the determinant of the matrix to be inverted, denoted as D_4, is after some algebraic manipulations obtained as

$$D_4 = a_1 x_4 + a_2 y_4 + a_3 z_4 + a_4 \tag{2.2-4}$$

where

$$\begin{aligned} a_1 &= a_{11} + a_{12} + a_{13}, \\ a_2 &= a_{21} + a_{22} + a_{23}, \\ a_3 &= a_{31} + a_{32} + a_{33}, \\ a_4 &= -(Z_1 a_{31} + Z_2 a_{32} + Z_3 a_{33}), \end{aligned} \tag{2.2-4a}$$

and where

$$a_{11} = -Y_2Z_3 + Y_3Z_2,\quad a_{12} = Y_1Z_3 - Y_3Z_1,\quad a_{13} = -Y_1Z_2 + Y_2Z_1,$$

$$a_{21} = X_2Z_3 - X_3Z_2,\quad a_{22} = -X_1Z_3 + X_3Z_1,\quad a_{23} = X_1Z_2 - X_2Z_1 \qquad (2.2\text{-}4b)$$

$$a_{31} = -X_2Y_3 + X_3Y_2,\quad a_{32} = X_1Y_3 - X_3Y_1,\quad a_{33} = -X_1Y_2 + X_2Y_1 .$$

The part of row j which is located in the last three column block will be denoted as r_j; it will have the form (after the row equivalence operations have been performed):

$$\underset{(1\times 3)}{r_j} = \frac{1}{D_4}\left\{ \begin{bmatrix} D_4\,(z_4\,Y_j - y_4\,Z_j)\,(X_j - x_3) \\ D_4\,(z_4\,Y_j - y_4\,Z_j)\,(Y_j - y_3) \\ D_4\,[\,z_4\,(X_j - Y_j\,\frac{x_3}{y_3}) - Z_j\,c_4\,]\,(X_j - x_2) \end{bmatrix}^{T} - \right.$$

$$- \begin{bmatrix} (z_4Y_1-y_4Z_1)(X_1-x_3)(-D_4k_1) & + (z_4Y_2-y_4Z_2)(X_2-x_3)(-D_4k_2) & + \\ (z_4Y_1-y_4Z_1)(Y_1-y_3)(-D_4k_1) & + (z_4Y_2-y_4Z_2)(Y_2-y_3)(-D_4k_2) & + \\ [z_4(X_1-Y_1\frac{x_3}{y_3})-Z_1c_4](X_1-x_2)(-D_4k_1) & + [z_4(X_2-Y_2\frac{x_3}{y_3})-Z_2c_4](X_2-x_2)(-D_4k_2) & + \end{bmatrix}$$

$$\left. \begin{matrix} + (z_4Y_3-y_4Z_3)(X_3-x_3)(-D_4k_3) \\ + (z_4Y_3-y_4Z_3)(Y_3-y_3)(-D_4k_3) \\ + [z_4(X_3-Y_3\frac{x_3}{y_3})-Z_3c_4](X_3-x_2)(-D_4k_3) \end{matrix} \right]^{T} \right\} \qquad (2.2\text{-}5)$$

where

$$c_4 = x_4 - y_4\frac{x_3}{y_3}.$$

The terms $(-D_4k_1)$, $(-D_4k_2)$, and $(-D_4k_3)$ can be expressed as

$$-D_4k_1 = \frac{1}{Z_1}\,[X_jZ_j \times B_1 + Y_jZ_j(-C_1) + Z_j^2 \times D_1 + Z_j \times E_1],$$

$$-D_4k_2 = \frac{1}{Z_2}\,[X_jZ_j(-B_2) + Y_jZ_j \times C_2 + Z_j^2\,(-D_2) + Z_j \times E_2], \qquad (2.2\text{-}6)$$

and

$$-D_4k_3 = \frac{1}{Z_3}\,[X_jZ_j \times B_3 + Y_jZ_j(-C_3) + Z_j^2 \times D_3 + Z_j \times E_3]$$

where

$$
\begin{aligned}
B_1 &= -a_{11} - (Y_2 - Y_3)\, z_4 + (Z_2 - Z_3)\, y_4 , \\
C_1 &= a_{21} - (X_2 - X_3)\, z_4 + (Z_2 - Z_3)\, x_4 , \\
D_1 &= -a_{31} - (X_2 - X_3)\, y_4 + (Y_2 - Y_3)\, x_4 , \\
E_1 &= a_{11} x_4 + a_{21} y_4 + a_{31} z_4 ,
\end{aligned}
\tag{2.2-6a}
$$

and

$$
\begin{aligned}
B_2 &= a_{12} - (Y_1 - Y_3)\, z_4 + (Z_1 - Z_3)\, y_4 , \\
C_2 &= -a_{22} - (X_1 - X_3)\, z_4 + (Z_1 - Z_3)\, x_4 , \\
D_2 &= a_{32} - (X_1 - X_3)\, y_4 + (Y_1 - Y_3)\, x_4 , \\
E_2 &= a_{12} x_4 + a_{22} y_4 + a_{32} z_4 ,
\end{aligned}
\tag{2.2-6b}
$$

and

$$
\begin{aligned}
B_3 &= -a_{13} - (Y_1 - Y_2)\, z_4 + (Z_1 - Z_2)\, y_4 , \\
C_3 &= a_{23} - (X_1 - X_2)\, z_4 + (Z_1 - Z_2)\, x_4 , \\
D_3 &= -a_{33} - (X_1 - X_2)\, y_4 + (Y_1 - Y_2)\, x_4 , \\
E_3 &= a_{13} x_4 + a_{23} y_4 + a_{33} z_4 .
\end{aligned}
\tag{2.2-6c}
$$

The three expressions in (2.2-6) are all equal to zero whenever $Z_j = 0$ (and so when target j is replaced by any of stations 1, 2, 3) and also if (X_j, Y_j, Z_j) is substituted for by (x_4, y_4, z_4). Since $D_4 \neq 0$, it must hold that

$$k_1 = k_2 = k_3 = 0 \qquad \text{whenever } j \equiv 1, 2, 3, \text{ or } 4; \tag{2.2-7a}$$

further,

$$
\begin{aligned}
k_1 &= -1, \quad k_2 = 0, \quad k_3 = 0 \quad \text{whenever } j \equiv \text{target } 1, \\
k_1 &= 0, \quad k_2 = -1, \quad k_3 = 0 \quad \text{whenever } j \equiv \text{target } 2, \\
k_1 &= 0, \quad k_2 = 0, \quad k_3 = -1 \quad \text{whenever } j \equiv \text{target } 3.
\end{aligned}
\tag{2.2-7b}
$$

The relations (2.2-7b) can be immediately found by inspection from (2.2-2); they follow also from (2.2-6)-(2.2-6c), which thus verifies all the above derivations.

The matrix $\overline{A}_4$ is such that it is composed of the rows r_j, $j = 4, 5, 6, \ldots$; if any target beyond targets 4 and 5 is denoted as a variable point (X, Y, Z) and its row by the letter r, then $\overline{A}_4$ can be written as

$$\overline{A}_4 = \begin{bmatrix} r_4 \\ r_5 \\ r \\ \vdots \end{bmatrix} . \tag{2.2-8}$$

Should $\overline{A}_4$ be singular then every row r would have to lie in the row space of r_4 and r_5, assumed independent (they would be dependent if it held that $z_4 = 0$, i.e., if all the observing stations were lying in a plane which is not true in this chapter); namely, it would hold for any row r that

$$\begin{vmatrix} r_4 \\ r_5 \\ r \end{vmatrix} = 0. \tag{2.2-9}$$

From the form of (2.2-5) and (2.2-6), it is clear that the row r contains the terms of first and second order in (X, Y, Z). Therefore, (2.2-9) expresses the condition that the variable point (X, Y, Z) lies on a second order surface passing through the origin (of the local coordinate system).

As usual, it will be useful to find nine points through which the second order surface passes and which could in general serve for its definition (see for instance Appendix 6). Clearly, whenever (X, Y, Z) is the same as (X_4, Y_4, Z_4) or (X_5, Y_5, Z_5), then r is the same as r_4 or r_5 and (2.2-9) holds. Further, whenever the variable point in (2.2-5) - there appearing as (X_j, Y_j, Z_j) - is substituted for by stations 1, 2, 3, or 4, that row becomes a zero row (for the first part in (2.2-5) it is seen directly and for the second part it follows when the conditions (2.2-7a) are considered). The same is true when the variable point is replaced by targets 1, 2, or 3;

this follows directly from (2.2-5) and (2.2-7b). Thus, the second order surface defined by (2.2-9) for any target (X, Y, Z) beyond target 5 passes through stations 1, 2, 3, 4 and targets 1, 2, 3, 4, and 5. It can be concluded that a problem is singular whenever all the (four) ground stations and all the targets are lying on one second order surface. This property has been demonstrated already in section 2.21 and in Appendix 8. However, the canonical approach from this section is the most important as it will be used for networks containing more than four ground stations as well.

2.23 Computations of Critical Surface for Four Ground Stations.

2.231 General Considerations.

A general equation of second degree in three variables (x, y, z) can be written as

$$Ax^2 + Hxy + Gxz + By^2 + Fyz + Cz^2 + Px + Qy + Rz + D = 0. \tag{2.2-10}$$

The solutions of this equation can be represented by a second degree surface. If the constant term D in (2.2-10) is equal to zero, the surface passes through the origin of the coordinate system. Some of the following notations and descriptions are taken from [4], p. 362. The (3 x 3) matrix containing the coefficients of the quadratic terms in (2.2-10) will be denoted by the letter A (not to be confused with the coefficient A in the above equation); its form is given as

$$A = \begin{bmatrix} A & \frac{1}{2}H & \frac{1}{2}G \\ \frac{1}{2}H & B & \frac{1}{2}F \\ \frac{1}{2}G & \frac{1}{2}F & C \end{bmatrix}. \tag{2.2-10a}$$

The matrix obtained from (2.2-10a) and denoted as E is defined in the following manner:

$$E = \begin{bmatrix} A & \frac{1}{2}H & \frac{1}{2}G & \frac{1}{2}P \\ \frac{1}{2}H & B & \frac{1}{2}F & \frac{1}{2}Q \\ \frac{1}{2}G & \frac{1}{2}F & C & \frac{1}{2}R \\ \frac{1}{2}P & \frac{1}{2}Q & \frac{1}{2}R & D \end{bmatrix}.$$

Its determinant is denoted by the symbol Δ, i.e.,

$$\Delta = |E|.$$

If further notations are introduced, namely

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad a = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \qquad c = D,$$

then the equation representing a second degree surface can be written as

$$x^T A x + x^T a + c = 0; \tag{2.2-11}$$

when the surface passes through the origin, its equation reduces to

$$x^T A x + x^T a = 0. \tag{2.2-11a}$$

A general case to be investigated is such that both A and E matrices have full rank. Its solution may be either real or complex. Of all possible cases, only those which have a real solution and whose matrices A and E both have full rank are of importance in this study. They are presented in Table (2.2-1). The fourth column there has the heading "signs of λ's"; since λ represents any of the three eigenvalues of the above A matrix, this column specifies whether all three λ's have the same sign or not. In the case of an ellipsoid, the signs which are the same must be positive (otherwise the ellipsoid would be imaginary).

Table (2.2-1)

Description of Pertinent Second Order Surfaces

Rank e	Rank E	Δ	Signs of λ's	Kind of Real Surface
3	4	< 0	Same	Ellipsoid
3	4	> 0	Different	Hyperboloid of one sheet
3	4	< 0	Different	Hyperboloid of two sheets

The distinctions specified in this table were used in Appendix 4, section A4.3, when dealing with solutions of second degree equations of the type (2.2-11).

In the present study, equations of the type (2.2-11a) are obtained with respect to the coordinate system in which they were derived, i.e., with respect to the local coordinate system. However, in general the points of a network are given in a different coordinate system, called basic. In order to obtain the critical surfaces in the basic coordinate system, transformations have to be applied between this and the local coordinate system. First, the following notations will be introduced:

$$X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \ldots$$ coordinates of a point in the basic coordinate system; thus in particular,

$$X_o = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} \quad \ldots$$ coordinates of the origin of the local coordinate system;

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \ldots$$ coordinates of a point in the local coordinate system.

According to (A4-4a) - (A4-4c) in Appendix 4,

$$x = P^T (X - X_o) \tag{2.2-12}$$

where x was identified with X' and P with R, a (3×3) orthogonal matrix. This formula gives the coordinates of a point in the local coordinate system computed from the coordinates given in the basic coordinate system; it is said to transform the coordinates from the basic to the local coordinate system. From (2.2-12) or (A4-4a) it is obtained that

$$X = X_o + P x \,, \qquad (2.2\text{-}13)$$

which is said to transform the coordinates from the local to the basic coordinate system. The orthogonal matrix P is made up of the directional cosines pertaining to the relative orientation of the two coordinate systems, namely

$$P = \begin{bmatrix} \cos(X, x) & \cos(X, y) & \cos(X, z) \\ \cos(Y, x) & \cos(Y, y) & \cos(Y, z) \\ \cos(Z, x) & \cos(Z, y) & \cos(Z, z) \end{bmatrix} . \qquad (2.2\text{-}14)$$

Here X, Y, Z represents the three axes of the basic coordinate system and x, y, z the three axes of the local coordinate system; thus (X, x) represents the angle between the X-axis and the x-axis, with similar description for the other angles.

The values needed for the transformation equations (2.2-12) and (2.2-13) are the vector X_o and the matrix P. The vector X_o is simply given as

$$X_o = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \quad \ldots \text{ coordinates of station 1 in the basic coordinate system.} \qquad (2.2\text{-}15)$$

(This notation should not be confused with (X_1, Y_1, Z_1) from all the other sections where the capital letters are reserved for targets' coordinates and the small letters are used for stations' coordinates; the capital letters are used for stations' coordinates exclusively in the problem of transformation of coordinates between the local and the basic coordinate systems involving only stations 1, 2, 3; in such a problem the targets' coordinates do not appear at all). The direction of the x-axis in terms of the basic coordinate system is given by the unit vector $\hat{i}$, where

$$\hat{i} \equiv \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} \cos(x, X) \\ \cos(x, Y) \\ \cos(x, Z) \end{bmatrix} . \qquad (2.2\text{-}16a)$$

Since the x-axis was defined as the line connecting stations 1 and 2, it holds that

$$i_1 = \frac{X_2 - X_1}{S_{12}}, \quad i_2 = \frac{Y_2 - Y_1}{S_{12}}, \quad i_3 = \frac{Z_2 - Z_1}{S_{12}}$$

where

$$S_{12} = \left[(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2\right]^{\frac{1}{2}}.$$

The z-axis was defined to be perpendicualr to the plane of stations 1, 2, 3; a vector v in this direction is given as a cross product of a vector in direction from station 1 to station 2 and a vector in direction from station 1 to station 3, which will be written as

$$v = (12) \times (13)$$

with appropriate vector interpretations. The vector v is then computed as

$$v \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (Y_2-Y_1)(Z_3-Z_1) - (Z_2-Z_1)(Y_3-Y_1) \\ (Z_2-Z_1)(X_3-X_1) - (X_2-X_1)(Z_3-Z_1) \\ (X_2-X_1)(Y_3-Y_1) - (Y_2-Y_1)(X_3-X_1) \end{bmatrix} .$$

With s denoting the norm of v, i.e., with

$$s = \left[v_1^2 + v_2^2 + v_3^2\right]^{\frac{1}{2}},$$

the direction of the z-axis can be given by the unit vector $\hat{k}$ such that

$$\hat{k} \equiv \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \cos(z, X) \\ \cos(z, Y) \\ \cos(z, Z) \end{bmatrix} \qquad (2.2\text{-}16b)$$

where

$$k_1 = \frac{v_1}{s}, \quad k_2 = \frac{v_2}{s}, \quad k_3 = \frac{v_3}{s}.$$

Finally, the unit vector of the direction of the y-axis is given as

$$\hat{j} \equiv \begin{bmatrix} j_1 \\ j_2 \\ j_3 \end{bmatrix} = \begin{bmatrix} \cos(y, X) \\ \cos(y, Y) \\ \cos(y, Z) \end{bmatrix} ; \tag{2.2-16c}$$

since $\hat{j}$ is a cross-product of $\hat{k}$ with $\hat{i}$, namely,

$$\hat{j} = \hat{k} \times \hat{i},$$

its components are given as follows:

$$j_1 = k_2 i_3 - k_3 i_2, \quad j_2 = k_3 i_1 - k_1 i_3, \quad j_3 = k_1 i_2 - k_2 i_1 .$$

Consequently, using (2.2-16a) - (2.2-16c) in the formula (2.2-14), the matrix P can be written as

$$P = \begin{bmatrix} i_1 & j_1 & k_1 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix} . \tag{2.2-17}$$

2.232 Critical Surface Algebraically, in Local Coordinate System.

The critical surface for four ground stations has been shown to be a second order surface passing through all four stations and (first) five targets. It could be found for instance by the fitting procedure described in Appendix 6. However, it will be helpful to find the form (2.2-11a) of this second order surface in the local coordinate system independently, using the canonical approach.

To do this, it will be necessary to obtain the three elements of the row r, appearing in (2.2-8) or (2.2-9), explicitly in terms of (X, Y, Z). It will be done using the relation (2.2-5) for r_j, where j holds for any target beyond 3. Thus for row r, corresponding to any target beyond 5, the index j will be simply dropped everywhere in (2.2-5). The three elements of this row will be denoted as r^1, r^2, and r^3, so that

$$r \equiv [r^1 \ \ r^2 \ \ r^3]. \tag{2.2-18}$$

Using the relations (2.2-6) together with (2.2-5), after some algebraic manipulations the following result, written conveniently in a matrix form, is obtained:

$$r^{\mathsf{T}} \equiv \begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} = \frac{1}{D_4} \underset{(3\times 9)}{M_4} \underset{(9\times 1)}{V} . \qquad (2.2\text{-}19)$$

the elements of M_4 are given for the first row as

$$m_{11} = 0, \quad m_{12} = z_4D_4, \quad m_{13} = -y_4D_4 - A_1B_1(X_1-x_3) + A_2B_2(X_2-x_3) - A_3B_3(X_3-x_3),$$

$$m_{14} = 0, \quad m_{15} = A_1C_1(X_1-x_3) - A_2C_2(X_2-x_3) + A_3C_3(X_3-x_3), \quad m_{16} = -A_1D_1(X_1-x_3)+$$

$$+A_2D_2(X_2-x_3) - A_3D_3(X_3-x_3), \quad m_{17} = 0, \quad m_{18} = -x_3z_4D_4, \quad m_{19} = x_3y_4D_4 - A_1E_1(X_1-x_3)-$$

$$-A_2E_2(X_2-x_3) - A_3E_3(X_3-x_3),$$

for the second row as

$$m_{21} = 0, \quad m_{22} = 0, \quad m_{23} = -A_1B_1(Y_1-y_3) + A_2B_2(Y_2-y_3) - A_3B_3(Y_3-y_3), \quad m_{24} = z_4D_4,$$

$$m_{25} = -y_4D_4 + A_1C_1(Y_1-y_3) - A_2C_2(Y_2-y_3) + A_3C_3(Y_3-y_3), \quad m_{26} = -A_1D_1(Y_1-y_3)+$$

$$+A_2D_2(Y_2-y_3) - A_3D_3(Y_3-y_3), \quad m_{27} = 0, \quad m_{28} = -y_3z_4D_4, \quad m_{29} = y_3y_4D_4 - A_1E_1(Y_1-y_3)-$$

$$-A_2E_2(Y_2-y_3) - A_3E_3(Y_3-y_3),$$

and for the third row as

$$m_{31} = z_4D_4, \quad m_{32} = -\frac{x_3}{y_3} z_4D_4, \quad m_{33} = -c_4D_4 - K_1B_1 + K_2B_2 - K_3B_3, \quad m_{34} = 0,$$

$$m_{35} = K_1C_1 - K_2C_2 + K_3C_3, \quad m_{36} = -K_1D_1 + K_2D_2 - K_3D_3, \quad m_{37} = -x_2z_4D_4,$$

$$m_{38} = x_2\frac{x_3}{y_3} z_4D_4, \quad m_{39} = x_2c_4D_4 - K_1E_1 - K_2E_2 - K_3E_3$$

where new notations have been introduced as follows:

$$A_1 = \frac{Y_1}{Z_1} z_4 - y_4, \quad A_2 = \frac{Y_2}{Z_2} z_4 - y_4, \quad A_3 = \frac{Y_3}{Z_3} z_4 - y_4, \qquad (2.2\text{-}19a)$$

and

$$K_1 = (X_1 - x_2)\left[z_4\left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right) - c_4\right],$$

$$K_2 = (X_2 - x_2)\left[z_4\left(\frac{X_2}{Z_2} - \frac{Y_2}{Z_2}\frac{x_3}{y_3}\right) - c_4\right], \qquad (2.2\text{-}19b)$$

$$K_3 = (X_3 - x_2)\left[z_4\left(\frac{X_3}{Z_3} - \frac{Y_3}{Z_3}\frac{x_3}{y_3}\right) - c_4\right].$$

The column vector V has the form:

$$V = [X^2,\ XY,\ XZ,\ Y^2,\ YZ,\ Z^2,\ X,\ Y,\ Z\,]^T. \qquad (2.2\text{-}20)$$

Clearly, for the rows r_4 and r_5 it can be written in analogy to (2.2-19):

$$r_4^T \equiv \begin{bmatrix} r_4^1 \\ r_4^2 \\ r_4^3 \end{bmatrix} = \frac{1}{D_4}\, M_4 V_4 \qquad (2.2\text{-}21)$$

where

$$V_4 = [X_4^2,\ X_4Y_4,\ X_4Z_4,\ Y_4^2,\ Y_4Z_4,\ Z_4^2,\ X_4,\ Y_4,\ Z_4]^T \qquad (2.2\text{-}21a)$$

and

$$r_5^T \equiv \begin{bmatrix} r_5^1 \\ r_5^2 \\ r_5^3 \end{bmatrix} = \frac{1}{D_4}\, M_4 V_5 \qquad (2.2\text{-}22)$$

where

$$V_5 = [X_5^2,\ X_5Y_5,\ X_5Z_5,\ Y_5^2,\ Y_5Z_5,\ Z_5^2,\ X_5,\ Y_5,\ Z_5]^T. \qquad (2.2\text{-}22a)$$

Using the latest notations, the expression (2.2-9) can be written as

$$\begin{bmatrix} r_4^1 & r_4^2 & r_4^3 \\ r_5^1 & r_5^2 & r_5^3 \\ r^1 & r^2 & r^3 \end{bmatrix} = 0, \qquad (2.2\text{-}23)$$

which gives

$$r^1 R_1 + r^2 R_2 + r^3 R_3 = 0 \qquad (2.2\text{-}23a)$$

where

$$R_1 = r_4^2 r_5^3 - r_4^3 r_5^2,$$

$$R_2 = r_4^3 r_5^1 - r_4^1 r_5^3, \qquad (2.2\text{-}23b)$$

$$R_3 = r_4^1 r_5^2 - r_4^2 r_5^1.$$

Using the expression (2.2-19) for r, it is now obtained from (2.2-23a):

$$\frac{1}{D_4}(X^2 b_1 + XY b_2 + XZ b_3 + Y^2 b_4 + YZ b_5 + Z^2 b_6 + X b_7 + Y b_8 + Z b_9) = 0 \qquad (2.2\text{-}24a)$$

where $b_1, b_2, \ldots, b_9$ can be determined from the relationship, which will also be useful later; namely,

$$B_4 = R M_4 \qquad (2.2\text{-}24b)$$

where

$$B_4 = [b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ b_8\ b_9]$$

and

$$R = [R_1 \quad R_2 \quad R_3]$$

(upon inspection of (2.2-19) and (2.2-23a) it is seen that (2.2-24a) is valid with $b_i = \sum_{j=1}^{3} m_{ji} R_j$, $i = 1, 2, \ldots, 9$, which is exactly (2.2-24b)). Since it holds in general that

$$b_1 \neq 0,$$

the equation (2.2-24) can be multiplied by $\frac{b_1}{D_4}$ and the result written in a matrix form as

$$x^T A x + x^T a = 0 \qquad (2.2\text{-}25)$$

where, with the usual notations for the elements of A-matrix and a-vector, it holds that

$$
\begin{aligned}
a_{11} &= 1, \\
a_{12} = a_{21} &= \frac{1}{2}\frac{b_2}{b_1}, \\
a_{13} = a_{31} &= \frac{1}{2}\frac{b_3}{b_1}, \\
a_{22} &= \frac{b_4}{b_1}, \\
a_{23} = a_{32} &= \frac{b_5}{b_1}, \\
a_{33} &= \frac{b_6}{b_1},
\end{aligned}
\tag{2.2-25a}
$$

and

$$
\begin{aligned}
a_1 &= \frac{b_7}{b_1}, \\
a_2 &= \frac{b_8}{b_1}, \\
a_3 &= \frac{b_9}{b_1}.
\end{aligned}
\tag{2.2-25b}
$$

The second order surface represented by (2.2-25) - (2.2-25b) is then the desired form (2.2-11a) given in the local coordinate system. Numerically (within round-off errors), the same values for A-matrix and a-vector were obtained as those computed by methods developed in Appendix 6 or Appendix 8.

2.233 Practical Computation of Critical Surface.

The critical surface for four ground stations can be computed in four steps as follows:

(1) Transformation of coordinates from the basic coordinate system to the local coordinate system of all the points (ground stations and targets).

(2) Computation of the critical surface, given in the local coordinate system by (2.2-25), in the canonical form

including the determination of its center and six main surface points; furthermore, computation of an approximate distance to the surface from any point beyond the four ground stations and the first five targets.

(3) Transformation of all the new points from the canonical coordinates to the local coordinate system.

(4) Transformation of all the new points from the local coordinate system to the basic coordinate system.

A more detailed description and explanation of these four steps is appropriate at this time.

The formula to be used in step (1) is (2.2-12); the values for X_0 and P used in it can be computed from (2.2-15) and (2.2-17).

The first part of the computations needed for step (2) can be carried out using the method presented in Appendix 4, section A4.3. To make the formula (2.2-25) representing the critical surface with respect to the local coordinate system complete, the coefficients listed in (2.2-25a) and (2.2-25b) have to be computed; this can be done when step (1) has been completed. The formula (2.2-25) corresponds to (A4-18) in Appendix 4. There, the local coordinate system is called "original coordinate system". In section A4.3, the approach to find the kind, size, and shape of the second order surface is given, which finally leads to determination of the center of the surface and its six "main surface points". At the same time, the computation of the second order surface yields the values for x_0 and R, necessary to determine its position and orientation with respect to the local coordinate system. In the second part of step (2), approximate distances from some points to the critical surface are required. A method to achieve this - with the precision improving when the point approaches the surface - was developed and described in Appendix 7. The computations of such distances are done in the canonical coordinate system. Furthermore, additional points on the second order surface are obtained; they correspond to certain projection of the above points onto the surface, as specified in Appendix 7. This feature can be useful in examining some nearly singular cases, because one can get a fairly good idea how close certain targets are to the critical surface. With all the ground stations in a plane such computations were unnecessary, since the critical curve in that case could be plotted and the distance measured.

Transformation from the canonical coordinates to the local coordinate system is also described in Appendix 4 and given by the formula (A4-8). The new points to be transformed are the center of the critical surface, its six main surface points and "projected points", if any. The old points can be also transformed to the local coordinate system and then back to the basic coordinate system for checking purposes.

The transformation of step (4) is performed using the formula (2.2-13) with X_o and P being the same as in step (1).

2.3 Range Observations from Any Number of Ground Stations with Three Stations Observing All Targets.

The stations observing all the targets will be denoted by numbers 1, 2, 3 as it was done in the first chapter; also other notations as well as the arrangement of observations in quads will remain the same. $\widetilde{A}$ matrix for any number of ground stations in general configuration is given in Table (1.2-2), section 1.2. The analysis of the critical surfaces in this section will be made using the same principles as those for four ground stations, described in previous sections, i.e., using the canonical approach.

2.31 Critical Surfaces Using Canonical Approach.

The same approach as in sections 2.22 will be now used with respect to all the three column blocks corresponding to all stations beyond 1, 2, 3 (i.e., for all except the last three column block). Each such block will be "cleared" by row equivalence operations of all except the first three of its rows. The remaining elements in such three rows will be again brought to zero by column equivalence operations. With the above row equivalence operations, the corresponding rows in the last three columns will be changed. The form of these rows for the first row block (corresponding to station 4), was described in the previous sections; they were denoted as rows r_j, which was later changed to r_4 and r_5 to denote the fourth and fifth row (corresponding to targets 4 and 5 observed from station 4), and to r to denote any further row beyond r_5. The submatrix composed of such rows r was denoted as $\overline{A}_4$. Exactly the same procedure with the same assumptions (i.e., singularity A) eliminated) will be used for other stations as well. Their submatrices in the last three column block will be similarly denoted as $\overline{A}_5$, $\overline{A}_6$, ... , etc., where the index specifies to which row block (or station) they refer. In analogy to D_4, the determinants associated with other row blocks will be denoted as D_5, D_6, ..., etc. After the outlined equivalence operations, the rows may be further arranged in such a way that $\overline{A}_4$, $\overline{A}_5$, $\overline{A}_6$, ..., etc., submatrices appear in the lower part of this modified $\widetilde{A}$ matrix. It now has the form:

$$\widetilde{A} \sim \begin{bmatrix} P_4 & 0 & 0 & \dots & 0 \\ 0 & P_5 & 0 & \dots & 0 \\ 0 & 0 & P_6 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \overline{A}_4 \\ 0 & 0 & 0 & \dots & \overline{A}_5 \\ 0 & 0 & 0 & \dots & \overline{A}_6 \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix} ,$$

which can be written as

$$\tilde{A} \sim \begin{bmatrix} P & 0 \\ 0 & \bar{A} \end{bmatrix} \tag{2.3-1}$$

where P is a non-singular matrix composed of non-singular (3 x 3) submatrices along the main diagonal and zeroes elsewhere, and where $\bar{A}$ is given as

$$\bar{A} = \begin{bmatrix} \bar{A}_4 \\ \bar{A}_5 \\ \bar{A}_6 \\ \vdots \end{bmatrix} . \tag{2.3-2}$$

Due to the form (2.3-1) and the property of P matrix, $\tilde{A}$ is singular if and only if $\bar{A}$ is singular. Thus, the problem has been again reduced to analyzing a matrix with only three columns. There has to be at least three rows in $\bar{A}$, or else $\tilde{A}$ would be automatically singular without any further considerations.

The rows in $\bar{A}_4$, denoted as r_4, r_5, $\bar{r}$, corresponded to the fourth, fifth, and any further target observed from station 4; they were given by (2.2-21), (2.2-22), and (2.2-19) respectively. The rows in $\bar{A}_5$, denoted as $\bar{r}_4$, $\bar{r}$, ... , are computed the same way, namely

$$\bar{r}_4^T \equiv \begin{bmatrix} \bar{r}_4^1 \\ \bar{r}_4^2 \\ \bar{r}_4^3 \end{bmatrix} = \frac{1}{D_5} M_5 V_4. \tag{2.3-3}$$

D_5 is computed according to (2.2-4) - (2.2-4b) with the coordinates of station 5 replacing the coordinates of station 4 and the coordinates of the first three satellites in j_5 taking place of the same coordinates in j_4; M_5 is computed from (2.2-19) through (2.2-19b) with exactly the same modifications as above; V_4, given by (2.2-21a), is associated with the fourth target in j_5 rather than in j_4. Any further row in $\overline{A}_5$, denoted as $\overline{r}$, is obtained as

$$\overline{r}^{T} \equiv \begin{bmatrix} \overline{r}^1 \\ \overline{r}^2 \\ \overline{r}^3 \end{bmatrix} = \frac{1}{D_5} M_5 V \qquad (2.3\text{-}4)$$

where V has the same form as represented by (2.2-20) and it is now associated with the critical surface for (the quad of) station 5 rather than station 4. Any row in $\overline{A}_6$ will be denoted as $\overline{\overline{r}}$; it is given as

$$\overline{\overline{r}}^{T} \equiv \begin{bmatrix} \overline{\overline{r}}^1 \\ \overline{\overline{r}}^2 \\ \overline{\overline{r}}^3 \end{bmatrix} = \frac{1}{D_6} M_6 V; \qquad (2.3\text{-}5)$$

it can be described exactly the same way as the row $\overline{r}$, except that all the quantities in (2.3-5) refer to station 6 rather than station 5 (including the corresponding satellite groups). Similar formulas and descriptions would apply for any further stations.

Since there are at least three rows in $\overline{A}$ matrix, each station has to observe at least three targets (otherwise singularity A) would automatically occur) and at least three more targets must be observed from one or more quads. (When the stations were considered lying in one plane at least three quads had to observe such additional targets, while here these targets may be observed by just one quad, so that theoretically one quad could observe six targets and the other quads only three targets each.) The problem will be singular if each additional row in $\overline{A}$ is lying in the row space of the first two rows, assumed independent. At this point the discussion will be divided into two cases: case (a), where j_4 is assumed to contain more than four targets, and case (b), rather theoretical, where j_4 is assumed to contain exactly four targets (otherwise the stations can be renumbered so that station 4 always observes more than three targets).

Case (a) is such that the first two rows in $\overline{A}$ are r_4 and r_5, corresponding to targets 4 and 5 observed by station 4; they are the same two rows as those used in section 2.232 and given by (2.2-21) and (2.2-22), needed to compute the critical surface for station 4. Consequently, the critical surface for station 4 in the local coordinate system is given by the formulas (2.2-25) - (2.2-25b) in which the b coefficients can be found from (2.2-24b) and (2.2-23b), with the elements of M_4 matrix given following the formula (2.2-19). At the end of section 2.22, this surface was shown to pass through stations 1, 2, 3, and 4 and through the first five targets observed by station 4. For station 5, the critical surface is represented by

$$\begin{vmatrix} r_4^1 & r_4^2 & r_4^3 \\ r_5^1 & r_5^2 & r_5^3 \\ \overline{r}^1 & \overline{r}^2 & \overline{r}^3 \end{vmatrix} = 0 \qquad (2.3\text{-}6)$$

where $\overline{r}$ corresponds to any row in $\overline{A}_5$, i.e., to $\overline{r}_4$ as well. This surface is seen to pass through stations 1, 2, 3, and 5 and through the first three targets observed by station 5; if the target corresponding to row $\overline{r}$ is substituted for by any of these points, then $\overline{r}$ is a zero row according to the same reasoning which followed (2.2-9) at the end of section 2.22. The equation (2.3-6) can be written as

$$\overline{r}^1 R_1 + \overline{r}^2 R_2 + \overline{r}^3 R_3 = 0 \qquad (2.3\text{-}6a)$$

where R_1, R_2, R_3 are given by (2.2-23b). From (2.3-6a) considering (2.3-4), the critical surface can be given by the formula (2.2-24a), where D_4 is to be replaced by D_5 and where the b terms are now computed from

$$B_5 = R M_5 \qquad (2.3\text{-}7)$$

with

$$B_5 = \lceil b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ b_8\ b_9 \rceil.$$

Thus, the critical surface for station 5 can be given again by (2.2-25) - (2.2-25b), using (2.3-7) for computation of the b-terms. In exactly the same way the critical surface for station 6 (and any further station) could be found; in this case the b-terms would be computed from

$$B_6 = RM_6. \tag{2.3-8}$$

This surface passes through stations 1, 2, 3, and 6 and the first three targets observed by station 6, analogous to the behavior of the critical surface for station 5.

Consequently, the problem is singular if all the targets beyond target 5 in j_4 and beyond target 3 in j_5, j_6, etc., are lying on the corresponding critical surfaces; these surfaces can be all computed using (2.2-25) - (2.2-25b) with the b-terms found respectively from (2.2-24b), (2.3-7), (2.3-8), etc. All the practical computations with respect to each of these critical surfaces are the same as those described in section 2.233.

Case (b) is such that the first two rows in $\overline{A}$ are r_4 and $\overline{r}_4$, corresponding to target 4 from the satellite group j_4 (observed by station 4) and target 4 from j_5. If there were some additional targets in j_4, represented by the row r, the critical surface for station 4 would be given as

$$\begin{vmatrix} r_4^1 & r_4^2 & r_4^3 \\ \overline{r}_4^1 & \overline{r}_4^2 & \overline{r}_4^3 \\ r^1 & r^2 & r^3 \end{vmatrix} = 0, \tag{2.3-9}$$

or

$$r^1 \overline{R}_1 + r^2 \overline{R}_2 + r^3 \overline{R}_3 = 0 \tag{2.3-9a}$$

where

$$\begin{aligned} \overline{R}_1 &= r_4^2 \overline{r}_4^3 - r_4^3 \overline{r}_4^2, \\ \overline{R}_2 &= r_4^3 \overline{r}_4^1 - r_4^1 \overline{r}_4^3, \\ \overline{R}_3 &= r_4^1 \overline{r}_4^2 - r_4^2 \overline{r}_4^1, \end{aligned} \tag{2.3-9b}$$

with the rows r_4 and $\overline{r}_4$ given by (2.21) and (2.3-3). This surface would pass through stations 1, 2, 3, and 4 and through the first four targets observed by station 4 (if the variable point corresponding to row r were replaced by any of the first seven points mentioned above then r would be a zero row, while for the eighth point - target 4 - r would be the same as the first row of the determinant in (2.3-9)). Comparing (2.3-9a) with (2.2-23a), it is clear that the critical surface could be represented by the equation (2.2-24a) with the b terms obtained from

$$\overline{B}_4 = \overline{R} M_4 \qquad (2.3\text{-}10)$$

where

$$\overline{R} = [\overline{R_1} \quad \overline{R_2} \quad \overline{R_3}]. \qquad (2.3\text{-}11)$$

Consequently, such critical surface for further targets in j_4 would be given again by (2.2-25) - (2.2-25b) with the b coefficients from (2.3-10). For station 5, the critical surface can be represented by the relation (2.3-9) where the row r is replaced by the row $\overline{r}$; this row is associated with any target in j_5 beyond target $\bar{4}$. This critical surface is seen to pass through stations 1, 2, 3, and 5 and through the first four targets observed by station 5. It is given by the relation

$$\overline{r}^1 \overline{R_1} + \overline{r}^2 \overline{R_2} + \overline{r}^3 \overline{R_3} = 0. \qquad (2.3\text{-}12)$$

Considering (2.3-4), this leads again to the formula (2.2-24a) where D_4 is to be replaced by D_5 and where the b terms are now computed from

$$\overline{B}_5 = \overline{R} M_5. \qquad (2.2\text{-}13)$$

Thus the critical surface for station 5 is given again by (2.2-25) - (2.2-25b), using (2.3-13) for computation of the b terms. The same formulas would also apply for the critical surface for station 6, except that the b terms would be computed from

$$\overline{B}_6 = \overline{R} M_6. \qquad (2.3\text{-}14)$$

This surface passes through stations 1, 2, 3, and 6 and through the first three targets observed by station 6, as it was already seen in case (a). One could continue the same way for any further station.

The problem would then be singular if all the targets beyond target 4 in j_4 (however, such targets are not assumed to exist in this case), beyond target 4 in j_5, beyond target 3 in j_6, etc., were lying on the corresponding critical surface; these surfaces can be all computed using (2.2-25) - (2.2-25b) with the b terms found respectively from (2.3-10), (2.3-13), (2.3-14), etc. All the practical computations with respect to each of these critical surfaces are again the same as those described in section 2.233.

2.32 Problem with Critical Surfaces Coinciding.

2.321 General Considerations.

The 9-vector V was given as

$$V = [X^2, XY, XZ, Y^2, YZ, Z^2, X, Y, Z]^T. \tag{2.3-15}$$

It will be now partitioned into two parts, according to absence or presence of any Z-coordinate; namely,

$$V^1 = [X^2, XY, Y^2, X, Y]^T \tag{2.3-15a}$$

and

$$V^2 = [XZ, YZ, Z^2, Z]^T. \tag{2.3-15b}$$

Any (1 x 9) vector B contained nine b terms; this vector was subscripted according to the corresponding station number; furthermore, in case (b) it was denoted as $\overline{B}$. The following derivations will be made for case (a); considering the critical surface for station i, the B vector is written as

$$B_i = [b_1 \; b_2 \; b_3 \; b_4 \; b_5 \; b_6 \; b_7 \; b_8 \; b_9]_i. \tag{2.3-16}$$

It will be partitioned correspondingly to vector V as

$$B_i^1 = [b_1 \;\; b_2 \;\; b_4 \;\; b_7 \;\; b_8]_i \tag{2.3-16a}$$

and

$$B_i^2 = [b_3 \;\; b_5 \;\; b_6 \;\; b_9]_i. \tag{2.3-16b}$$

Matrix M_4 and any further matrix M_i (corresponding to the same station i as B_i) will also be partitioned; from the form of M_4 it is seen that

$$M_4^1 = \begin{bmatrix} 0 & z_4 & 0 & 0 & -x_3 z_4 \\ 0 & 0 & z_4 & 0 & -y_3 z_4 \\ z_4 & -\frac{x_3}{y_3} z_4 & 0 & -x_2 z_4 & x_2 \frac{x_3}{y_3} z_4 \end{bmatrix}, \tag{2.3-17a}$$

which is of rank three; (3 x 4) matrix M_4^2 is much more complicated. Similarly,

$$M_i^1 = \begin{bmatrix} 0 & z_i & 0 & 0 & -x_3 z_i \\ 0 & 0 & z_i & 0 & -y_3 z_i \\ z_i & -\frac{x_3}{y_3} z_i & 0 & -x_2 z_i & x_2 \frac{x_3}{y_3} z_i \end{bmatrix}. \tag{2.3-17b}$$

It has been assumed that $z_4 \neq 0$ and $z_i \neq 0$. From (2.3-17a) and (2.3-17b) one can see that

$$M_4^1 z_i = M_i^1 z_4. \tag{2.3-18}$$

Since B_4, B_i were computed as

$$B_4 = RM_4, \qquad B_i = RM_i$$

and consequently

$$B_4^1 = RM_4^1, \qquad B_i^1 = RM_i^1,$$

it also holds that

$$B_4^1 z_i = B_i^1 z_4. \tag{2.3-19}$$

But then $\frac{b_2}{b_1}$, $\frac{b_4}{b_1}$, $\frac{b_7}{b_1}$, and $\frac{b_8}{b_1}$ must be the same for station 4 and any station i.

Clearly, all these results hold in case (b) as well. Consequently, the parts of A-matrix and a-vector from (2.2-25), such as

$$\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

must be the same for all critical surfaces in one network. This property served as a useful check in numerical computations in both case (a) and case (b).

A plausible geometric interpretation of this result is such that each of the critical surfaces intersects the plane of station 1, 2, and 3 (i.e., plane $z = 0$) in a second order curve and that all these second order curves coincide. Necessarily, this one second order curve, common to all the critical surfaces, passes through the stations 1, 2, and 3 (further stations and all targets are in general assumed not to be lying in the plane $z = 0$). An illustration of this result is presented in Figure 6.

The critical surface for any station was given by (2.2-25) as

$$x^T A x + x^T a = 0,$$

which can be also written as

$$BV = B^1 V^1 + B^2 V^2 = 0; \tag{2.3-20}$$

here the subscripts have been omitted. Whenever non-zero A-matrix and a-vector for two second order surfaces are the same, these two surfaces coincide. In terms of the equation (2.3-20) this means that whenever it holds that

$$B_4^2 z_1 = B_1^2 z_4, \tag{2.3-21}$$

then, due to

$$B_4^1 z_1 = B_1^1 z_4,$$

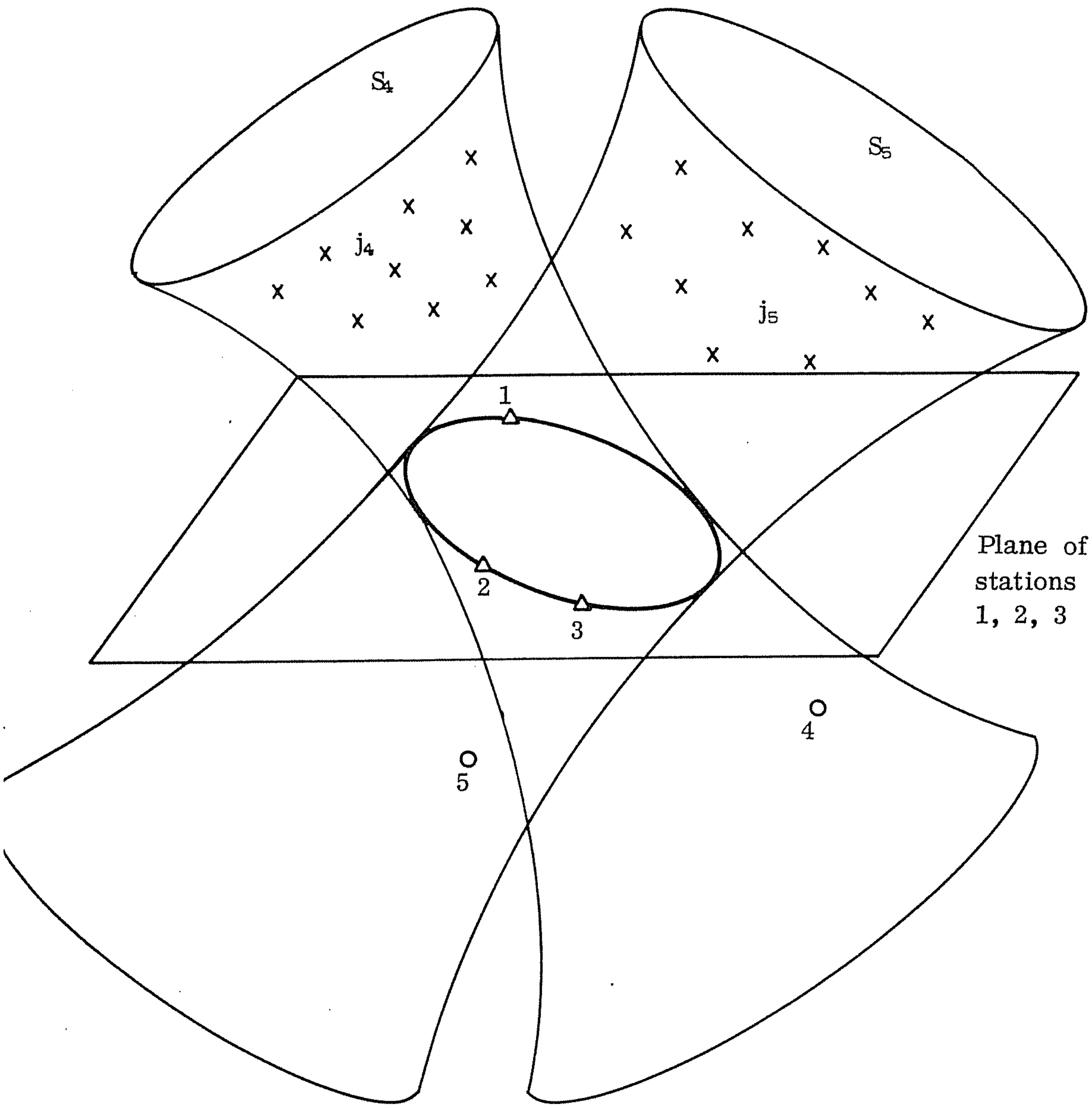


Figure 6

ILLUSTRATION OF CRITICAL SURFACES: Stations 1, 2, 3 observe all targets; stations 4 and 5 together with their satellite groups j_4 and j_5 are on the second order surfaces S_4 and S_5, respectively; stations 1, 2, 3 are on the second order intersection curve of surfaces S_4 and S_5.

it also holds that

$$B_4 z_i = B_i z_4 \tag{2.3-22}$$

and, therefore, all the terms in A-matrix and a-vector for station 4 and station i are the same (see what was said following (2.3-19)). Consequently, if (2.3-21) holds, then the critical surfaces for station 4 and station i coincide.

Next, it will be shown when (2.3-21) can hold. Let $\bar{1}, \bar{2}, \bar{3}$ denote the first three targets observed by station i. Assume that station i and targets $\bar{1}, \bar{2}, \bar{3}$, lie on the critical surface of station 4. This is expressed by the relations of the type (2.3-20); namely,

$$B_4 V_i = 0, \quad B_4 V_{\bar{1}} = 0, \quad B_4 V_{\bar{2}} = 0, \quad B_4 V_{\bar{3}} = 0. \tag{2.3-23a}$$

For the critical surface of station i it holds that

$$B_i V_i = 0, \quad B_i V_{\bar{1}} = 0, \quad B_i V_{\bar{2}} = 0, \quad B_i V_{\bar{3}} = 0, \tag{2.3-23b}$$

since points i and $\bar{1}, \bar{2}, \bar{3}$ always lie on this surface as it was shown for both case (a) and case (b) in section 2.31. Considering (2.3-20), it follows from (2.3-23b) and (2.3-23a) respectively:

$$B_i^2 V_i^2 = -B_i^1 V_i^1, \quad B_i^2 V_{\bar{1}}^2 = -B_i^1 V_{\bar{1}}^1, \quad B_i^2 V_{\bar{2}}^2 = -B_i^1 V_{\bar{2}}^1, \quad B_i^2 V_{\bar{3}}^2 = -B_i^1 V_{\bar{3}}^1.$$

$$B_4^2 V_i^2 = -B_4^1 V_i^1, \quad B_4^2 V_{\bar{1}}^2 = -B_4^1 V_{\bar{1}}^1, \quad B_4^2 V_{\bar{2}}^2 = -B_4^1 V_{\bar{2}}^1, \quad B_4^2 V_{\bar{3}}^2 = -B_4^1 V_{\bar{3}}^1.$$

These last relations can be written in matrix form as

$$B_i^2 [V_i^2 \quad V_{\bar{1}}^2 \quad V_{\bar{2}}^2 \quad V_{\bar{3}}^2] = -B_i^1 [V_i^1 \quad V_{\bar{1}}^1 \quad V_{\bar{2}}^1 \quad V_{\bar{3}}^1]$$

and

$$B_4^2 [V_i^2 \quad V_{\bar{1}}^2 \quad V_{\bar{2}}^2 \quad V_{\bar{3}}^2] = -B_4^1 [V_i^1 \quad V_{\bar{1}}^1 \quad V_{\bar{2}}^1 \quad V_{\bar{3}}^1].$$

Denoting

$$\underset{(4\times 4)}{N} = [V_i^2 \quad V_{\bar{1}}^2 \quad V_{\bar{2}}^2 \quad V_{\bar{3}}^2]$$

$$\underset{(5\times 4)}{M} = [V_1^1 \quad V_{\bar{1}}^1 \quad V_{\bar{2}}^1 \quad V_{\bar{3}}^1],$$

...equations can be written as

$$B_1^2 N = B_1^1 M$$

$$B_4^2 N = B_4^1 M,$$

$$B_1^2 = B_1^1 M N^{-1} \tag{2.3-24a}$$

$$B_4^2 = B_4^1 M N^{-1}, \tag{2.3-24b}$$

...d N is non-singular. Using the relation (2.3-19), i.e., using

$$B_1^1 = \frac{z_1}{z_4} B_4^1$$

...-24a), it follows that

$$B_1^2 = \frac{z_1}{z_4} B_4^1 M N^{-1};$$

...gehter with (2.3-24b) gives

$$B_4^2 z_1 = B_1^2 z_4,$$

...is exactly the equation (2.3-21).

...the last step in this derivation, it will be proved that under the earlier ...tion of singularity A) eliminated, the above matrix N is indeed non-singular. ...ere singular, it would have to hold that

$$\begin{vmatrix} x_1 z_1 & X_{\bar{1}} Z_{\bar{1}} & X_{\bar{2}} Z_{\bar{2}} & X_{\bar{3}} Z_{\bar{3}} \\ y_1 z_1 & Y_{\bar{1}} Z_{\bar{1}} & Y_{\bar{2}} Z_{\bar{2}} & Y_{\bar{3}} Z_{\bar{3}} \\ z_1^2 & Z_{\bar{1}}^2 & Z_{\bar{2}}^2 & Z_{\bar{3}}^2 \\ z_1 & Z_{\bar{1}} & Z_{\bar{2}} & Z_{\bar{3}} \end{vmatrix} = 0. \tag{2.3-25}$$

Factoring out z_1, $Z_{\bar{1}}, Z_{\bar{2}}$, $Z_{\bar{3}}$ from the above determinant (all assumed non-zero), transposing it and subtracting its first row from the other three rows, (2.3-25) can be written, using the development by the first row, as

$$\begin{vmatrix} X_{\bar{1}} - x_1 & Y_{\bar{1}} - y_1 & Z_{\bar{1}} - z_1 \\ X_{\bar{2}} - x_1 & Y_{\bar{2}} - y_1 & Z_{\bar{2}} - z_1 \\ X_{\bar{3}} - x_1 & Y_{\bar{3}} - y_1 & Z_{\bar{3}} - z_1 \end{vmatrix} = 0.$$

However, this is the equation of a plane passing through the points i, $\bar{1}, \bar{2}, \bar{3}$. But since singularity A) was excluded for the first three targets, the above relation cannot hold and so $|N| \neq 0$, which completes the discussion.

It can be concluded in general, that if the critical surface of one station in a network contains another station and its (first) three targets, then the critical surfaces of both stations coincide. Next, suppose that target $\bar{4}$ lies on the critical surfaces of both station 4 and station i. The same conclusion would hold for target $\bar{4}$ (observed by station i) replacing station i in (2.3-23b) and the following derivations, provided $Z_{\bar{4}} \neq 0$ and provided the four targets ($\bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$) are not lying in one plane. Consequently, if four targets lying on one critical surface are also lying on another critical surface, then these two surfaces in general coincide. This can be explained by the fact that all except four values in A-matrix and a-vector for different critical surfaces in a network are the same from the beginning, without any conditions.

Also this result has a plausible geometric interpretation. Since all the critical surfaces (one corresponding to each station beyond station 3) intersect in a second order curve in the plane of stations 1, 2, and 3, they can be considered to have five points in common in that plane (which is equivalent to having five parameters in common). Consequently, four more points in general configuration are needed (none of them in that plane) for each critical surface to be determined. If all four of them were lying in one plane, the second order surface would degenerate into two intersecting planes. If these four points lie on the ıme second order surface, then the corresponding critical surfaces coincide, nce in general nine points are needed for the determination of a second order ırface. This also implies that if all ground stations observe the same four rgets then all the critical surfaces coincide and such a network is singular only all its points (stations and targets) lie on one second order surface.

322 Critical Surface if All Ground Stations Co-observe.

Should a network be singular, all the targets would have to lie on the corre- onding second order surfaces. If all the stations co-observe, all the existing oups of targets coincide. Therefore, for singularity to occur, all the targets general more than four) would have to be lying simultaneously on all the criti- l surfaces. But then all these surfaces would coincide as demonstrated above. nsequently, with all the ground station co-observing the problem is in general ıgular only if all the points of a network, i.e. all the stations and all the tar- s, lie on one second order surface. One exception is "reverse singularity B)", ich occurs when all the targets are in a plane on a second order curve. This e of singularity is described in section 2.21. (Singularity A) could occur only ler similar conditions as described in 1.34, namely, if all the targets lay in traight line. However, this is only a special case of "reverse singularity B)" en a second order curve degenerates into two coincident lines.) If all the tar- ts lie in a plane containing any ground station the solution is singular whether l stations co-observe or not; however, for the reasons given in section 1.34 is is not singularity A) in the usual sense for all stations co-observing, even ough geometrically it is closely related to it. To show that the solution is ıgular in this case one can argue that an "observing" set of points lying in a ne (here all targets) contains a member of the "observed" set of points. Under s condition the solution is singular according to the first chapter. This is the ly other exception to the rule stated above.

Naturally, with all the points of a network lying on a second order surface, any arrangement of observations (e.g., exactly three stations observing all the targets, replacing of certain stations, or finally "leapfrogging") makes an adjustment singular, as it differs from the case when all stations co-observe only by absence of certain observations. A geometric illustration of such a configuration is presented in Figure 7.

2.33 Independent Derivation of Singularity C) When All Ground Stations Are Lying in Plane.

Assuming $z = 0$ for all ground stations, important simplifications will take place in all the formulas leading to and including M matrix which first appeared in (2.2-19). For instance, if station i is considered, some of the simplifications taking place are:

$$A_1 = A_2 = A_3 = -y_i;$$

$$K_1 = -c_i(X_1 - x_2), \quad K_2 = -c_i(X_2 - x_2), \quad K_3 = -c_i(X_3 - x_2)$$

(c_i is obtained by substituting x_i, y_i for x_4, y_4 in c_4);

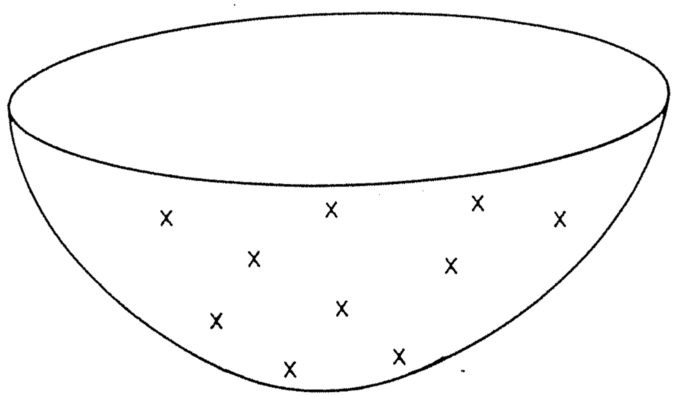
$$D_i = a_1 x_i + a_2 y_i + a_4$$

where the a-coefficients become also simplified. Finally, it is obtained for M matrix:

$$M_i = \frac{1}{D_i}\begin{bmatrix} 0 & 0 & -y_i(x_i-x_3)a_1 & 0 & -y_i(x_i-x_3)a_2 & -y_i(x_i-x_3)a_3 & 0 & 0 & -y_i(x_i-x_3)a_4 \\ 0 & 0 & -y_i(y_i-y_3)a_1 & 0 & -y_i(y_i-y_3)a_2 & -y_i(y_i-y_3)a_3 & 0 & 0 & -y_i(y_i-y_3)a_4 \\ 0 & 0 & -c_i(x_i-x_2)a_1 & 0 & -c_i(x_i-x_2)a_2 & -c_i(x_i-x_2)a_3 & 0 & 0 & -c_i(x_i-x_2)a_4 \end{bmatrix}$$

(2.3-26)

where the coordinates of targets 1, 2, 3 present in the a-coefficients necessarily refer to the targets observed from station i. With these simplifications, it is obtained for any row r in the submatrix $\overline{A}_i$ of $\overline{A}$:



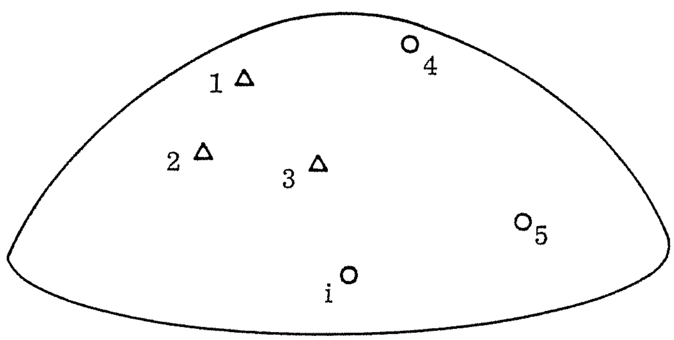


Figure 7

ILLUSTRATION OF CRITICAL SURFACES: All stations observe all targets; all stations and all targets are on a second order surface.

$$r^{T} \equiv \begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} = \frac{1}{D_i} Z\ (a_1 X + a_2 Y + a_3 Z + a_4) \begin{bmatrix} -y_i(x_i - x_3) \\ -y_i(y_i - y_3) \\ -c_i(x_i - x_2) \end{bmatrix}. \tag{2.3-27}$$

Since $Z \neq 0$ was stipulated from the beginning, this row will be a zero row if

$$a_1 X + a_2 Y + a_3 Z + a_4 = 0. \tag{2.3-28}$$

This is an equation of a plane in X, Y, Z which passes through the first three targets observed from the quad of station i; upon substituting the coordinates of these three targets for X, Y, Z, the equation (2.3-28) can be shown to hold after some algebraic manipulations. Therefore, for each further target lying in a plane with the first three targets, the corresponding row in $\overline{A}$ matrix is a zero row. Suppose that station i observes two or more targets not in plane with the first three targets. It is seen from (2.3-27) that all such rows are linearly dependet, since the (3×1) vector on the right-hand side is the same for all targets observed from the same station. Consequently, only one non-zero row r can be obtained from one station when the corresponding target is off-plane with respect to the first three targets. Thus, the necessary condition to prevent singularity C) was obtained: at least three stations must observe off-plane targets.

Next, the sufficient conditions for singularity C) to be eliminated will be derived. Suppose that only stations 4, 5, i observe their targets off-plane. Let the corresponding non-zero rows in $\overline{A}$ matrix be denoted as r_j, r_k, r_m; they are associated with targets j, k, and m. Denote further

$$T = a_1 X + a_2 Y + a_3 Z + a_4,$$

which is non-zero for any variable point being off-plane with respect to the first three targets in its satellite group; consequently,

$$T_j \neq 0, \qquad T_k \neq 0, \qquad T_m \neq 0.$$

he submatrix of $\overline{A}$, formed by r_j, r_k, r_m, is seen from (2.3-27) to have the from:

$$L = \begin{bmatrix} -\frac{1}{D_4} Z_j T_j & 0 & 0 \\ 0 & -\frac{1}{D_5} Z_k T_k & 0 \\ 0 & 0 & -\frac{1}{D_i} Z_m T_m \end{bmatrix} \begin{bmatrix} y_4(x_4-x_3) & y_4(y_4-y_3) & c_4(x_4-x_2) \\ y_5(x_5-x_3) & y_5(y_5-y_3) & c_5(x_5-x_2) \\ y_i(x_i-x_3) & y_i(y_i-y_3) & c_i(x_i-x_2) \end{bmatrix} .$$

(2.3-29)

f this matrix is singular or non-singular, then $\overline{A}$ matrix and consequently $\widetilde{A}$ atrix is singular or non-singular (singularity A) had been assumed eliminated). ince the first matrix in (2.3-29) is non-singular, L is singular only if the second atrix is singular. But this occurs exactly when stations 4, 5, i lie on a second rder curve with station 1, 2, 3, as one may see from (1.3-2), section 1.32. herefore, the necessary and sufficient conditions for singularity C) eliminated re: there must be (at least three) off-curve stations making off-plane observa-ions. But this is exactly the result of section 1.33.

2.4 Brief Discussion Concerning Replacing of Stations.

The principle of replacing of stations when the ground stations are general configuration is simpler, but similar to the same discussion with all e ground stations in a plane. The simplicity of this problem for most cases onsists in the fact, that four ground stations could be sufficient to form the ndamental unit (provided all the points do not lie on one second order surface). onsequently, any three "old" stations co-observing with a new station contribute the expansion of a non-singular network; the sufficient conditions that it be so re: no target in the new satellite group is lying in a plane with the three "old" oserving stations, and the new station is not lying in a plane with these new rgets. The first condition guarantees the unique determination of each new rget using the three unique stations (they are part of a non-singular network), and the second condition guarantees the unique determination of the new station from these targets.

More sophisticated considerations are necessary in one special case which will be only discussed without expressing it analytically. Suppose that the smallest non-singular network consists of six stations. This may happen when each quad observes only four satellite points. As mentioned previously, the necessary condition to have non-singular $\widetilde{A}$ is that there be at least three additional targets in a network beyond three targets per quad. When the ground stations are not lying in a plane, such three targets may be distributed over one, two, or three quads; if they belong to three quads, then a fundamental unit consists of six stations; otherwise it consists of four or five stations and in each of these two cases some three stations are observing all the targets. In the case of six stations forming a fundamental unit, replacement of observations may occur. After the first replacement, the sufficient conditions to expand the network, mentioned previously, will apply also in this case. It is assumed that station k replaces station 3, as it was done in section 1.41. $\widetilde{A}$ matrix with general distribution of the ground stations for this replacement was presented in Table (1.4-1). Clearing the column blocks in this matrix from the non-zero elements beyond the first three rows in each row block up to the row block "From s" is done in exactly the same way as described in section 2.22. In the three column block for station k, "clearing" has to continue also for the rows "From s", using the same first three rows of the row block "From k". However, it is seen from Table (1.4-1) that the three coefficients k_1, k_2, and k_3 for each of these additional rows will again contain second degree terms in the coordinates of the corresponding target (in j_s). Consequently, after this step is completed, some second degree terms will have been added to all the rows "From s" of the last three column block. Therefore, the nature of the rows "From s" in this block is the same as that of the previous rows before the row equivalence operations were started (each row contains some second degree rms). Finally, the "clearing" of the three column block for s leads to some efficients k_1, k_2, k_3 and to rows r in the last three column block which again ntain second degree terms in the coordinates of the corresponding targets. onsequently, the structure of $\widetilde{A}$ matrix and of its $\overline{A}$ submatrix in particular ter these operations is the same as when three stations were observing all the rgets. As a matter of fact, the only rows r in $\overline{A}$ which changed are the rows orresponding to station s. Consequently, the critical surfaces would be again epresented by second order surfaces; for stations 4 through k these surfaces ould be the same as when three stations observed all the targets, while for tation s the critical surface would be different.

2.5 Numerical Examples and Verifications of Theory.

Example 1. In this example, points with their coordinates given in Table 2.5-1) are used to define a second order surface.

Table (2.5-1)

Coordinates of Nine Points to Define a Second Order Surface

Point	x	y	z
1	1,000,000	0	1,412,000
2	707,000	706,000	1,410,000
3	0	1,002,000	1,414,000
11	500,000	600,000	1,269,000
12	-400,000	200,000	1,095,000
13	250,000	320,000	1,080,000
14	650,000	275,000	1,224,000
15	105,000	-520,000	1,132,000
16	0	0	997,000

The resulting surface, obtained numerically according to the method of Appendix

6, is a hyperboloid of one sheet with the coordinates for the six main surface points and the center given in Table (2.5-2). Next, point 4 is added to the nine points of Table (2.5-1) so as to form a quad with points 1, 2, 3; an adjustment is made for this quad observing points 11 through 16; the coordinates of point 4 are varied which results in six cases. With the points 11-16 as targets there are no redundant observations present in the adjustment. This experiment represents category 1. Adding one more target at the location +b brings one redundant observation into the adjustment. With the same six cases as previously, this experiment represents category 2. The result of the adjustment of the six cases in both categories are given in Table (2.5-3). From this table it can be verified that the problem is singular if all the points lie on one second order surface. The best results are obtained for station 4 occupying the center of this surface. Adding one further satellite point to the network improves significantly all the non-singular cases.

Table (2.5-2)

Coordinates of Six Main Surface Points and Center of the Hyperboloid of One Sheet Defined by Nine Points

Point	x	y	z
+a	413,256	1,378,290	1,049,000
-a	527,127	665,999	925,920
+b	238,082	984,345	991,475
-b	702,300	1,059,940	983,450
+c	479,634	986,799	1,200,740
-c	460,748	1,057,490	774,180
x_0	470,191	1,022,140	987,462

Table (2.5-3)

Results of the Adjustment of One Quad

in General Configuration.

Case	Point 4	$Tr(N^{-1})$ Category 1	Category 2	Note
1	x_o	15,170	383	Best
2	$+\frac{a}{2}$	88,140	1,629	
3	$+ a$	$-.4 \times 10^8$	$-.5 \times 10^8$	Singular
4	$+\frac{3a}{2}$	191,400	12,500	
5	$+ 2a$	68,930	6,731	
6	$+\frac{5a}{2}$	43,320	5,580	

Example 2. In this example six ground stations and twelve satellite positions are used for observations in three quads, so that no redundancy is present. The $\bar{x}$ and $\bar{y}$ coordinates of stations 1 through 5 are given in Table (1.5-1) of section 1.5; for station 6, they are taken such as presented in case (1) of Table (1.5-7), namely

$$x_6 = 340,000, \qquad y_6 = 790,000.$$

The z-coordinates of these six stations are given as follows:

$$z_1 = 0, \qquad z_2 = 30,000, \qquad z_3 = 70,000,$$

$$z_4 = 20,000, \qquad z_5 = 40,000, \qquad z_6 = 110,000.$$

The three targets are divided into groups, j_1, j_2, j_3, exactly the same way as it was done in (1.5-3) of Example 3, section 1.5. The observations are taken according to five different arrangements. The first three arrangements, denoted as (a), (b), (c), are such that three stations observe all the targets. They are presented as follows:

$$
\text{(a)} \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & & & \ldots & j_1 \\
 & 2 & 3 & 4 & 5 & & \ldots & j_2 \\
 & 2 & 3 & 4 & & 6 & \ldots & j_3
\end{array}
$$

$$
\text{(b)} \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & & & \ldots & j_1 \\
1 & 2 & 3 & & 5 & & \ldots & j_2 \\
1 & 2 & 3 & & & 6 & \ldots & j_3
\end{array} \tag{2.5-1}
$$

$$
\text{(c)} \quad \begin{array}{cccccccc}
1 & & 3 & 4 & 5 & & \ldots & j_1 \\
 & 2 & 3 & 4 & 5 & & \ldots & j_2 \\
 & & 3 & 4 & 5 & 6 & \ldots & j_3 \, .
\end{array}
$$

The next two arrangements use replacing of observations; they are denoted as (ab) and (ac). The arrangement (ab) represents an intermediate step between (a) and (b); namely, station 4 replaced station 1 in (b) for observations of j_2. Thus,

$$
\text{(ab)} \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & & & \ldots & j_1 \\
 & 2 & 3 & 4 & 5 & & \ldots & j_2 \\
1 & 2 & 3 & & & 6 & \ldots & j_3 \, .
\end{array}
$$

Similarly, arrangement (ac) is an intermediate step between (a) and (c); namely, station 5 replaced station 2 in (a) for observations of j_3. Thus,

$$
\text{(ac)} \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & & & \ldots & j_1 \\
 & 2 & 3 & 4 & 5 & & \ldots & j_2 \\
 & & 3 & 4 & 5 & 6 & \ldots & j_3 .
\end{array}
$$

The critical surfaces for (a) were computed using all the points appearing in (2.5-1) except for target 182 of the group j_3. The critical surface for j_3 was found to be a hyperboloid of two sheets. Target 182 was then chosen to be exactly on the critical surface; it remained in this position also for all the other arrangements. In both (b) and (c), this location of target 182 happened to be near the critical surface for j_3, namely 290.5 m and 4,841.2 m, respectively; these distances were computed using the method of Appendix 7. As a matter of fact the chosen location of target 182, denoted as 182′, was computed as a "projection" of the original point 182 onto the critical surface of j_3 in arrangement (a); its coordinates are given as:

182′ ... 575,704 943,123 1,522,290.

This constitutes category 1 in the present example. Next, target 182 was chosen to be located at x_o of the critical surface for j_3 of (a) and it was denoted as 182_o. This location gave the distances to j_3 of (a), (b), and (c) as 104,604 m, 198,231 m, and 4,336 m, respectively. The coordinates 182_o were found to be

182_o ... 612,197 1,729,180 150,488.

This experiment falls in category 2. In the third experiment, target 182 was chosen to be located at the point +2a of the critical surface for j_3 of (a) and it was denoted as 182_{2a}. The distances to j_3 of (a), (b), and (c) were computed as 104,604 m, 13,787 m, and 187,874 m, respectively. The coordinates of 182_{2a} are given as:

182_{2a} ... 569,195 1,823,300 332,398.

This part constitutes category 3. Finally, target 182 was chosen to be located at the point +5 a of the critical surface for j_3 of (a), and it was denoted as 182_{5a}. The distances to j_3 of (a), (b), and (c) were computed to be 418,416 m, 290,624 m, and 479,917 m, respectively. The coordinates of 182_{5a} were obtained as

$$182_{5a} \ldots \quad 504,542 \qquad 1,964,480 \qquad 605,013.$$

This experiment falls into category 4. The results of all arrangement in all categories are given in Table (2.5-4).

Table (2.5-4)

Results of Experiments in Example 2.

Arrangement	$Tr(N^{-1})$			
	Category 1	Category 2	Category 3	Category 4
(a)	$-.4 \times 10^8$	474,100	95,880	53,920
(b)	$-.3 \times 10^8$	86,360	2,750,000	37,850
(c)	$+.3 \times 10^8$	$-.5 \times 10^9$	89,650	58,120
(ab)	$+.7 \times 10^8$	362,800	2,337,000	52,300
(ac)	$+.1 \times 10^8$	$-.2 \times 10^9$	75,250	54,420
Distances 182 - (a)	0	104,604	104,604	418,416
Distances 182 - (b)	290.5	198,231	13,787	290,624
Distances 182 - (c)	4,841.2	4,336.0	187,874	479,917
182:	182′	182_o	182_{2a}	182_{5a}

The results in this table indicate for the arrangements (a), (b), (c), for which the critical surfaces were computed, that the problem is indeed singular when target 182 is on or very near its critical surface. The best results were obtained in category 4 where target 182 was the farthest apart from the critical surface of j_3. Further significant improvement is to be expected when additional targets are added to the present system (with d.f. = 0) and/or when more than four stations observe simultaneously.

Example 3. In category 1 of this example all the points of a network are chosen to be lying on one second order surface, a hyperboloid of one sheet. All the arrangements in all categories of this example are be the same as in Example 2. The points of the network lying on the above second order surface consist of stations 1, 2, 3, 4, having the same coordinates as in Example 2 and targets 151, 152, 161, 171, and 183, given in Table (1.5-1), and further of the points from the same table whose coordinates have been changed; the latter are given in Table (2.5-5), as well as the center and the six main surface points of the

Table (2.5-5)

Coordinates of Some Points Related to Second Order Surface of Example 3

Point	x	y	z
5	1,333,100	1,294,770	-25,514
6	204,600	839,635	270,346
153	678,155	838,141	1,520,680
162	455,046	520,662	1,581,940
163	376,898	339,908	1,617,060
172	680,973	980,757	1,762,480
173	1,286,180	1,319,310	2,097,610
181	577,858	703,845	1,502,890
182	1,392,680	89,704	1,958,720
x_o	698,913	283,076	1,037,120
+ a	559,891	-236,543	985,553
- a	837,934	802,695	1,088,700
+ b	936,198	222,390	1,008,930
- b	461,627	343,762	1,065,320
+ c	660,968	336,280	603,339
- c	736,857	229,871	1,470,910

critical surface (coordinates of all the points of interest were printed to [illegible] six digits). The lengths of its two real axes in order of magnitude and of the imaginary axis are given as 540,361 m, 246,541 m, and 438,680 m, respectively. In category 2, point 182 is chosen to coincide with x_o. In category 3, it is [illegible] to coincide with the point - 2b, and in category 4 to coincide with the point [illegible] its coordinates in these two categories are given as

224, 341   404, 448   1, 093, 520

and

-487, 517   586, 506   1, 178, 120,

respectively. The adjustment results of the five arrangements in four categories are given in Table (2.5-6). From this table it can be verified that the problem must be singular for any arrangement of observations when all the points of a network are lying on one second order surface. In general, results of this example are somewhat inferior to those of Example 2. In particular, category [illegible] is very weak. Here again, much better adjustment is to be expected by adding of redundant observations. It can be illustrated in the following experiment.

Table (2.5-6)

Results of Experiments in Example 3.

Arrangement	$Tr(N^{-1})$			
	Category 1	Category 2	Category 3	Category 4
(a)	$-.1 \times 10^9$	$.29 \times 10^7$	136, 600	67, 960
(b)	$-.7 \times 10^8$	$.15 \times 10^7$	191, 900	79, 880
(c)	$-.1 \times 10^9$	$.67 \times 10^7$	79, 210	75, 460
(ab)	$-.8 \times 10^8$	$.14 \times 10^7$	116, 100	51, 760
(ac)	$-.8 \times 10^8$	$.69 \times 10^7$	56, 140	69, 640
182-(a), (b), (c)	0	246, 541	246, 541	986, 164

Example 4. This example is exactly the same as Example 3, except that all the six ground stations are co-observing. The results for $Tr(N^{-1})$ in category 1 through category 4 are given respectively as

$$-.3 \times 10^{8} \qquad .36 \times 10^{5} \qquad 3,927 \qquad 643.5.$$

These values indicate highly significant improvement for all non-singular cases as compared with Example 3, due to the fact that all the ground stations are co-observing. Also, singularity due to all points lying on one second order surface is clearly evident.

Example 5. In the first experiment of this example, all the ground stations are again co-observing and all the points of the network are lying on one second order surface. In contrast with Example 4 where the critical surface was quite general and where the points were generally distributed on that surface, the second order surface is now represented by a sphere (centered at the origin and having the radius equal to 1,000,000 m) and most of the points are symmetrically distributed. Altogether, there are eight ground stations (numbered as 51, 52, . . . , 58) and eight targets (numbered as 501, 502, . . . , 508), all lying on the sphere. Their coordinates are given in Table (2.5-7). This configuration yields

$$Tr(N^{-1}) = -.2 \times 10^{8}$$

from an adjustment. Clearly, the problem is singular.

Next, the location of station 58 is varied. Altogether, there are nine locations occupied by station 58, denoted as 58_1 through 58_9. Point 58_1 is located at the center of the sphere, points 58_2 through 58_6 inside the sphere, point 58_7 on the sphere, and points 58_8, 58_9 are located outside the sphere. All the other points are unchanged and all the stations are again assumed co-observing. The varying coordinates of station 58, its distance from the sphere (on which all the other points are lying) and $Tr(N^{-1})$ obtained from an adjustment are presented

Table (2.5-7)

Coordinates of Sixteen Points of Example 5.

Point	x	y	z
51	0	0	- 1,000,000.00
52	0	- 707,106.78	- 707,106.78
53	612,372.43	- 353,553.39	- 707,106.78
54	612,372.43	353,553.39	- 707,106.78
55	0	707,106.78	- 707,106.78
56	- 612,372.43	353,553.39	- 707.106.78
57	- 612,372.43	- 353,553.39	- 707,106.78
58	0	- 939,692.62	- 342,020.14
501	0	0	1,000,000.00
502	0	- 707,106.78	707,106.78
503	612,372.43	- 353,553.39	707,106.78
504	612,372.43	353,553.39	707,106.78
505	0	707,106.78	707,106.78
506	- 612,372.43	353,553.39	707,106.78
507	- 612,372.43	- 353,553.39	707,106.78
508	0	- 984,807.76	173,648.18

in Table (2.5-8). The results from this table again verify that a singular solution is obtained when all the points are lying on one second order surface (here sphere) and that very strong solutions can be obtained with all stations co-observing.

Table (2.5-8)

Nine Experiments Corresponding to Location of Station 58.

Point	x	y	z	Distance from Sphere	$Tr(N^{-1})$
58_1	0	0	0	1,000,000	135.5
58_2	0	- 200,000	0	800,000	137.0
58_3	0	- 400,000	0	600,000	148.3
58_4	0	- 600,000	0	400,000	169.7
58_5	0	- 800,000	0	200,000	207.0
58_6	0	- 900,000	0	100,000	263.5
58_7	0	- 1,000,000	0	0	$-.6 \times 10^8$
58_8	0	- 1,100,000	0	100,000	319.4
58_9	0	- 1,200,000	0	200,000	287.3

2.6 Conclusions

Perhaps the most important theoretical result in this chapter is that whenever all the points (ground stations and targets) of a network lie on one second order surface the network is necessarily singular. An illustration of such a configuration appears in Figure 7.

Some special cases of singular solution arise when all the targets observed by a certain station (they can be in one or more satellite groups) are in a plane which contains this station (mostly called singularity A)), or when all the targets of a network are in a plane on a second order curve (this was called "reversed singularity B)"). When all its points lie on a second degree surface, the network is singular even if all the ground stations co-observe; this is the only case of a singular problem when all the stations co-observe, except for the special cases when all the targets in a network are in a plane containing one ground station, or when they are all on a second order (plane) curve. Naturally, when all the points are on one second order surface, the network is singular no matter how the observations are arranged ("leapfrogging", etc.).

When only a limited number of stations co-observe, the situation is somewhat more complicated. In practice, four stations forming quads may co-observe a set of targets. With three stations observing all the targets, it was found that an adjustment of range observations is singular if for each quad the stations and the corresponding targets lie on a specific second order critical surface. In sections 2.23 and 2.31 the method is given to compute such critical surfaces explicitly. All these critical surfaces intersect in one second order (plane) curve containing the above three stations. This geometric property is illustrated in Figure 6. If the special singular cases due to singularity A) or "reverse singularity B)" do not exist the network has a non-singular solution if there is at least one (satellite) point located outside the corresponding critical surface.

When utilizing the concept of station replacement, it was found that besides the above two special cases singular solutions would again be associated with specific second order surfaces. However, these were not expressed explicitly. In this case sufficient conditions for non-singular networks stipulate that after an expansion of a non-singular network the new network is still non-singular if the targets of any "new" satellite group do not lie in a plane with the "old" three stations and that the fourth, "new" station does not lie in one plane with these targets.

The main results of this section are summarized in Table (2.6-1).

For the reasons mentioned in section 2.1, geodetic networks are not likely to be singular when the ground stations are generally distributed in space. However, when the number of redundant observations is small or zero, an adjustment may be sometimes quite weak. Adding extra observations can significantly improve the quality of the solution. Several computer runs indicated that some very good results can be expected with more than four ground stations co-observing.

Table (2.6-1)

Necessary and Sufficient Conditions to Avoid Singular Solutions

When Ground Stations Are Generally Distributed

Type of Singularity	Arrangement of Observations	Necessary Conditions to Prevent Singularity	Sufficient Conditions to Prevent Singularity	Note
Singularity A) (or closely related singularity)	Any	No station should be in a plane with all its observed targets (distributed over one or more satellite groups)	No station should be in a plane with the corresponding satellite group	This singularity is assumed non-existent in analysis of global singularity
Reversed Singularity B)	Any	Targets should not be all in a plane on a second order curve	The same as the necessary conditions	This singularity is assumed non-existent in analysis of global singularity
Global Singularity	Stations 1, 2, 3 Observe all targets	Avoid all satellite groups (one group per quad) containing targets lying all on the corresponding second order critical surfaces (one surface per quad). Always: Avoid all points lying on a second order surface	The same as the necessary conditions	All the critical surfaces can be computed explicitly. They all intersect in the plane of stations 1, 2, 3 on a second order curve containing the three stations. If four points outside this plane are common to some critical surfaces then these surfaces coincide
	Station replacement (e.g., leap-frogging)	Always: Avoid all points lying on a second order surface	Avoid certain second order surfaces not expressed explicitly	
	All stations observe all targets (all stations co-observe)	Avoid all points lying on a second order surface	The same as the necessary conditions	

APPENDICES

APPENDIX 1

EFFECT OF ADDITIONAL OBSERVATIONS ON VARIANCE-COVARIANCE MATRIX OF THE SAME SET OF PARAMETERS

It is a plausible statement that a weight-coefficient matrix Q_x of the unknown parameters X "improves" with additional observations, and if these are affected by random errors coming from the same population as the original set of observations, the variance of unit weight being thus the same, then also the variance-covariance matrix Σ_x of the unknown parameters "improves". The word "improves" is used to describe the fact, that $Q_x^1 - Q_x$ is a positive (semi-) definite matrix, denoted also as $Q_x^1 - Q_x \geq 0$; here Q_x^1 stands for the case when only original observations were considered and Q_x when also additional observations were included in the least squares adjustment. This also means that

$$\mathrm{Tr}(Q_x^1) - \mathrm{Tr}(Q_x) \geq 0$$

or

$$\mathrm{Tr}(Q_x^1) \geq \mathrm{Tr}(Q_x).$$

This last expression can indeed be interpreted as an improvement in a weight coefficient matrix Q_x due to the additional observations.

The asserted statment will be proved using "A method" of the least squares adjustment, such as treated in [5], since this method has been used throughout in treating ground stations - satellites range observations. For the original group of observations, which can be considered as consisting of the minimum number of observations and thus d.f. =0 (degrees of freedom), it holds that

$$V_1 = A_1 X + L_1 \qquad \text{(A1-1a)}$$

and

$$N_1 = A_1^T P_1 A_1 , \tag{A1-1b}$$

where V_1 stands for the residual vector, A_1 for the coefficient matrix of the observation equations (A1-1a), L_1 for the constant vector, N_1 for the coefficient matrix of normal equations and P_1 for the weight matrix, assumed to be positive-definite, all for this group, denoted by the index $_1$. When all the observations are included, it holds that

$$V = AX + L$$

and

$$N = A^T P A \tag{A1-2}$$

with similar description of this group. It is assumed that P_1 and P_2, which compose the P matrix, are uncorrelated. Furthermore,

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

Thus, it follows from (A1-2) that

$$N = N_1 + N_2$$

where

$$N_2 = A_2^T P_2 A_2 , \tag{A1-3}$$

similar to (A1-1b).

The normal equations for the original observations are

$$N_1 X + U_1 = 0$$

with U_1 given as

$$U_1 = A_1^T P_1 L_1$$

and for all the observations

$$NX + U = 0$$

with

$$U = A_1^T P_1 L_1 + A_2^T P_2 L_2 .$$

Now, two cases arise. In the first case, N_1^{-1} is considered to exist, in which case N_1 is of full rank and no constraints for the parameters are needed. This, in practice, could be equivalent to fixing of at least six coordinates for range observations alone, which would mean the the corresponding rows and columns are deleted from N_1.

The second case is of practical significance for fundamental networks using range observations alone. The rank deficiency of both N_1 and N is such a network is six in general which means that at least six constraints have to be used to make an adjustment possible. Then the augmented coefficient matrix of normal equations is assumed to have the full rank. For fundamental networks exactly six linearly independent constraints are used.

1. According to [5], when N_1 is invertible the weight-coefficient matrix for the parameters is given as

$$Q_x^1 = N_1^{-1}$$

while for all the observations it holds that

$$Q_x = (N_1 + N_2)^{-1} .$$

Making use of (A1-3), this last equation can be developed as

$$Q_x = N_1^{-1} - N_1^{-1} A_2^T \ (P_2^{-1} + A_2 N_1^{-1} A_2^T)^{-1} A_2 N_1^{-1} .$$

Since P_2, N_1 are both positive-definite here, $(P_2^{-1} + A_2 N_1^{-1} A_2^T)^{-1}$ is positive-definite and S positive (semi-) definite, where

$$S = N_1^{-1} A_2^T \ (P_2^{-1} + A_2 N_1^{-1} A_2^T)^{-1} A_2 N_1^{-1} .$$

Due to N_1 positive definite and N_2 positve (semi-) definite, N and so also $N^{-1} = Q_x$ are positive definite matrices.

Now

$$Q_x = Q_x^1 - S$$

and

$$\mathrm{Tr}\,(Q_x) = \mathrm{Tr}\,(Q_x^1) - \mathrm{Tr}\,(S)$$

where

$$\mathrm{Tr}\,(Q_x) > 0, \quad \mathrm{Tr}\,(Q_x^1) > 0, \quad \mathrm{Tr}\,(S) \geq 0.$$

Thus,

$$\mathrm{Tr}\,(Q_x) \leq \mathrm{Tr}\,(Q_x^1)$$

indicating an increase in accuracy due to added observations to the original set.

2. Using six constraints among the parameters, the following system is to be solved when all the observations are considered:

$$NX + U = 0$$
$$CX + W_c = 0$$

where the parameters can be rearranged in such a way that

$$CX = [\,C_a \; C_b\,] \begin{bmatrix} X_a \\ X_b \end{bmatrix}$$

where C_a is a (6×6), non-singular matrix. Then

$$X_a = -C_a^{-1} C_b X_b - C_a^{-1} W_c \tag{A1-4}$$

from which

$$Q_{x_a} = T Q_{x_b} T^T \tag{A1-5}$$

where Q_{x_a} is positive (semi-) definite and $T = C_a^{-1} C_b$.

With A_1 , A_2 partitioned in the same way as C, it holds for the observation equations where all the observations are included, that

$$V = \begin{bmatrix} A_{1a} & A_{1b} \\ A_{2a} & A_{2b} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

Upon plugging X_a from (A1-4) into these equations, the following observation equations from which the parameters X_a have been eliminated, are obtained:

$$V = \begin{bmatrix} A_{1b} - A_{1a} C_a^{-1} C_b \\ A_{2b} - A_{2a} C_a^{-1} C_b \end{bmatrix} X_b + \begin{bmatrix} L_1 - A_{1a} C_a^{-1} W_c \\ L_2 - A_{2a} C_a^{-1} W_c \end{bmatrix} .$$

This can be written as

$$V = \widetilde{A} X_b + \widetilde{L} ,$$

where

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{bmatrix} .$$

As stated before, due to the constraints, the solution of such a system now exists, whether for the original observations (to which $\widetilde{A}_1$ is pertaining) or for all the observations included (to which $\widetilde{A}$ is pertaining). This system is analogous to the one investigated in Part 1., and so the conclusions are the same, namely

$$Q^1_{x_b} - Q_{x_b} \text{ is positive (semi-definite)} \tag{A1-6a}$$

or

$$\mathrm{Tr}\,(Q_{x_b}) \leq \mathrm{Tr}\,(Q^1_{x_b}) . \tag{A1-6b}$$

Now, analogically to (A1-5), it holds that

$$Q^1_{x_a} = T Q^1_{x_b} T^T . \tag{A1-7}$$

From (A1-5) and (A1-7) it follows that

$$Q^1_{x_a} - Q_{x_a} = T\,(Q^1_{x_b} - Q_{x_b})\,T^T$$

which is positive (semi-) definite, due to (A1-6a). Thus,

$$\mathrm{Tr}\,(Q_{x_a}) \leq \mathrm{Tr}\,(Q^1_{x_a}) ,$$

which together with (A1-6b) yields

$$\mathrm{Tr}(Q_x) \leq \mathrm{Tr}(Q_x^1),$$

the same result as in the Part 1., indicating an increase in accuracy due to added observations to the original set, while preserving the same parameters as unknowns.

These and similar aspects are considered in different publications, for instance in [12]. An interesting treatment connected with the adding of observations and/or constraints to an original set of observations is presented in [11].

APPENDIX 2

BEST FITTING PLANE

An equation of a plane can have the form

$$(r - r_0) \cdot n = 0,$$

or

$$r \cdot n - r_0 \cdot n = 0,$$

where $r_0 = (x_0, y_0, z_0)^T$ represents a radius-vector to a certain point in the plane, $r = (x, y, z)^T$ a radius-vector to a general point in the plane and $n = (a, b, c)^T$ a unit normal vector to the plane, in which case $\sqrt{a^2 + b^2 + c^2} = 1$ must hold. In absolute value, $r_0 \cdot n = d$ represents a perpendicular distance of the plane from the origin, and since $r \cdot n = xa + yb + zc$, the equation of a plane is written as

$$xa + yb + zc + d = 0.$$

In the following each of the points to which the plane is fitted will be considered as lying in the plane after the adjustment. Thus the deviations from the plane will be regarded as due to "errors" in the "observations", which will be represented here by the actual cartesian coordinates of each points, leading thus to three "observations" per point. Denoting all the adjusted values (parameters, observations) by superscript a, it will hold for a point i:

$$x_i^a a^a + y_i^a b^a + z_i^a c^a + d^a = 0 \qquad \text{(A2-1a)}$$

and

$$(a^a)^2 + (b^a)^2 + (c^a)^2 - 1 = 0. \qquad \text{(A2-1b)}$$

There will be as many equations of the type (A2-1a) as there are points involved, say r, and one equation of the type (A2-1b). This corresponds to the mathematical structure of the general L.S. method with constrains such as described in [5], namely

$$F(X^a, L^a) = \underset{(r \times 1)}{0} \qquad \text{(A2-2a)}$$

and

$$G(X^a) = \underset{(s \times 1)}{0} \qquad \text{(A2-2b)}$$

where X^a are adjusted parameters, which in the present case are $(a^a, b^a, c^a, d^a)^T$, and L^a adjusted observed quantities, here $(x_1^a, y_1^a, z_1^a; x_2^a, y_2^a, z_2^a; \ldots; x_i^a, y_i^a, z_i^a; \ldots)^T$. In the above mathematical model, (A2-2a) is represented by r equations of the type (A2-1a) and (A2-2b) by one equation (A2-1b).

After the linearization, (A2-2a) and (A2-2b) become

$$AX + BV + W = 0 \qquad \text{(A2-3a)}$$

$$CX \qquad + W_c = 0, \qquad \text{(A2-3b)}$$

where all the notations of [5] are preserved, namely:

$X \equiv dX \ldots$ u corrections to the u approximate values of parameters, X^O.
$V \ldots$ n residuals, corrections to the n observed quantities, L^b.

$A = \left[\frac{\partial F}{\partial X}\right]_{o,b}$ ... $(r \times u)$ matrix of coefficients

$B = \left[\frac{\partial F}{\partial L}\right]_{o,b}$ ... $(r \times n)$ matrix of coefficients

$C = \left[\frac{\partial G}{\partial X}\right]_{o}$ ... $(s \times u)$ matrix of coefficients due to the constraints

$W = F(X^O, L^b)$ ... r-vector of misclosures

$W_c = G(X^O)$ ... s-vector of misclosures due to the constraints.

In the present case the dimensions are as follows:

r ... number of poi ts to be used for fitting
u = 4 ... for parameters a, b, c, d
n = 3r ... number of "observables", i. e. x, y, z coordinates of the points used
s = 1 ... one constraint, namely (A2-1b).

Accordingly, using the structure (A2-1a) and (A2-1b) the matrices and vectors in (A2-3a) and (A2-3b) will be:

$$\underset{(r\times 4)}{A} = \begin{bmatrix} & \vdots & \\ x_i^b & y_i^b \; z_i^b & 1 \\ & \vdots & \end{bmatrix}; \quad \underset{(4\times 1)}{X} = \begin{bmatrix} da \\ db \\ dc \\ dd \end{bmatrix}; \quad \underset{(n\times 1)}{V} = \begin{bmatrix} \vdots \\ v_{x\,i} \\ v_{y\,i} \\ v_{z\,i} \\ \vdots \end{bmatrix};$$

$$\underset{(r\times n)}{B} = \begin{bmatrix} a^o\, b^o\, c^o & 0\;0\;0 & \cdots & 0\;0\;0 \\ 0\;\;0\;\;0 & a^o\, b^o\, c^o & \cdots & 0\;0\;0 \\ \vdots & \vdots & \vdots & \vdots \\ 0\;\;0\;\;0 & 0\;0\;0 & \cdots & a^o\, b^o\, c^o \end{bmatrix}; \quad \underset{(1\times 4)}{C} = [2a^o\; 2b^o\; 2c^o\; 0]; \qquad \text{(A2-4)}$$

$$\underset{(r\times 1)}{W} = \begin{bmatrix} \vdots \\ x_i^b\, a^o + y_i^b\, b^o + z_i^b\, c^o + d^o \\ \vdots \end{bmatrix}; \quad \underset{(1\times 1)}{W_c} = (a^o)^2 + (b^o)^2 + (c^o)^2 - 1,$$

with a^o, b^o, c^o, d^o as approximate values of the parameters. Furthermore, the weight matrix P will be taken as a unit matrix, as there is no reason why some coordinates should be weighed more heavily than others; also, with this $P = I$, the adjustment will eventually render Σ (distance from the plane)^2 to be a minimum such as demonstrated in A2.2, a condition which is indeed desirable.

A2.1 Transformation of General Method Adjustment into the "A Method"

From the mathematical structure for the "A method", $F(X^a) - L^a = 0$, it follows for error considerations that $A\,dX - dL = 0$, or

$$dL = A\,dX, \qquad \text{(A2-5)}$$

where dL can represent errors, coming from a certain population, which affect L^b, observed quantities. The variance-covariance matrix of dL, which is identified with the variance-covariance matrix of L^b, denoted Σ_{L^b}, is a measure of uncertainties in the observations. The weight matrix P is then taken as

$$P = \Sigma_{Lb}^{-1}. \qquad (A2\text{-}6)$$

The observation equations

$$V = AX + L,$$

subject to the condition

$$V^T PV = \min.,$$

yield for the parameters:

$$X = -(A^T PA)^{-1} A^T PL, \qquad (A2\text{-}7)$$

as presented in [5].

From the mathematical structure for the general method, $F(X^a, L^a) = 0$, it follows anologically that

$$-B\,dL = A\,dX. \qquad (A2\text{-}8)$$

Here $\Sigma_{BdL} = B\,\Sigma_{Lb}\,B^T$, which using the same P as in (A2-6), gives

$$\Sigma_{BdL} = B\,P^{-1}B^T. \qquad (A2\text{-}9)$$

Linearization of the general method model gives

$$-BV = AX + W. \qquad (A2\text{-}10)$$

Subject to the condition

$$V^T PV = \min.,$$

this yields for the parameters:

$$X = -[A^T(BP^{-1}B^T)^{-1}A]^{-1} A^T(BP^{-1}B^T)^{-1}W, \qquad (A2\text{-}11)$$

as derived in [5].

If the notation

$$\widetilde{P} = (BP^{-1}B^T)^{-1} \qquad (A2\text{-}12)$$

is introduced, (A2-11) reads as

$$X = -(A^T\widetilde{P}A)^{-1} A^T\widetilde{P}W. \qquad (A2\text{-}13)$$

But this is the result which will be obtained from (A2-10), if the equations are written as "transformed observation equations"

$$\widetilde{V} = AX + W \tag{A2-14}$$

and with the associated weight matrix, $\widetilde{P}$; in other words, the equations

$$\widetilde{V} = -BV = AX + W \tag{A2-14a}$$

are given weights

$$\widetilde{P} = (\Sigma_{9dL})^{-1}, \tag{A2-15}$$

while in "A method" equations $V = AX + W$ were given weights $P = (\Sigma_{dL})^{-1}$.

It remains to be shown that in the general method

$$\widetilde{V}^T \widetilde{P} \widetilde{V}^T = V^T P V, \tag{A2-16}$$

so that the correct use of "transformed observation equations" (A2-14) with the weights (A2-15) be verified. It holds for the general method that

$$V = P^{-1} B^T (B P^{-1} B^T)^{-1} B V$$

and, consequently,

$$V^T P V = V^T B^T (B P^{-1} B^T)^{-1} B V.$$

On the other hand, making use of (A2-14a) and (A2-12), it holds that

$$\widetilde{V}^T \widetilde{P} \widetilde{V} = V^T B^T (B P^{-1} B^T)^{-1} B V$$

which proves (A2-16).

As a conclusion, it follows that whenever it appears advantageous (reducing of dimensions etc.), the "transformed observation equations" and appropriate weights may be used for "A method" adjustment as given by (A2-14a) and (A2-15). The A matrix is thus the same in both methods and so is the constant vector, $W \equiv L$. Also all the results, namely X, $V^T P V$, $\hat{\sigma}$, Σ_x (as seen from (A2-11) and (A2-13)) are the same whether the general method or "A method" with the "transformed observation equations" is used.

A2.2 "Transformed Observation Equations" in the Best Fitting Plane Problem.

A2.21 General Considerations.

Due to the special form of the B matrix in (A2-4), $\widetilde{P}$ matrix will be particularly simple. Since $P = I$, then $B P^{-1} B^T$ will be of dimensions $(r \times r)$ with zeroes as off-diagonal elements. Furthermore, the diagonal elements will be all equal to $(a^0)^2 + (b^0)^2 + (c^0)^2$ which should be close to one (in the first iteration and in last several iterations — when the iterative procedure is used — it should be equal to one for all practical purposes). Thus

$$\widetilde{P} = \underset{(r \times r)}{I}$$

and

$$\underset{(r \times 1)}{\widetilde{V}} = AX + W$$

with

$$CX + W_c = \underset{(1 \times 1)}{0}$$

with A, X, W, C, W_c the same as in (A2-4).

Next, the vector $\widetilde{V}$ will be interpreted. The original vector V was composed of three vectors, such as

$$v_i = \begin{bmatrix} v_{xi} \\ v_{yi} \\ v_{zi} \end{bmatrix} ,$$

connecting the "measured point i" with its "adjusted position". Then $n \cdot v_i$ is the projection of this vector on n, normal to the plane, or the (perpendicular) distance of the existing point i from the plane. Due to $\widetilde{V} = -BV$ and due to the special form of the B matrix it follows that

$$\widetilde{V} = -\begin{bmatrix} n \cdot v_1 \\ n \cdot v_2 \\ \vdots \\ n \cdot v_r \end{bmatrix} .$$

Thus the absolute values in $\widetilde{V}$ represent distances between all points in consideration and the best fitting plane. $\widetilde{V}^T \widetilde{P} \widetilde{V}$ thus represents Σ (distance from plane)^2 and it is a minimum, which is the desired property.

A2.22 Approximate Values of Parameters.

The starting approximate values of the parameters may be obtained by fitting a plane to the first three points, which have r_1, r_2, r_3 as radius-vectors. Formulas to be used:

(a) $r_{12} = r_2 - r_1$; $r_{13} = r_3 - r_1$

(b) $$n = \frac{r_{12} \times r_{13}}{|r_{12} \times r_{13}|} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} ; \quad \sqrt{a^2 + b^2 + c^2} = 1$$

(c) $d = -r_1 \cdot n = -r_2 \cdot n = -r_3 \cdot n.$

These values of a, b, c, d are used for the first iteration.

The misclosure for point i, w_i, will be according to (A2-4):

$$w_i = x_i a + y_i b + z_i c + d, \tag{A2-17}$$

where the superscripts have been omitted. Because of (c), $w_1 = w_2 = w_3 = 0$. Let $\overline{P} \equiv (\bar{x}, \bar{y}, \bar{z})^T$ denote the projection of $P \equiv (x, y, z)^T$ on the plane. Thus

$$(\overline{x}, \overline{y}, \overline{z})^T = (x, y, z)^T + \ell\, n,$$

where $|\ell|$ is the distance from the plane through 1, 2, 3, to point P. Since $\overline{P}$ is in the plane, it must hold:

$$(ax + a^2\ell) + (by + b^2\ell) + (cz + c^2\ell) + d = 0,$$

or

$$-\ell\,(a^2 + b^2 + c^2) = ax + by + cz + d.$$

But the right side is exactly the misclosure for point i, as seen from (A2-17), while $(a^2 + b^2 + c^2) = 1$. Thus $\ell = -w_i$ and the distance of any point from the plane through 1, 2, 3 is determined by the absolute value of its misclosure, $|w_i|$, in the first iteration.

A2.3 Summary of Formulas and Sequence of Operations for the Best Fitting Plane Program

The program for the best fitting plane will use the "A method" adjustment procedure such as described in [5], taking advantage of the particular features inherent in this problem, namely "transformation of observation equations" (as shown in section A2.2). Iterative procedure will be used until the values of the parameters (or Σpvv) remain practically constant. Summary of the steps in this adjustment:

1) Data consists of (x, y, z) coordinates of all the r points used for the best fitting plane. The coordinates may by scaled, depending on their nominal values.

2) Using (first) three points, compute a^o, b^o, c^o, d^o as outlined in (a), (b), (c) of A2.22.

3) Carry out the "A method" adjustment with constraints, where the observation equations have been "transformed", by plugging for standard matrices:

$$\underset{(r\times 4)}{A} = \begin{bmatrix} \vdots & & & \\ x_i^b & y_i^b & z_i^b & 1 \\ \vdots & & & \end{bmatrix}$$

$$\underset{(r\times 1)}{L} = \begin{bmatrix} \vdots \\ x_i^b\, a^o + y_i^b\, b^o + z_i^b\, c^o + d^o \\ \vdots \end{bmatrix}$$

$$\underset{(1\times 4)}{C} = \begin{bmatrix} 2a^o & 2b^o & 2c^o & 0 \end{bmatrix}$$

$$\underset{(1\times 1)}{W_c} = (a^o)^2 + (b^o)^2 + (c^o)^2 - 1,$$

with the unit matrix ($r \times r$) as the weight matrix.

4) Iterate as long as desired; at each step the values in W, C, W_c, change, due to changing values of the parameters.

Since A (and P) does not change, N and N^{-1} matrices are the same throughout.

5) The final absolute values of the residuals are (perpendicular) distances of the points from the best fitting plane. The final Σpvv represents Σ (distance from plane)^2. In practical computations the average distance from the best fitting plane was used as a reasonable measure as to the closeness of all the considered points from a plane.

6) The last step is optional. It makes it possible to compute the adjusted "observations", i.e., the projections of all the considered points on the best fitting plane. It can be shown that they are obtained from

$$L^a = L^b - B^T \widetilde{V},$$

or

$$x_i^a = x_i^b - a\widetilde{v}_i$$

$$y_i^a = y_i^b - b\widetilde{v}_i$$

$$z_i^a = z_i^b - c\widetilde{v}_i$$

where $\widetilde{v}_i$, corresponding to the i th element of $\widetilde{V}$, and a, b, c, are all taken from the last iteration. All these projected points can be further transformed into the "local coordinate system" (which has the first point at its origin, the second point on its x-axis and the third point in its xy plane), using the procedure outlined in section 2.231, in (2.2-12) through (2.2-17). Necessarily, all the projected points are in the xy plane of the local coordinate system. Using the procedure of section 1.322, it can be determined whether they are all lying on or near a second order curve. Since they are given in the xy plane of the local coordinate system, the basic and the local coordinate systems of section 1.322 now coincide.

APPENDIX 3

CRITICAL CURVE IN LOCAL COORDINATES AS OBTAINED ANALYTICALLY BY FITTING SECOND ORDER CURVE TO STATIONS 1 THROUGH 5

A general second degree curve such as ellipse or hyperbola has five parameters to be determined, namely x_o, y_o specifying its position, α specifying its orientation and a, b determining its size and shape. These five unknowns can be solved for from five equations which may be furnished by five distinct points lying on the curve. It will now be shown that fitting of the second order curve to the stations 1 through 5 (which lie on the critical curve as it was shown in the part following (1.3-2)) will furnish the same A, a as expressed in (1.3-5) through (1.3-5c).

For the proof, the same local coordinate system will be chosen, having station 1 at the origin and station 2 on the x axis, stations 3, 4, and 5, being generally distributed in the x, y plane. The equation of a second degree curve can be written as

$$g_{11}x^2 + 2g_{12}xy + g_{22}y^2 + h_1x + h_2y + k = 0. \qquad \text{(A3-1)}$$

Using the fact that the station 1 lies on the curve yields $k = 0$. Upon division by g_{11} (assumed non-zero) (A3-1) becomes:

$$x^2 + 2b_{12}xy + b_{22}y^2 + b_1x + b_2y = 0. \qquad \text{(A3-2)}$$

Assumption $g_{11} \neq 0$ is reasonable, since, of the stations 1, 2, 3, any one can be chosen as the origin of the local coordinate system and any one as determining the direction of the x-axis. In the matrix notation (A3-2) becomes

$$x^T Bx + x^T b = 0$$

where

$$x \equiv [x, y, z]^T$$

and where B, b are analogous to A, a of (1.3-5). Since station 2 lies on the curve, $(x_2, 0)$ can be plugged for (x, y) in (A3-2), giving $b_1 = -x_2$, upon which it becomes

$$b_{22}y^2 + 2b_{12}xy + b_2 y = x(x_2 - x).$$

This last equation should be satisfied for stations 3, 4, 5, yielding thus three equations in three unknowns, namely

$$\begin{bmatrix} y_3^2 & 2x_3y_3 & y_3 \\ y_4^2 & 2x_4y_4 & y_4 \\ y_5^2 & 2x_5y_5 & y_5 \end{bmatrix} \times \begin{bmatrix} b_{22} \\ b_{12} \\ b_2 \end{bmatrix} = \begin{bmatrix} x_3(x_2 - x_3) \\ x_4(x_2 - x_4) \\ x_5(x_2 - x_5) \end{bmatrix}. \qquad \text{(A3-3)}$$

The last step consists of solving for b_{22}, b_{12}, b_2 from (A3-3), for which the inverse of the (3×3) coefficient matrix will be used in the form

$$\frac{1}{D}\begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$$

where D represents the determinant of the coefficient matrix and α_{ij}, the cofactor of its ij th element. Thus,

$$b_{22} = \frac{1}{D}[\alpha_{11}x_3(x_2 - x_3) + \alpha_{21}x_4(x_2 - x_4) + \alpha_{31}x_5(x_2 - x_5)],$$

$$b_{12} = \frac{1}{D}[\alpha_{12}x_3(x_2 - x_3) + \alpha_{22}x_4(x_2 - x_4) + \alpha_{32}x_5(x_2 - x_5)],$$

$$b_2 = \frac{1}{D}[\alpha_{13}x_3(x_2 - x_3) + \alpha_{23}x_4(x_2 - x_4) + \alpha_{33}x_5(x_2 - x_5)]$$

where it can be shown that with the notations of (1.3-4a) through (1.3-4c),

$$D = -2y_3 C.$$

Furthermore,

$$b_{22} = \frac{1}{C}\left[-\frac{y_4 y_5}{y_3}(x_4 - x_5)\,x_3(x_2 - x_3) - y_5(x_5 - x_3)\,x_4(x_2 - x_4) - y_4(x_3 - x_4)x_5(x_2 - x_5)\right], \tag{A3-4a}$$

$$b_{12} = \frac{-1}{2y_3 C}\left[y_4 y_5 (y_5 - y_4)\,x_3(x_2 - x_3) + y_3 y_5 (y_3 - y_5)\,x_4 (x_2 - x_4) + y_3 y_4 (y_4 - y_3) \times \right.$$
$$\left. \times\, x_5(x_2 - x_5)\right], \tag{A3-4b}$$

$$b_2 = \frac{-1}{y_3 C}\left[y_4 y_5 (y_4 x_5 - x_4 y_5)\,x_3(x_2 - x_3) + y_3 y_5(x_3 y_5 - y_3 x_5)\,x_4(x_2 - x_4) + \right.$$
$$\left. +\, y_3 y_4(y_3 x_4 - x_3 y_4)\,x_5(x_2 - x_5)\right]. \tag{A3-4c}$$

After some algebraic manipulations, it is seen that the expression in brackets of (A3-4a) is equal to B, the one of (A3-4b) to $x_3 C - y_3 A$ and the one of (Ae-4c) to $-x_2 x_3 C + x_3 y_3 A + y_3^2 B$. Thus, besides $b_{11} = 1$ and $b_1 = -x_2$, it also holds that

$$b_{22} = \frac{B}{C},$$

$$b_{12} = \frac{1}{2}\left(\frac{A}{C} - \frac{x_3}{y_3}\right),$$

$$b_2 = x_2\frac{x_3}{y_3} - x_3\frac{A}{C} - y_3\frac{B}{C},$$

which are exactly the same values as those for A, a in (1.3-5b) and (1.3-5d).

Thus, fitting of the second order curve to the ground stations 1 through 5 is equivalent to determining the critical loci for any further ground stations causing singularity B) to occur.

APPENDIX 4

COMPUTATION OF CANONICAL FORM OF SECOND ORDER (HYPER-) SURFACE, GIVEN EXPLICITLY

A4.1 Preliminary Transformation of Coordinates.

Whether dealing with the n-dimensional, three-dimensional or two-dimensional spaces, the equations transforming the coordinates of a vector from one basis into another are essentially the same. In the subsequent derivations only orthonormal bases will be considered. If X' denotes an array of coordinates of a vector in a new basis and Y an array of coordinates of the same vector in the original basis, the following relation holds:

$$Y = RX' \tag{A4-1}$$

where R is an orthogonal matrix whose ijth entry is equal to the dot product of the ith basis vector in the original basis and the jth basis vector in the new basis, Y and X being written as column vectors.

Should the (coordinate) vector X' representing a physical point P, emanate from the origin of the new coordinate system (with the axes in the direction of "new basis vectors") which is different from the origin of the original coordinate system, then Y is to be written as

$$Y = X - X_o ,$$

where X and X_o, both in the original coordinate system, represent the radius-vector of the point P and the radius-vector of the origin of the new coordinate system respectively. (A4-1) is thus written as

$$X - X_o = RX' \tag{A4-2}$$

where R can be written as

$$R = [t_1 \ t_2 \ \cdot \ \cdot \ \cdot \ \cdot \ ] \, . \tag{A4-3}$$

Here t_i, a column vector, represents the i th "new basis vector" in the original coordinate system, and is orthogonal to every t_i vector, $j \neq i$.

In particular, three and two-dimensional spaces will be of main interest here. Thus, for the three-dimensional space, when the original coordinate system has the axes represented by unit vectors (orthonormal basis) i, j, k, and the new system by i', j', k', both being the right-handed coordinate systems, (A4-2) can be written as

$$X - X_0 = \begin{bmatrix} \cos(i\,i') & \cos(i\,j') & \cos(i\,k') \\ \cos(j\,i') & \cos(j\,j') & \cos(j\,k') \\ \cos(k\,i') & \cos(kj') & \cos(k\,k') \end{bmatrix} X' , \tag{A4-4a}$$

indicating that for the orthogonal (3×3) R matrix, it holds that

$$R = [t_1 \quad t_2 \quad t_3] \tag{A4-4b}$$

where

$$t_1 = \begin{bmatrix} \cos(i\,i') \\ \cos(j\,i') \\ \cos(k\,i') \end{bmatrix}, \quad t_2 = \begin{bmatrix} \cos(i\,j') \\ \cos(j\,j') \\ \cos(kj') \end{bmatrix}, \quad t_3 = \begin{bmatrix} \cos(i\,k') \\ \cos(j\,k') \\ \cos(k\,k') \end{bmatrix} . \tag{A 4-4c}$$

Analogous relations will hold for the two-dimensional space, i.e., plane, with the k, k' coordinate axes eliminated and angles measured counter-clockwise as to be compatible with the above system. If α denotes the angle between the axes represented by i' and i (or j' and j) in this order, then as counterparts to (A4-4a) - (A4-4c) there will be:

$$X - X_o = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} X' \qquad \text{(A4-5a)}$$

and

$$R = [t_1 \ t_2], \qquad \text{(A4-5b)}$$

with

$$t_1 = \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix}, \quad t_2 = \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix} . \qquad \text{(A4-5c)}$$

A4.2 Canonical Form of Second Order (Hyper-) Surface.

As a starting point in this discussion, the second degree (hyper-) surface will be considered such as

$$x^T A x + x^T a + c = 0 , \qquad \text{(A4-6)}$$

where A matrix, a vector and constant term c are given explicitly, x being the coordinate vector of a variable point.

A new coordinate system, called canonical, is desired such that after a transformation of coordinates, the equation (A4-6) will be of the form

$$x'^T \Lambda \ x' = 1 , \qquad \text{(A4-7)}$$

where x' is the coordinate vector of a variable point in the canonical coordinate system, and Λ is a diagonal matrix

$$\Lambda = \text{diag.} (\lambda_1, \lambda_2 \dots) . \qquad \text{(A4-7a)}$$

Similarly to (A4-2), it holds that

$$x - x_o = R x',$$

or

$$x = x_o + R x' \qquad \text{(A4-8)}$$

where x_o, origin of the canonical coordinates, and R are as yet unknown. Since R is an orthogonal matrix, x' can be expressed from (A4-8) as

$$x' = R^T (x - x_o) , \qquad \text{(A4-8a)}$$

which plugged into (A4-7) gives

$$(x - x_o)^T R \Lambda R^T (x - x_o) = 1 . \qquad \text{(A4-9)}$$

Denoting

$$R \Lambda R^T = M , \qquad \text{(A4-10a)}$$

for which

$$\Lambda = R^T M R , \qquad \text{(A4-10b)}$$

(A4-9) becomes

$$x^T M x - x^T 2 M x_o + x_o^T M x_o - 1 = 0. \qquad \text{(A4-11)}$$

Once M is known, it will be possible to determine Λ and R matrices. In order to determine M matrix and x_o vector, comparison between (A4-6) and (A4-11), expressing the same surface, will be made, taking into account that the equation (A4-6) can be multiplied by any constant k, as yet unknown. This gives raise to the following relations:

$$M = kA , \qquad \text{(A4-12a)}$$

$$-2 M x_o = ka , \qquad \text{(A4-12b)}$$

and

$$x_o^T M x_o - 1 = kc . \qquad \text{(A4-12c)}$$

Substituting (A4-12a) into (A4-12b) gives

$$x_o = -\tfrac{1}{2} A^{-1} a , \qquad \text{(A4-13)}$$

which has a unique solution for any second degree (hyper-) surface with non-singular A. Upon substituting (A4-12a) into (A4-12c), the expression

$$k = \frac{1}{x_o^T A x_o - c} \tag{A4-14}$$

is obtained, which substituted into (A4-12a) gives

$$M = kA = \frac{1}{x_o^T A x_o - c} A . \tag{A4-15}$$

Whenever the (hyper-) surface contains the origin of the coordinate system to which (A4-6) refers, called here the original coordinate system, then $c = 0$ as it can be seen from (A4-6); (A4-14) with (A4-15) then become

$$k = \frac{1}{x_o^T A x_o} \tag{A4-14a}$$

and

$$M = kA = \frac{1}{x_o^T A x_o} A, \tag{A4-15a}$$

while (A4-13) remains unchanged.

Although this approach and notations are somewhat similar to the derivations presented in [1], Annex J4, it is different in that here the (hyper-) surface is given by an equation such as (A4-6) rather than by a set of given (errorless) points.

To compute Λ, R from (A4-10a) or (A4-10b), standard procedures for finding eigenvectors and eigenvalues of a real symmetric matrix, which are always real, can be used as outlined in [6], Chapters 19 and 21, or [7], Chapter 9. The solution for $(\lambda_1, \lambda_2, \ldots)$ in (A4-7a) is obtained by solving

$$| M - I\lambda | = 0$$

and the matrix R in (A4-3) is obtained by solving for $(t_1, t_2, \ldots)$ vectors from

$$(M - I\lambda_i)\, t_i = 0$$

for every i, subject to the condition that each t_i has a unit norm. However, due to $M = kA$, eigenvalues and eigenvectors for A matrix may be computed, from which Λ and R are easily obtained. If $\Lambda_A = \text{diag.} (\bar{\lambda}_1, \bar{\lambda}_2, \ldots)$ and $R_A = (\bar{t}_1, \bar{t}_2, \ldots)$ constitute the eigenvalues and eigenvectors of the A matrix, then

$$\Lambda = k \Lambda_A \tag{A4-16a}$$

and

$$R = R_A . \tag{A4-16b}$$

From the equation (A4-7) it is seen that

$$\lambda_i = \frac{1}{a_i^2},$$

or

$$a_i = \frac{1}{\sqrt{|\lambda_i|}} \tag{A4-17}$$

where a_i represents the length of the i th axis in the canonical coordinates. Direction of this axis with respect to the orignial coordinate system is given by the corresponding vector t_i.

For computation of eigenvalues and eigenvectors of a real symmetric matrix A (or M), an iterative method such as Jacobi method can be advantageously applied, namely when digital computers are used; it is described in [8], pp. 487-492. However, when working with three and two-dimensional spaces, eigenvalues can be quickly found in closed form by solving for the roots of third and second degree equations, respectively. Such procedure may be desirable when a digital computer is not available, and is described in detail in the next sections. When this approach and the Jacobi method were both used and compared for checking purposes, the results agreed to six decimal places in practically all investigated cases.

Note: Due to (A4-13) from which

$$A x_o = -\tfrac{1}{2} a ,$$

it follows that

$$\frac{1}{k} = x_o^T A x_o - c = -(\tfrac{1}{2} x_o^T a + c) ,$$

and since by (A4-16a)

$$\lambda_1 = k \bar{\lambda}_1 ,$$

it finally holds that

$$a_1 = \sqrt{\frac{-(\frac{1}{2} x_o^T a + c)}{\bar{\lambda}_1}} .$$

A4.3 Canonical Form of Second Order Surface (in Three-Dimensional Space).

As already mentioned following (A4-13), it will be assumed that A is of full rank, i.e. $|A| \neq 0$, in which case none of its eigenvalues can be 0; in addition, it will be assumed that $\Delta \neq 0$, where Δ is the determinant of the "augmented A matrix", described in section 2.231. Thus, of all the real cases, ellipsoid, hyperboloid of one sheet, and hyperboloid of two sheets will be dealt with. As the first step leading to a canonical form of a second degree surface expressed by (A4-6) in three-dimensional space, i.e., as

$$x^T A x + x^T a + c = 0 , \tag{A4-18}$$

the eigenvalues of A will be computed and then used in equations of type (A4-16a) and (A4-17). This leads to solving for the roots of a cubic equation, procedure used for practical computations which will be described in detail. For obtaining eigenvalues of A, denoted as $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$,

$$|\bar{\lambda} I - A| = 0$$

will be solved, which amounts to solving

$$\bar{\lambda}^3 + p\bar{\lambda}^2 + q\bar{\lambda} + r = 0 \qquad \text{(A4-18a)}$$

where p, q, r are all real. Moreover, $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, will be all real since A is a real symmetric matrix. If a_{ij} denotes an ij th entry of A matrix, the values of p, q, r can be computed according to [6], p. 151, as follows:

$$p = -\left\{ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right\}$$

$$q = a_{11} + a_{22} + a_{33} \qquad \text{(A4-18a')}$$

$$r = -|A| .$$

Upon substitution

$$\bar{\lambda} = x - \frac{p}{3} , \qquad \text{(A4-18b)}$$

(A4-18a) becomes

$$x^3 + ax + b = 0 , \qquad \text{(A4-18c)}$$

with

$$a = \frac{1}{3}(3q - p^2) \qquad \text{(A4-18d)}$$

and

$$b = \frac{1}{27}(2p^3 - 9pq + 27r) , \qquad \text{(A4-18e)}$$

both real. If the notation

$$c = \left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3 \qquad \text{(A4-19)}$$

is introduced, then due to the fact that all three roots are real, c can be only

$$c = 0 \qquad \text{(A4-19a)}$$

or

$$c < 0 . \qquad \text{(A4-19b)}$$

The condition (A4-19a) means that at least two roots will be equal and (A4-19b) that the three real roots will be unequal. With the notations

$$A = \left[-\frac{b}{2} + \sqrt{c} \right]^{\frac{1}{3}}$$

and

$$B = \left[-\frac{b}{2} - \sqrt{c} \right]^{\frac{1}{3}} ,$$

the solutions of (A4-18c) are:

$$x_1 = A + B$$

$$x_2 = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3} \qquad \text{(A4-20)}$$

$$x_3 = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3} ,$$

as presented in [4], p. 93.

Whenever conditon (A4-19a) occurs, then $A = B$ and

$$x_1 = 2A$$
$$x_2 = -A$$
$$x_3 = -A .$$

However, due to round-off errors, this condition will not be fulfilled exactly and so only (A4-19b) will be of practical interest when using digital computers. First, a constant k will be introduced such that

$$k = (-\frac{a}{3})^3 . \tag{A4-21}$$

Due to (A4-19) and (A4-19b) it holds that

$$k > 0$$

and

$$|c| = k - (\frac{b}{2})^2 > 0. \tag{A4-22}$$

A and B thus become:

$$A = z_1^{\frac{1}{3}}$$

$$B = z_2^{\frac{1}{3}}$$

where

$$z_1 = -\frac{b}{2} + \sqrt{|c|}\, i$$

$$z_2 = -\frac{b}{2} - \sqrt{|c|}\, i \quad .$$

Since z_1 and z_2 are complex conjugates,

$$\rho = |z_1| = |z_2|$$

holds, where

$$\rho^2 = (\frac{b}{2})^2 - c = k$$

and thus

$$\rho = \sqrt{k} > 0 \; .$$

With the following notation

$$\cos\Phi = -\frac{b}{2\rho}$$

and

$$\sin\Phi = \frac{\sqrt{|c|}}{\rho} > 0 \quad ,$$

meaning that Φ was restricted to the interval $0 < \Phi < \pi$, z_1 and z_2 will become:

$$z_1 = \rho(\cos\Phi + i\sin\Phi)$$

$$z_2 = \rho(\cos\Phi - i\sin\Phi)$$

and A, B will then be

$$A = \rho^{\frac{1}{3}} \left(\cos\frac{\Phi}{3} + i\sin\frac{\Phi}{3}\right)$$

$$B = \rho^{\frac{1}{3}} \left(\cos\frac{\Phi}{3} - i\sin\frac{\Phi}{3}\right)$$

or

$$A = e + di$$

$$B = e - di$$

where

$$e = \rho^{\frac{1}{3}} \cos\frac{\Phi}{3} > 0 \qquad \text{(A4-24a)}$$

$$d = \rho^{\frac{1}{3}} \sin\frac{\Phi}{3} > 0 \; , \qquad \text{(A4-24b)}$$

both positive due to the restriction on Φ.

To compute Φ, the formula

$$\Phi = \text{arc tg}\left[\sqrt{c}\,/\left(-\frac{b}{2}\right)\right] \qquad \text{(A4-25)}$$

will be used. Finally,

$$A + B = 2e$$

and

$$A - B = 2di,$$

from which

$$\frac{A - B}{2}\sqrt{-3} = -d\sqrt{3}\,.$$

Consequently, from (A4-20):

$$\begin{aligned} x_1 &= 2e \\ x_2 &= -e - d\sqrt{3} \\ x_3 &= -e + d\sqrt{3}\,. \end{aligned} \qquad \text{(A4-26)}$$

All the expressions needed to compute x_1, x_2, x_3 from (A4-26) are given as follows:

e, d . . . . in (A4-24a), (A4-24b);
ρ . . . . in (A4-23)
Φ. . . . . in (A4-25)
k . . . . in (A4-21)
$|c|$ . . . . in (A4-22)
a, b . . . . in (A4-18d), (A4-18e)
p, q, r . . such as in (A4-18a′).

Finally, using (A4-18b) the eigenvalues of A are found:

$$\bar{\lambda}_i = x_i - \frac{p}{3}, \quad i = 1, 2, 3. \qquad \text{(A4-27)}$$

The eigenvalues for M can then be found using (A4-16a) together with (A4-27) as being

$$\lambda_i = k\,\bar{\lambda}_i, \qquad i = 1, 2, 3. \qquad \text{(A4-28)}$$

Next, eigenvectors of A, which are the same as those of M according to (A4-16b), will be found. They will be again denoted by the letter t_i, $i = 1, 2, 3$, and represented by a column vector with three entires. They will be computed by standard procedures outlined in [7], Chapter 9. Namely, for an eigenvalue $\bar{\lambda}_i$, a corresponding eigenvector subject to the condition of having unit norm, accomplished by scaling, will be computed from

$$(A - I\bar{\lambda}_i)\, t_i = 0. \tag{A4-29}$$

This relation represents three equations in three unknowns, the unknowns being the coordinates of t_i vector, which, however, has the rank two (since $|A - I\bar{\lambda}_i| = 0$); thus only two equations of (A4-29) will be used, one coordinate of t_i being chosen arbitrarily. In the present case the first two equations of (A4-29) will be used and the third coordinate of t_i as yet unscaled, set as $t'_{i3} = 1$. With a_{ij} as the ij th entry of A, D_i will be computed as

$$D_i = \begin{vmatrix} a_{11} - \bar{\lambda}_i & a_{12} \\ a_{21} & a_{22} - \bar{\lambda}_i \end{vmatrix},$$

and the first two coordinates of t as yet unscaled, computed from the two equations of (A4-29) according to the formulas

$$t'_{i1} = \frac{1}{D_i} \begin{vmatrix} -a_{13} & a_{12} \\ -a_{23} & a_{22} - \bar{\lambda}_i \end{vmatrix}$$

and

$$t'_{i2} = \frac{1}{D_i} \begin{vmatrix} a_{11} - \bar{\lambda}_i & -a_{13} \\ a_{21} & -a_{23} \end{vmatrix}.$$

The scale factor

$$s_i = \sqrt{(t'_{i1})^2 + (t'_{i2})^2 + 1}$$

will be used to compute

$$t_{i1} = \frac{1}{s_i}\, t'_{i1},$$

$$t_{i2} = \frac{1}{s_i}\, t'_{i2},$$

$$t_{i3} = \frac{1}{s_i},$$

which are the coordinates of an eigenvector t_i, associated with the eigenvalue $\overline{\lambda}_i$. All three eigenvectors arranged as columns in a (3×3) matrix constitute the orthogonal R matrix presented in (A4-4b).

In canonical coordinates, points at a distance a_i in both directions with respect to the origin on the i coordinate axis, which are the end points of the ith second order surface axis, will be called a pair of "main surface points." In (A4-17) and (A4-16a) it was shown that $a_i = 1/\sqrt{|\lambda_i|}$ where $\lambda_i = k\,\overline{\lambda}_i$. There are six such main surface points, a pair for each coordinate axis, which together with the center of the surface, coinciding with the origin of the canonical coordinate system, and the name of the second order surface give a good idea about it. Thus, in canonical coordinates, the center of the surface and the six main surface points have the coordinates respectively:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -a_1 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -a_2 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -a_3 \end{bmatrix}.$$

In the original coordinate system, the coordinates of these points will be computed according to (A4-8) as

$$x = x_0 + Rx',$$

plugging the above seven coordinate vectors for x' and taking

$$x_0 = -\frac{1}{2} A^{-1} a,$$

according to (A4-13).

In the forthcoming discussion the kind of second order surface will be determined, namely whether the investigated case is:

(a) an ellipsoid

(b) a hyperboloid of one sheet

(c) a hyperboloid of two sheets.

If the values of λ_i such as computed by (A4-28) are arranged in the descending order of magnitude, then for all three values being positive the kind of surface is (a), for $\lambda_3 < 0$ it is (b) and for $\lambda_2 < 0$, $\lambda_3 < 0$ it is (c), provided all the remaining eigenvalues λ are positive. If in the case (a) the letters a, b, c denote the semi-axes in order of magnitude, all real, then

$$a = 1/\sqrt{\lambda_3}$$

$$b = 1/\sqrt{\lambda_2}$$

$$c = 1/\sqrt{\lambda_1} \ .$$

When in the case (b) the letters a, b denote the real semi-axes in order of magnitude and c the imaginary semi-axis, then

$$a = 1/\sqrt{\lambda_2}$$

$$b = 1/\sqrt{\lambda_1}$$

$$c = 1/\sqrt{-\lambda_3} \ .$$

If finally in the case (c) a denotes the real semi-axis and b, c denote the imaginary semi-axes in order of magnitude, then

$$a = 1/\sqrt{\lambda_1}$$

$$b = 1/\sqrt{-\lambda_2}$$

$$c = 1/\sqrt{-\lambda_3} \ .$$

If the renumbering of a_i, λ_i and corresponding t_i is carried out such a way that for the above semi-axes "a" corresponds to the index $_1$, "b" to the index $_2$, "c" to the index $_3$, then also the six main surface points will have a similar plausible interpretation as the above semi-axes a, b, c. This appears to be convenient and helpful in visualizing the surface and it was included in the program dealing with second order surfaces.

Note: There are other ways to determine the kind of second order surface. First, due to the fact that $\overline{\lambda}_i$, $i = 1, 2, 3$ are the eigenvalues of A, it holds that

$$|A| = \bar{\lambda}_1 \times \bar{\lambda}_2 \times \bar{\lambda}_3,$$

which can also serve as a numerical check. Since in the present discussion $c = 0$ (due to the fact that the surface passes through the origin of the original coordinate system which is actually the "local coordinate system" in the main discussion), (A4-14a) holds, namely

$$k = \frac{1}{x_0^T A x_0}.$$

Under the earlier assumption of non-singular A, none of $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, is zero. If all three of $\bar{\lambda}_i$ are positive, A is a positive definite matrix and $k > 0$ for each non-zero x_0. Thus also M is a positive definite matrix, since by (A4-12a)

$$M = k A.$$

If all three of $\bar{\lambda}_i$ are negative, A is a negative definite matrix and $k < 0$ for each non-zero x_0. Consequently, M is again a positive definite matrix. This shows that not all three of λ_i can be negative and thus not all three axes a, b, c imaginary (this would represent an imaginary ellipsoid and it would not pass through the origin of the "local coordinate system", which is real). Thus in the two above cases the surface is (a), a (real) ellipsoid. If $\bar{\lambda}_i$, $i = 1, 2, 3$ have different signs, so do λ_i and both A and M are indefinite matrices. Suppose first that $\bar{\lambda}_1 > 0$, $\bar{\lambda}_2 > 0$, $\bar{\lambda}_3 < 0$; then $|A| < 0$.

If now

$$k > 0,$$

then

$$\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_3 < 0,$$

which represents (b), a hyperboloid of one sheet;

also

$$|A| k < 0.$$

On the other hand, if

$$k < 0,$$

then $\lambda_1 < 0,\ \lambda_2 < 0,\ \lambda_3 > 0$ (before renumbering) which represents (c), a hyperboloid of two sheets; also, $|A|\,k > 0$. Next, suppose that

$$\bar{\lambda}_1 > 0,\ \bar{\lambda}_2 < 0,\ \bar{\lambda}_3 < 0.$$

Similar results will hold, from which it can be concluded, together with the preceding part:

if $|A|\,k > 0$, it is case (a) or (c);

if $|A|\,k < 0$, it is case (b).

The above criteria have been used in the existing program for second order surfaces. Further check may be used, considering the determinant of the "augmented A matrix", denoted as Δ:

if $\Delta < 0$ it is case (a) or (c);

if $\Delta > 0$ it is case (b).

A4.4 Canonical Form of Second Order Curve.

This section will be similar to the section A4.3, but much simpler, because a plane (two-dimensional space) will now replace the three-dimensional space considered earlier. Here also A matrix and "augmented A matrix" are assumed to be non-singular, which is expressed by $|A| = J \neq 0$ and $\Delta \neq 0$, using notations of (1.3-3b) and (1.3-3c). Thus, only ellipse and hyperbola will be delt with. Again, in the first step, leading to a canonical form of a second degree curve expressed by (A4-6) in two-dimensional space, i.e., as

$$x^T A x + x^T a + c = 0, \qquad \text{(A4-30)}$$

the eigenvalues of A will be computed and then used in (A4-16a) and (A4-17). This leads to solving for roots of a quadratic equation similar to the type (A4-18) in three variables. Now the relation

$$|\bar{\lambda} I - A| = 0$$

leads to the equation

$$\bar{\lambda}^2 - (a_{11} + a_{22}) + |A| = 0$$

where

$$|A| = a_{11} a_{22} - a_{12}^2,$$

and where a_{ij} denotes the i j th element of the symmetric A matrix. The two roots of this equation are obtained as

$$\bar{\lambda}_1 = \frac{a_{11} + a_{22}}{2} + \sqrt{\left(\frac{a_{11} + a_{22}}{2}\right)^2 - |A|}$$

$$\bar{\lambda}_2 = \frac{a_{11} + a_{22}}{2} - \sqrt{\left(\frac{a_{11} + a_{22}}{2}\right)^2 - |A|}$$

and are again both real. Then the eigenvalues for M can be formed using (A4-16a) as

$$\lambda_1 = k\bar{\lambda}_1$$

$$\lambda_2 = k\bar{\lambda}_2 \ .$$

Next, eigenvectors of A, being the same as eigenvectors of M, will be found; they are denoted as t_1 and t_2 each being a column vector with two entries. For an eigenvector t_i, associated with the eigenvalue $\bar{\lambda}_i$, the matrix equation

$$(A - I\bar{\lambda}_i)\, t_i = 0 \qquad \text{(A4-31)}$$

will be solved. Here again, one of the coordinates of t_i will be chosen arbitrarily since the two equations of (A4-31) are not independent. The condition of unit norm for t_i will be then used for scaling. Thus the second coordinate, as yet unscaled, will be chosen as $t_{i2}' = 1$ and the first equation of (A4-31) will be used to compute t_{i1}', namely

$$t_{i1}' = -\frac{a_{12}}{a_{11} - \bar{\lambda}_i} \ .$$

Then

$$s_i = \sqrt{(t_{i1}')^2 + 1}$$

will be used to compute

$$t_{11} = \frac{1}{s_1} \; t'_{11}$$

and

$$t_{12} = \frac{1}{s_1} \quad .$$

These are the coordinates of an eigenvector t_1, which together with the other eigenvector will constitute the (2×2) orthogonal R matrix, presented in (A4-4b).

Analogous to section 4. 3, the "main curve points" will be obtained, from which the curve can be drawn. Thus, in canonical coordinates, the center of the curve and the four main curve points have the coordinates respectively:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} a_1 \\ 0 \end{bmatrix}, \begin{bmatrix} -a_1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ a_2 \end{bmatrix}, \begin{bmatrix} 0 \\ -a_2 \end{bmatrix}, \qquad \text{(A4-32)}$$

here again $a_i = 1/\sqrt{|\lambda_i|}$. In the original coordinate system, the coordinates of these points will be computed according to (A4. 8) as

$$x = x_0 + Rx',$$

plugging the vectors of (A4-32) for x' and taking

$$x_0 = -\frac{1}{2} A^{-1} a,$$

according to (A4-13).

If the values of λ_i, such as computed by (A4-28), are arranged in descending order of magnitude, the second order curve will be (a) an ellipse if both λ_1 and λ_2 are positive, and (b) a hyperbola if $\lambda_1 > 0$ and $\lambda_2 < 0$. If in the case (a) the letters a, b denote the semi-major and semi-minor axes respectively, then

$$a = 1/\sqrt{\lambda_2},$$

$$b = 1/\sqrt{\lambda_1}.$$

If in the case (b) the letters a, b denote the real and imaginary semi-axes respectively, then

$$a = 1/\sqrt{\lambda_1}\,,$$

$$b = 1/\sqrt{-\lambda_2}\,.$$

If the renumbering of a_i, λ_i and corresponding t_i is carried out so that the (first) axis "a" corresponds always to the index $_1$ and the (second) axis "b" to the index $_2$, then the ellipse or hyperbola can be drawn from the four main curve points without further investigations; such renumbering was included in the program dealing with second order curves.

Note: Similar to what was said for the three-dimensional cases, here

$$|A| = \bar{\lambda}_1 \times \bar{\lambda}_2$$

and

$$k = \frac{1}{x_0^T A x_0}\,,$$

since c = 0 (due to curve's passing through the origin). Again, if λ_1 and λ_2 are both positive or negative, M is a positive definite matrix and λ_1, λ_2 are both positive, characterizing an ellipse. Also, $|A| > 0$. Thus, not both λ_1 and λ_2 can be negative (and both semi-axes a, b imaginary). If $\bar{\lambda}_1 > 0$ and $\bar{\lambda}_2 < 0$, then both A and M are indefinite and consequently λ_1, λ_2 have different signs, characterizing a hyperbola. Also, $|A| < 0$. So it can be concluded:

if $|A| > 0$ the curve is an ellipse, and
if $|A| < 0$ the curve is a hyperbola.

APPENDIX 5

SOME SPECIAL CASES OF SINGULARITY B)

The singularity of the last three column block in $\approx\!A$ such as shown in Table (1.3-1) can be caused by some special relations which were not treated in section 1.32. Only a few will be demonstrated here as an illustration of special cases which may result from configurations among the ground stations not dealt with previously, such as two straight lines. For the sake of simplicity, the last three columns of $\overset{\approx}{A}$ matrix in Table (1.3-1) will be called here as "a", "b", "c" columns and each ground station's contribution will be limited to one row only, as all the other rows for that station are the same. These new columns are presented in Table (A5-1).

Table (A5-1)

Representation of Columns a, b, c Associated with $\overset{\approx}{A}$ Matrix

a	b	c
$y_4(x_4 - x_3)$	$y_4(y_4 - y_3)$	$(x_4 - y_4\frac{x_3}{y_3})\ (x_4 - x_2)$
$y_5(x_5 - x_3)$	$y_5(y_5 - y_3)$	$(x_5 - y_5\frac{x_3}{y_3})\ (x_5 - x_2)$
.	.	.
.	.	.
.	.	.
.	.	.
.	.	.
$y_i(x_i - x_3)$	$y_i(y_i - y_3)$	$(x_i - y_i\frac{x_3}{y_3})\ (x_i - x_2)$

The following simple cases will be illustrated:

1. Column "a" equal to zero.

2. Column "b" equal to zero.

3. Column "c" equal to zero.

4. Column "b" being a (non-zero) multiple of column "a".

As stated in (1.2-8) and assumed throughout, $x_2 \neq 0$ and $y_3 \neq 0$.

When Column "a" contains only zeros, it must hold that

$$y_i(x_i - x_3) = 0, \quad i = 4, 5, \ldots,$$

which is possible only if some of the "i" stations have $y = 0$ and all the rest of the "i" stations have $x = x_3$. This represents a case with all the ground stations lying in two straight lines, one passing through the stations 1 and 2, i.e., coincident with the x - axis (representing $y = 0$) and the other passing through the station 3 perpendicular to the first line (representing $x = x_3 = \text{const.}$). Thus, the case 1. represents two perpendicular lines.

When column "b" contains only zeros, the relation

$$y_i(y_i - y_3) = 0, \quad i = 4, 5, \ldots,$$

must hold, which means that a part of the "i" stations has $y = 0$ and the remaining part has $y = y_3$. The first line is again coincident with the x-axis while the second is parallel to it, passing through the station 3 (representing $y = y_3 = \text{const.}$). Thus, the case 2. represents two parallel lines.

When column "c" contains only zeros, it must hold that

$$(x_i - y_i\frac{x_3}{y_3})(x_i - x_2) = 0, \quad i = 4, 5, \ldots,$$

which occurs if a part of the "i" stations have

$$x = y\,\frac{x_3}{y_3},$$

representing a straight line through the station 1 (origin) and the station 3, while the remaining part has to satisfy $x = x_2$, representing a straight line through the station 2, perpendicular to the line connecting stations 1 and 2 (x - axis). Thus, the case 3. represents two intersecting lines, which have the property, that a connecting line between two of stations 1, 2, 3 (one lying on the first and the other on the second line) is perpendicular to one of those two lines.

Finally, if column "b" is a multiple of column "a", the following relation holds:

$$y_i (x_i - x_3) = c\, y_i (y_i - y_3), \quad i = 4, 5, \ldots,$$

where it is assumed that $c \neq 0$ (otherwise it would be case 1.). It is also assumed that for at least one station, say the station 4, $y_4 \neq 0$ and $y_4 \neq y_3$ (otherwise it would be again case 1.); the same station has to have $x_4 \neq x_3$, otherwise the above relation would not hold. Thus the station 4 has a general location and the above relation is valid for a part of stations having $y = 0$, representing a straight line through the stations 1 and 2, while for the other part, having $y \neq 0$, the following holds:

$$y - y_3 = \frac{y_4 - y_3}{x_4 - x_3} (x - x_3).$$

It represents a straight line through the stations 3 and 4. Consequently, case 4. represents two (intersecting) straight lines with no further condition.

APPENDIX 6

RITICAL SURFACE IN LOCAL COORDINATES AS OBTAINED NUMERICALLY BY FITTING SECOND ORDER SURFACE TO NINE POINTS

As in Appendix 4, an equation of the second degree surface may be ıt of (A4-6), which is

$$x^T A x + x^T a + c = 0. \tag{A6-1}$$

the case $c \neq 0$, this equation may be divided by c to obtain

$$x^T \overline{A} x + x^T \overline{a} + 1 = 0, \tag{A6-2a}$$

ıich contains nine unknowns: six in the symmetric (3×3) matrix $\overline{A}$ and ree in the vector $\overline{a}$. For any point on the surface, the equation (A6-2a) ın be written explicitly, with a_{ij} being the ij th element of $\overline{A}$, a_i being e th element of $\overline{a}$, and a variable point having the coordinates $= (x\ y\ z)^T$, as

$$\begin{aligned} x^2 a_{11} + y^2 a_{22} + z^2 a_{33} + 2xy a_{12} + 2xy a_{13} + 2yz a_{23} + \\ + a_1 x + a_2 y + a_3 z + 1 = 0. \end{aligned} \tag{A6-2b}$$

or the general cases such as ellipsoid, hyperboloid of one sheet and yperboloid of two sheets, considered at the beginning of Section A4.3 .e., with non-singular A and "augmented A" matrices expressed there y $|A| \neq 0$ and $\Delta \neq 0$), nine points of the surface, furnishing nine equations f the type (A6-2b), will be necessary in order to solve for the nine unnowns. In the matrix notation, it is thus obtained that

$$RX + S = 0, \tag{A6-3}$$

from which

$$X = -R^{-1}S$$ (A6-[illegible])

where, for $i = 1, 2, \ldots, 9$ it holds that

$$R = \begin{bmatrix} \vdots \\ x_i^2 \; y_i^2 \; z_i^2 \; 2x_i y_i \; 2x_i z_i \; 2y_i z_i \; x_i \; y_i \; z_i \\ \vdots \end{bmatrix},$$

$$X = [a_{11} \; a_{22} \; a_{33} \; a_{12} \; a_{13} \; a_{23} \; a_1 \; a_2 \; a_3]^T,$$

$$S = [1\;1\;1\;1\;1\;1\;1\;1\;1]^T.$$

Thus, having $\bar{A}$ and $\bar{a}$ by solving nine equations in nine unknowns in (A6-3a), the equation of the second degree surface determined by nine points is known in form (A6-2a). From there the canonical form and all the necessary information may be computed, using the approach of Section A4.3, where A, a, and c are substituted for by $\bar{A}$, $\bar{a}$ and 1, respectively.

Dealing with critical surfaces for range observational mode yields $c = 0$ in the equation (A6-1), due to the fact that the surface passes through the origin of the coordinate system of (A6-1), which is the "local coordinate system" in the main discussion; since any of the ground stations 1, 2, 3 (observing all the satellites) may determine origin of this coordinate system and any of the other two the direction of its x-axis, it may be assumed that the first entry in A matrix is non-zero.

Dividing (A6-1) in which $c = 0$ by this element yields an equation for the second degree surface as

$$x^T \bar{A} x + x^T \bar{a} = 0$$ (A6-[illegible])

with similar description of the elements in $\overline{A}$ and $\bar{a}$ as given for (A6-2a). However, only eight unknowns are now to be determined using eight further points (the origin having been used). This will lead to solving of eight equations in eight unknowns (a_{11} element is now a constant, $a_{11} = 1$). Analogous to the equation (A6-2b), the equation (A6-4a) can be now written for a variable point as

$$x^2 + y^2 a_{22} + z^2 a_{33} + 2xy\,a_{12} + 2xz\,a_{13} + 2yz\,a_{23} + a_1 x + a_2 y + a_3 z = 0 . \tag{A6-4b}$$

Similarly to what was said following the equation (A6-2b), it is assumed again that $|A| \neq 0$ and $\Delta \neq 0$. Using eight further surface points in (A6-4b), it can be written in a matrix form similar to (A6-3):

$$RX + S = 0 \tag{A6-5}$$

where, for i = 2, 3, ..., 9 it holds that

$$R = \begin{bmatrix} & & & & \vdots & & & \\ y_i^2 & z_i^2 & 2x_iy_i & 2x_iz_i & 2y_iz_i & x_i & y_i & z_i \\ & & & & \vdots & & & \end{bmatrix} ,$$

$$X = [a_{22} \; a_{33} \; a_{12} \; a_{13} \; a_{23} \; a_1 \; a_2 \; a_3]^T ,$$

and

$$S = [\, . \; . \; . \; . \; . \; . \; . \; x_i^2 \; . \; . \; . \; . \; . \; . \; . \,]^T .$$

With the conventional numbering of points such a way that i = 1 denoted the station 1 at the origin (of the local coordinate system), i = 2 the station 2 on the x-axis and i = 3 the station 3 in the xy plane, the equations (A6-5) will be simplified due to $y_2 = z_2 = 0$ and $z_3 = 0$; this gives for the first equation of (A6-5):

$$x_2 a_1 + x_2^2 = 0 .$$

Since $x_2 \neq 0$, it follows that

$$a_1 = -x_2 .$$

Plugging this into (A6-5) the following is obtained:

$$\widetilde{R}\widetilde{X} + \widetilde{S} = 0 ,$$

from which

$$\widetilde{X} = -\widetilde{R}^{-1}\widetilde{S}.$$

With i = 4, 5, . . . , 9, it is seen that

$$\widetilde{R} = \begin{bmatrix} y_3^2 & 0 & 2x_3 y_3 & 0 & 0 & y_3 & 0 \\ \hline & & & \vdots & & & \\ y_i^2 & z_i^2 & 2x_i y_i & 2x_i z_i & 2y_i z_i & y_i & z_i \\ & & & \vdots & & & \end{bmatrix},$$

$$X = [\, a_{22} \;\; a_{33} \;\; a_{12} \;\; a_{13} \;\; a_{23} \;\; a_2 \;\; a_3 \,]^T,$$

and

$$\widetilde{S} = [\, x_3(x_2 - x_3) \;\big|\; \ldots \;\; x_i(x_2 - x_i) \;\; \ldots \,]^T .$$

This amounts to solving seven equations in seven unknowns, which, in addition to $a_{11} = 1$ and $a_1 = -x_2$, determines the matrix $\overline{A}$ and the vector $\bar{a}$. These can then be used to compute the canonical form of the second order surface using the approach of section A4.3, where A, a, and c are substituted for by $\bar{A}$, $\bar{a}$ and 0, respectively.

APPENDIX 7

APPROXIMATE DISTANCE OF A POINT FROM A GIVEN SECOND ORDER SURFACE

Dealing with critical loci for the range observational mode brings about problems connected with second order surfaces. Specifically, if all points of a network lie exactly on certain second order surfaces, the unique solution for unknown parameters is impossible and the problem is said to be singular. Therefore, in order to determine whether singularity or near-singularity could be caused by certain distribution of points (ground stations and satellites), it will be necessary to determine whether all the points in consideration lie on or near their critical second order surfaces. With some points far from these loci, the above mentioned singularity does not occur and it is not of particular interest to know exactly how far these points are from any surface. On the other hand it is important to detect cases when all the points are exactly on or very near their critical surfaces. In these cases it is desirable to have a fairly good idea about the distances of the points from such second order surfaces.

A7.1 General Approach

A second order surface can be expressed as

$$(x - x_o)^T M(x - x_o) = 1, \qquad \text{(A7-1)}$$

as seen in Appendix 4, using equations (A4-9) and (A4-10a). When the surface is given explicitly, the symmetric matrix M, as well as the vector x_o are given; x denotes the coordinate vector of a variable point lying on the surface. As a function of x, the equation (A7-1) can be written as

$$f(x) = \text{constant},$$

for which ∇f, the gradient, represents a vector perpendicular to the surface at point x. It is computed as

$$\nabla f(x) = \partial f(x) / \partial x = 2M(x - x_o).$$

With n denoting the unit vector perpendicular to the surface (A7-1) at the point represented by x, which will be hereafter called point x, it follows:

$$n = M(x - x_o)/s \tag{A7-2a}$$

where

$$s = |M(x - x_o)|. \tag{A7-2b}$$

The following relation between x and x_1 holds (with n properly oriented):

$$x_1 = x + dn \tag{A7-3}$$

where x_1 represents a known point, while x represents a point on the surface, which is as yet unknown; d, also unknown, is then the desired distance, connecting x and x_1, perpendicular to the surface. If is seen that (A7-3) represents three equations in four unknowns, namely the three coordinates of x and the distance d. The fourth equation is then represented by (A7-1). Using (A7-2a) and (A7-2b), the system of four equations in four unknowns may be now written as

$$x_1 = x + M(\lambda x - \lambda x_o) / |M(x - x_o)| \tag{A7-4a}$$

together with

$$(x - x_o)^T M(x - x_o) = 1. \tag{A7-4b}$$

The three equations (A7-4a) are of second order in the unknowns d and x, since in this approach n was also a function of the unknown vector x. Thus, a simple substitution for x from (A7-4a) into (A7-4b) is not possible and a different approach would have to take place in order to solve the system of (A7-4a) and (A7-4b), quadratic in the unknowns. One such approach will be described in the following section, taking advantage of the specification that the distances are not needed to a great accuracy, namely for points far from the second order surface, for which the distances are desired only approximately or not at all.

A7.2 Specific Approach

The starting equation in this approach will be similar to (A7-1), namely

$$(x - x_o)^T M (x - x_o) = k, \tag{A7-5}$$

representing a family of the second order surfaces, which are said to be similar. The surface described by (A7-1) is one of these surfaces, when $k = 1$. It will be called the critical surface.

As in the previous section, the distance d will pertain to a known point represented by x_1, or point x_1. However, a new concept permitting great simplifications will be introduced: the distance d will be measured perpendicular to that surface of the family (A7-5), which passes through the point x_1. Thus the distance will be measured perpendicularly to the critical surface only when the two surfaces are infinitesimally close and therefore only in these cases will the distance be exact. With the point x_1 moving further from the critical surface the separation between the two surfaces will be greater and the angular difference between the normals to both surfaces will also grow, depending further on the location of x_1; thus the computed distance will be decreasing in precision and could eventually become completely false, or the real solution may not exist at all. However, even in these cases the purpose of this approach would be fulfilled, namely, the results would indicate that the point x_1 is not on or very close to the critical surface, which is the desired information. On the other hand, when x_1 is on or very near the critical surface, not only would this be detected, but also a fairly precise nominal value of the distance in question from the surface would be obtained. This was also supported by the computer runs with generated points x_1. For the above reasons the present method seems to be suitable for detecting of singularity or near-singularity connected with the second order surfaces and it gives a good idea which points and to what extent (depending on their closeness to the surface) could be responsible for it.

For the surface passing through x_1 which does not lie on the critical surface it holds that $k \neq 1$; it will be called "k-surface". The constant k for this particular surface will be computed as

$$k = (x_1 - x_0)^T M (x_1 - x_0), \qquad (A7\text{-}6)$$

since it is a surface from the family of (A7-5) and x_1 lies on this surface. M and x_0 are again considered to be known as in section A7.2; this makes the determination of k possible. For the unit normal to the k-surface the same approach and the same formulas as (A7-2a) and (A7-2b) are used, except that x_1 will replace x, since that is the point at which the unit normal is desired (the difference between the critical surface and the k-surface rests in the right hand-side of (A7-5), the constant, which does not alter the formula for the gradient). Denoting it again as n, it holds that

$$n = M (x_1 - x_0)/s \qquad (A7\text{-}7a)$$

where

$$s = | M(x_1 - x_0) | . \qquad (A7\text{-}7b)$$

The difference between this and the previous section is that n is now a known vector. Similar to (A7-3) or (A7-4a) it holds that

$$x_1 = x + d\,n$$

where x is again an unknown point on the critical surface and d the desired distance, n being properly oriented (having now the opposite sense with respect to (A7-3)). The main simplification consists now is expressing x as

$$x = x_1 - d\,n \qquad (A7\text{-}8a)$$

and substituting it into (A7-4b), i.e., in

$$(x - x_0)^T M (x - x_0) = 1, \qquad (A7\text{-}8b)$$

which holds for point x lying on the critical surface. With this substitution (A7-8b) will give:

$$(x_1 - x_o)^T M (x_1 - x_o) - 2dn^T M (x_1 - x_o) + d^2 n^T Mn = 1, \quad \text{(A7-8c)}$$

which is a quadratic equation in one unknown, d. The first term on the left side is equal to k, according to (A7-6). Thus (A7-8c) yields

$$d^2 + 2pd + q = 0$$

where

$$p = -n^T M (x_1 - x_o)/n^T Mn \quad \text{(A7-9a)}$$

and

$$q = (k - 1)/n^T Mn. \quad \text{(A7-9b)}$$

The two solutions for d are given by

$$d_{1,2} = -p \pm \sqrt{p^2 - q} \ , \quad \text{(A7-9c)}$$

indicating, that in general two intersections of the line passing through point x_1 perpendicular to the k-surface with the critical surface will exist. If the signs of d_1 and d_2 are different, the intersections will take place on different sides of the line with respect to x_1. The absolute value of d will indicate the distance (in chosen units) between the points x_1 and x, whose relation to the distance of x_1 from the critical surface has been discussed. The shortest of the two computed distances will be associated with the closest intersection, which represents the desired information. No real solution for d will indicate that the above line does not intersect the critical surface, thus in general indicating that the two surfaces are "far apart" with no further specification, which, however, is in itself also valuable information.

With d_1 and d_2 known, some numerical checks may be performed and additional information pertaining to the critical surface extracted; namely, the positions of two additional points on the critical surface for each point x_1

can be obtained. Position of the two additional points on the critical surface is computed by substituting both values of d into (A7-8a). As a numerical check, the equation (A7-8b) must hold for each of such additional points. Furthermore, it is possible to compute the unit normal to the critical surface, denoted as v, at any such additional point x (which is now known) by (A7-2a) and (A7-2b), as

$$v = M(x - x_o)/|M(x - x_o)|. \qquad \text{(A7-10)}$$

With φ denoting the angle between n and v in the interval $< 0, \pi >$, it holds that

$$\varphi = \text{arc tg}(\sin\varphi/\cos\varphi) \qquad \text{(A7-11a)}$$

where

$$\sin\varphi = \sqrt{1 - \cos^2\varphi} \qquad \text{(A7-11b)}$$

and

$$\cos\varphi = n^T v. \qquad \text{(A7-11c)}$$

With φ approaching zero the computed distance will approach the distance of point x_1 from the critical surface (measured perpendicularly to the critical surface).

A7.3 Practical Computations with Critical Surface in Canonical Form

Substantial simplifications in computations are made when the family of second order surfaces (A7-5) is given in canonical form. This procedure will be used in practice, after a particular second order surface has been obtained in its canonical form according to section A4.3; $x_1 = [x_1, y_1, z_1]^T$ is assumed to have been transformed from the original local coordinate system to the canonical coordinate system using (A4-8a). Accordingly,

$$M = \text{diag.}(\lambda_1, \lambda_2, \lambda_3),$$

$$x_o = [0 \ 0 \ 0]^T;$$

(A7-5) reads then as

$$x^2\lambda_1 + y^2\lambda_2 + z^2\lambda_3 = k \quad \text{(A7-12)}$$

with

$$x = [x \ y \ z]^T.$$

For computation of k, (A7-6) now yields

$$k = x_1^2\lambda_1 + y_1^2\lambda_2 + z_1^2\lambda_3; \quad \text{(A7-13)}$$

n, according to (A7-7a) and (A7-7b), is given as

$$n = [x_1\lambda_1 \ \ y_1\lambda_2 \ \ z_1\lambda_3]^T \times \frac{1}{s} \quad \text{(A7-14a)}$$

where

$$= \sqrt{s_1 + s_2 + s_3} \quad \text{(A7-14b)}$$

with

$$s_1 = x_1^2\lambda_1^2, \quad s_2 = y_1^2\lambda_2^2, \text{ and } s_3 = z_1^2\lambda_3^2; \quad \text{(A7-14c)}$$

this can be also seen directly from (A7-12).

Due to

$$n^T M = [x_1\lambda_1^2 \ \ y_1\lambda_2^2 \ \ z_1\lambda_3^2] \times \frac{1}{s}$$

it follows that

$$n^T M(x_1 - x_o) = n^T M x_1 = s \quad \text{(A7-15a)}$$

and

$$n^T M n = (\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3)/s^2. \quad \text{(A7-15b)}$$

For the sake of clearness the two solutions for d will be denoted as d_a and d_b rather than d_1 and d_2. The two additional points on the critical surface will be denoted as $x_a = [x_a \ y_a \ z_a]^T$ and $x_b = [x_b \ y_b \ z_b]^T$, associated with d_a and d_b respectively.

Thus, according to (A7-9a) through (A7-9c) together with (A7-15a) and (A7-15b), it is obtained:

$$p = -s^3/(\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3),$$

$$q = s^2(k-1)/(\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3),$$

$$d_a = -p + \sqrt{p^2 - q},$$

and

$$d_b = -p - \sqrt{p^2 - q}.$$

In these expressions, k was given by (A7-13), s by (A7-14b), and s_1, s_2, and s_3 by (A7-14c). Thus d_a and d_b, representing the main outcome of these computations, have been obtained.

Next, the two additional points on the critical surface will be computed, according to (A7-8a), as

$$x_a = x_1 - d_a n$$

and

$$x_b = x_1 - d_b n;$$

this, together with (A7-14a) yields

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \begin{bmatrix} x_1(1 + \lambda_1 d_a/s) \\ y_1(1 + \lambda_2 d_a/s) \\ z_1(1 + \lambda_3 d_a/s) \end{bmatrix}$$

and

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} x_1(1 + \lambda_1 d_b/s) \\ y_1(1 + \lambda_2 d_b/s) \\ z_1(1 + \lambda_3 d_b/s) \end{bmatrix}.$$

For any such points lying on the critical surface the equation (A7-8b), here as

$$x_i^2 \lambda_1 + y_i^2 \lambda_2 + z_i^2 \lambda_3 = 1,$$

serving as a numerical check of computations, must hold; here x_a, y_a, z_a, x_b, y_b, z_b, or the coordinates of any other point lying on the critical surface can be substituted for x_i, y_i, z_i. If the points x_a and x_b are of further interest, they can be transformed back to the original (local) coordinate system, according to (A4-8).

Finally, the unit normals to the critical surface at x_a, x_b, or any other point lying on it, can be computed according to (A7-10), with the coordinates of any of these points substituted for x_i, y_i, z_i:

$$v_i = [x_i\lambda_1 \; y_i\lambda_2 \; z_i\lambda_3] \frac{1}{s_i}$$

where

$$s_i = \sqrt{x_i^2\lambda_1^2 + y_i^2\lambda_2^2 + z_i^2\lambda_3^2}.$$

For computation of the angles between the unit normal to the k-surface at x_1 and the unit normal to the critical surface at any point, namely at x_a and x_b, may be computed according to (A7-11a) through (A7-11c):

$$\cos\varphi_i = n^T v_i ,$$

$$\sin\varphi_i = \sqrt{1 - \cos^2\varphi_i}$$

and

$$\varphi_i = \text{arc tg}\,(\sin\varphi_i / \cos\varphi_i),$$

which gives

$$\varphi_i \text{ in the interval } < 0, \pi >.$$

APPENDIX 8

CRITICAL SURFACE FOR FOUR GROUND STATIONS

A8.1 Critical Surface for Four Ground Stations in Local Coordinates Using Taylor Expansion of Determinant, as Function of Ground Station Number Four.

When only four ground stations observe ranges to satellite points, the $\widetilde{A}$ matrix of Table (1.2-1) will contain only o first rows, corresponding to "for station 4" of that table and six non-zero columns corresponding to these o rows. Thus, these columns will have heading ∂x_4, ∂y_4, ∂z_4, $\frac{\partial x_3}{y_3}$, $\frac{\partial y_3}{y_3}$, and $\frac{\partial x_2}{x_2}$, respectively. As in section 1.2, the adjustment problem wil be singular if

$$\operatorname{rank} \widetilde{A} < 6. \tag{A8-1}$$

From $(o \times 6)$ $\widetilde{A}$ matrix, (6×6) matrices A can be formed using all combinations of six rows in $\widetilde{A}$. If (A8-1) holds, then

$$|A| = 0 \tag{A8-2}$$

will be true for all matrices A. Conversely, it (A8-2) holds for any A, then also (A8-1) holds. Consequently, (A8-1) and (A8-2) for all matrices A are equivalent statements. In the forthcoming investigation only one A matrix will be considered, its rows corresponding to satellite points 1, 2, ..., 6. Then the same conclusions will be drawn for all possible combinations of six satellites.

For j = 1, 2, ..., 6, A matrix can be read from Table (1.2-1) as

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_j & b_j & c_j & d_j & e_j & f_j \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{A8-3}$$

where

$$a_j = Z_j (X_j - x_4), \tag{A8-3a}$$

$$b_j = Z_j (Y_j - y_4), \tag{A8-3b}$$

$$c_j = Z_j (Z_j - z_4), \tag{A8-3c}$$

$$d_j = (z_4 Y_j - y_4 Z_j)(X_j - x_3), \tag{A8-3d}$$

$$e_j = (z_4 Y_j - y_4 Z_j)(Y_j - y_3),$$

and

$$f_j = \left[z_4 (X_j - Y_j \frac{x_3}{y_3}) - Z_j (x_4 - y_4 \frac{x_3}{y_3}) \right] (X_j - x_2). \qquad \text{(A8-3e)} \quad \text{(A8-3f)}$$

It will be assumed throughout that

$$z_4 \neq 0, \qquad \text{(A8-4)}$$

since for $z_4 = 0$ the four ground stations would lie in a plane. Configurations with all stations in a plane were investigated in section 1.3. From what was stated there it holds for four ground stations that

$$|A| = 0 \text{ whenever } z_4 = 0. \qquad \text{(A8-5)}$$

As seen from (A8-3) - (A8-3f), $|A|$ can be expressed as a function of x_4, y_4, z_4 coordinates; the relation (A8-2) can be thus viewed as an equation of a surface in x_4, y_4, z_4, the order of which will be now examined. Using Laplace expansion for the last (or first) three columns, $|A|$ is given as a sum of signed products of (3×3) minors and their (3×3) algebraic complements in all combinations, such as described in [6], p. 33. One such product will be sufficient in determining the order of the surface represented by $|A| = 0$. Let it be denoted by $E \equiv PT$ where

$$P = \begin{vmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{vmatrix} \quad \text{and} \quad T = \begin{vmatrix} d_e & e_e & f_e \\ d_m & e_m & f_m \\ d_n & e_n & f_n \end{vmatrix} .$$

Considering (A8-3a) - (A8-3c), it is seen that Z_i, Z_j, Z_k can be factored out of the first determinant and P can be written as

$$P = Z_i Z_j Z_k \begin{vmatrix} (X_i - x_4) & (Y_i - y_4) & (Z_i - z_4) \\ (X_j - x_4) & (Y_j - y_4) & (Z_j - z_4) \\ (X_k - x_4) & (Y_k - y_4) & (Z_k - z_4) \end{vmatrix} ;$$

upon subtracting the first row in this determinant from the second and third rows, the only row which will still contain any of x_4, y_4, z_4 will be the first row and thus P will be of order at most one in x_4, y_4, z_4. No such simplifications are possible for T, each row of the corresponding determinant being of order exactly one in x_4, y_4, z_4; consequently, T is of order exactly three in x_4, y_4, z_4. It can be then concluded that $|A|$ is of order at most four and at least three in x_4, y_4, z_4, which means that (A8-2) represents a fourth order surface in x_4, y_4, z_4. Furthermore, this surface passes through satellite points 1, 2, ... 6 associated with A, and ground stations 1, 2, 3. This is easy to see, since whenever X_j, Y_j, Z_j are plugged for x_4, y_4, z_4, then

$$a_j = b_j = c_j = d_j = e_j = f_j = 0 \qquad \text{(A8-5a)}$$

as seen from (A8-3a) - (A8-3f), and when any ground station's coordinates are plugged for x_4, y_4, z_4, then $|A| = 0$ holds as well; this is due to the fact that z_4 becomes zero by this substitution and then (A8-5) is applied.

Next, an explicit form of $|A|$ will be obtained by expanding it in Taylor series as a function of z_4, at the point $z_4 = 0$. Since $|A|$ represents a polynomial in z_4 of order at most four, this expansion will have the form:

$$|A| = |A|_{z_4=0} + \left[\frac{d|A|}{dz_4}\right]_{z_4=0} \times z_4 + \frac{1}{2!}\left[\frac{d^2|A|}{dz_4^2}\right]_{z_4=0} \times z_4^2 + \frac{1}{3!}\left[\frac{d^3|A|}{dz_4^2}\right]_{z_4=0} \times z_4^3 +$$

$$+ \frac{1}{4!}\left[\frac{d^4|A|}{dz_4^4}\right]_{z_4=0} \times z_4^4. \qquad \text{(A8-6)}$$

The derivative of a determinant with respect to z_4 will be taken as a sum of six determinants (when dealing with (6×6) matrices), by replacing in all possible ways the elements of one column of this determinant by their derivatives with respect to z_4, according to [6], p. 34. As stated in (A8-5),

$$|A|_{z_4=0} = 0. \qquad \text{(A8-7)}$$

With the notations $\frac{da_j}{dz_4} = a_j'$ etc., it is obtained from (A8-3a) - (A8-3f):

$$a_j' = b_j' = 0,$$

$$c_j' = -Z_j,$$

$$d_j' = Y_j(X_j - x_3),$$

$$e_j' = Y_j(Y_j - y_3),$$

and

$$f_j' = (X_j - Y_j \frac{x_3}{y_3})(X_j - x_2).$$

Thus

$$\frac{d|A|}{dz_4} = a + b + c + d \tag{A8-8}$$

and

$$\left[\frac{d\,|A|}{dz_4}\right]_{z_4=o} = \bar{a} + \bar{b} + \bar{c} + \bar{d}$$

where

$$\bar{a} = [a]_{z_4=o},$$

$$\bar{b} = [b]_{z_4=o},$$

$$\bar{c} = [c]_{z_4=o},$$

and

$$\bar{d} = [d]_{z_4=o},$$

with

$$a = |a_j \ b_j \ c_j' \ d_j \ e_j \ f_j|,$$

$$b = |a_j \ b_j \ c_j \ d_j' \ e_j \ f_j|,$$

$$c = |a_j \ b_j \ c_j \ d_j \ e_j' \ f_j|,$$

$$d = |a_j \ b_j \ c_j \ d_j \ e_j \ f_j'|.$$

(The dots inside the determinants have been omitted.) Upon plugging $z_4 = 0$ for a_j, b_j, c_j', d_j, e_j, f_j, it follows that

$$\bar{a} = |Z_j(X_j - x_4),\ Z_j(Y_j - y_4),\ -Z_j,\ -y_4Z_j(X_j - x_3),\ -y_4Z_j(Y_j - y_3),\ -Z_j(x_4 - y_4\frac{x_3}{y_3})(X_j - x_2)$$

when Z_j in each row ($j = 1, 2, \ldots, 6$) is factored out together with $-y_4^2(x_4 - y_4\frac{x_3}{y_3})$ from columns 4, 5, and 6, then the above expression becomes

$$\bar{a} = -Z_1Z_2\ldots Z_6\, y_4^2(x_4 - y_4\frac{x_3}{y_3})\ |X_j - x_4,\ Y_j - y_4,\ -1,\ X_j - x_3,\ Y_j - y_3,\ X_j - x_2| = 0,$$

since there are several ways in which to bring other columns besides the third column to be constant (for instance, subtracting column one from column four leaves each element of column four to be $(x_4 - x_3)$; multiplying column three by $(x_4 - x_3)$ and adding it to column four brings each element of column four to zero, from which the asserted relation follows). Similarly, upon factoring out Z_J for each row and $y_4(x_4 - y_4 \frac{x_3}{y_3})$ for columns five and six, it is obtained that

$$\bar{b} = Z_1 Z_2 \ldots Z_6 y_4(x_4 - y_4 \frac{x_3}{y_3}) \mid X_J - x_4,\ Y_J - y_4,\ Z_J,\ \frac{Y_J}{Z_J}(X_J - x_3),\ Y_J - y_3,\ X_J - x_2 \mid = 0,$$

since subtracting column two form column five and subtracting column one from column six brings columns five and six to be constant columns. It also follows that

$$\bar{c} = Z_J Z_2 \ldots Z_6 y_4(x_4 - y_4 \frac{x_3}{y_3}) \mid X_J - x_4,\ Y_J - y_4,\ Z_J,\ X_J - x_3,\ \frac{Y_J}{Z_J}(Y_J - y_3),\ X_J - x_2 \mid = 0,$$

upon performing similar equivalence operations for columns one and four and one and six. Finally,

$$\bar{d} = Z_1 Z_2 \ldots Z_6 y_4^2 \mid X_J - x_4,\ Y_J - y_4,\ Z_J,\ X_J - x_3,\ Y_J - y_3,\ (\frac{X_J}{Z_J} - \frac{Y_J}{Z_J}\frac{x_3}{y_3})(X_J - x_2) \mid = 0,$$

upon using columns one and four and two and five for equivalence operations. Thus $\bar{a} = \bar{b} = \bar{c} = \bar{d} = 0$, which yields

$$\left[\frac{d\,|A|}{dz_4}\right]_{z_4=0} = 0. \qquad \text{(A8-9)}$$

Due to

$$a_J' = b_J' = 0 \qquad \text{(A8-10a)}$$

and

$$c_J'' = d_J'' = e_J'' = f_J'' = 0, \qquad \text{(A8-10b)}$$

it is obtained that

$$\frac{da}{dz_4} = a_1 + a_2 + a_3 \qquad \text{(A8-11a)}$$

where

$$a_1 = |a_j\ b_j\ c_j'\ d_j'\ e_j\ f_j|$$

$$a_2 = |a_j\ b_j\ c_j'\ d_j\ e_j'\ f_j|$$

$$a_3 = |a_j\ b_j\ c_j'\ d_j\ e_j\ f_j'|\ .$$

Further,

$$\frac{db}{dz_4} = b_1 + b_2 + b_3 \tag{A8-11b}$$

where

$$b_1 = |a_j\ b_j\ c_j'\ d_j'\ e_j\ f_j| = a_1,$$

$$b_2 = |a_j\ b_j\ c_j\ d_j'\ e_j'\ f_j|,$$

and

$$b_3 = |a_j\ b_j\ c_j\ d_j'\ e_j\ f_j'|;$$

also,

$$\frac{dc}{dz_4} = c_1 + c_2 + c_3 \tag{A8-11c}$$

where

$$c_1 = |a_j\ b_j\ c_j'\ d_j\ e_j'\ f_j| = a_2,$$

$$c_2 = |a_j\ b_j\ c_j\ d_j'\ e_j'\ f_j| = b_2,$$

$$c_3 = |a_j\ b_j\ c_j\ d_j\ e_j'\ f_j'|;$$

finally,

$$\frac{dd}{dz_4} = d_1 + d_2 + d_3 \tag{A8-11d}$$

where

$$d_1 = |a_j\ b_j\ c_j'\ d_j\ e_j\ f_j'| = a_3,$$

$$d_2 = |a_j\ b_j\ c_j\ d_j'\ e_j\ f_j'| = b_3,$$

and

$$d_3 = |a_j\ b_j\ c_j\ d_j\ e_j'\ f_j'| = c_3.$$

From (A8-11a) - (A8-11d) together with (A8-8), it follows that

$$\frac{d^2|A|}{dz_4^2} = 2(a_1 + a_2 + a_3 + b_2 + b_3 + c_3); \qquad \text{(A8-12)}$$

thus

$$\frac{1}{2!}\left[\frac{d^2|A|}{dz^2}\right]_{z_4=0} = \bar{a}_1 + \bar{a}_2 + \bar{a}_3 + \bar{b}_2 + \bar{b}_3 + \bar{c}_3, \qquad \text{(A8-12a)}$$

where the bar will indicate from now on that such a value was obtained by plugging $z_4 = 0$ in its unbarred counterpart. Proceeding the same way which led to the obtaining of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, it is now found that

$$\bar{a}_1 = Z_1Z_2\ldots Z_6y_4(x_4 - y_4\frac{x_3}{y_3})\,|\,X_j - x_4,\ Y_j - y_4,\ -1,\ \frac{Y_j}{Z_j}(X_j - x_3),\ Y_j - y_3,\ X_j - x_2\,| = 0,$$

$$\bar{a}_2 = Z_1Z_2\ldots Z_6y_4(x_4 - y_4\frac{x_3}{y_3})\,|\,X_j - x_4,\ Y_j - y_4,\ -1,\ X_j - x_3,\ \frac{Y_j}{Z_j}(Y_j - y_3),\ X_j - x_2\,| = 0,$$

$$\bar{a}_3 = Z_1Z_2\ldots Z_6y_4^2\,|\,X_j - x_4,\ Y_j - y_4,\ -1,\ X_j - x_3,\ Y_j - y_3,\ (\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3})(X_j - x_2)\,| = 0,$$

$$\bar{b}_2 = -Z_1Z_2\ldots Z_6(x_4 - y_4\frac{x_3}{y_3})\,|\,X_j - x_4,\ Y_j - y_4,\ Z_j,\ \frac{Y_j}{Z_j}(X_j - x_3),\ \frac{Y_j}{Z_j}(Y_j - y_3),\ (X_j - x_2)\,| =$$

$$= Z_1Z_2\ldots Z_6(x_4 - y_4\frac{x_3}{y_3})(x_4 - x_2)\,|\,1,\ Y_j - y_4,\ Z_j,\ \frac{Y_j}{Z_j}(X_j - x_3),\ \frac{Y_j}{Z_j}(Y_j - y_3),\ (X_j - x_2)\,|,$$

(A8-13a)

$$\bar{b}_3 = -Z_1Z_2\ldots Z_6y_4\,|\,X_j - x_4,\ Y_j - y_4,\ Z_j,\ \frac{Y_j}{Z_j}(X_j - x_3),\ Y_j - y_3,\ (\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3})(X_j - x_2)\,| =$$

$$= Z_1Z_2\ldots Z_6y_4(y_4 - y_3)\,|\,X_j - x_4,\ 1,\ Z_j,\ \frac{Y_j}{Z_j}(X_j - x_3),\ Y_j - y_3,\ (\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3})(X_j - x_2)\,|,$$

(A8-13b)

and

$$\bar{c}_3 = -Z_1Z_2\ldots Z_6y_4\,|\,X_j - x_4,\ Y_j - y_4,\ Z_j,\ X_j - x_3,\ \frac{Y_j}{Z_j}(Y_j - y_3),\ (\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3})(X_j - x_2)\,| =$$

$$= Z_1Z_2\ldots Z_6y_4(x_4 - x_3)\,|\,1,\ Y_j - y_4,\ Z_j,\ X_j - x_3,\ \frac{Y_j}{Z_j}(Y_j - y_3),\ (\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3})(X_j - x_2)\,|.$$

(A8-13c)

Thus, the using of (A8-12a) yields:

$$\frac{1}{2!}\left[\frac{d^2|A|}{dz_4^2}\right]_{z_4=0} = \bar{b}_2 + \bar{b}_3 + \bar{c}_3. \tag{A8-14}$$

Due to (A8-10a) and (A8-10b), it follows that

$$\frac{da_1}{dz_4} = a_{11} + a_{12} \tag{A8-14a}$$

where

$$a_{11} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j' \;\; e_j' \;\; f_j \,|$$

and

$$a_{12} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j' \;\; e_j \;\; f_j' \,|;$$

$$\frac{da_2}{dz_4} = a_{21} + a_{22} \tag{A8-14b}$$

where

$$a_{21} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j' \;\; e_j' \;\; f_j \,| = a_{11}$$

and

$$a_{22} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j \;\; e_j' \;\; f_j' \,|;$$

$$\frac{da_3}{dz_4} = a_{31} + a_{32} \tag{A8-14c}$$

where

$$a_{31} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j' \;\; e_j \;\; f_j' \,| = a_{12}$$

and

$$a_{32} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j \;\; e_j' \;\; f_j' \,| = a_{22};$$

$$\frac{db_2}{dz_4} = b_{21} + b_{22} \tag{A8-14d}$$

where

$$b_{21} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j' \;\; e_j' \;\; f_j \,| = a_{11}$$

and

$$b_{22} = |\, a_j \;\; b_j \;\; c_j \;\; d_j' \;\; e_j' \;\; f_j' \,| \, ;$$

$$\frac{db_3}{dz_4} = b_{31} + b_{32} \tag{A8-14e}$$

here

$$b_{31} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j' \;\; e_j \;\; f_j' \,| = a_{12}$$

and

$$b_{32} = |\, a_j \;\; b_j \;\; c_j \;\; d_j' \;\; e_j' \;\; f_j' \,| = b_{22} \, ;$$

finally,

$$\frac{dc_3}{dz_4} = c_{31} + c_{32} \tag{A8-14f}$$

where

$$c_{31} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j \;\; e_j' \;\; f_j' \,| = a_{22}$$

and

$$c_{32} = |\, a_j \;\; b_j \;\; c_j \;\; d_j' \;\; e_j' \;\; f_j' \,| = b_{22} .$$

From (A8-14a) - (A8-14f) together with (A8-12), it follows that

$$\frac{d^3 |A|}{dz_4^3} = 2 \cdot 3 (a_{11} + a_{12} + a_{22} + b_{22}); \tag{A8-15}$$

thus

$$\frac{1}{3!} \left[\frac{d^3 |A|}{dz_4^3} \right]_{z_4 = 0} = \bar{a}_{11} + \bar{a}_{12} + \bar{a}_{22} + \bar{b}_{22} . \tag{A8-15a}$$

Now, proceeding as in the previous parts, it is seen that

$$\bar{a}_{11} = -Z_1 Z_2 \ldots Z_5 (x_4 - y_4 \frac{x_3}{y_3}) \,|\, X_j - x_4,\; Y_j - y_4,\; -1,\; \frac{Y_j}{Z_j}(X_j - x_3),\; \frac{Y_j}{Z_j}(Y_j - y_3),\; X_j - x_2 \,| = 0,$$

$$\bar{a}_{12} = 0,$$

$$\bar{a}_{22} = 0,$$

and

$$\bar{b}_{22} = Z_1 Z_2 \ldots Z_5 \,|\, X_j - x_4,\; Y_j - y_4,\; Z_j,\; \frac{Y_j}{Z_j}(X_j - x_3),\; \frac{Y_j}{Z_j}(Y_j - y_3),\; (\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3})(X_j - x_2) \,| . \tag{A8-16a}$$

Thus, from (A8-15a):

$$\frac{1}{3!}\left[\frac{d^3|A|}{dz_4^{\,3}}\right]_{z_4=0} = \overline{b}_{22}. \tag{A8-16}$$

From (A8-15), due to (A8-10a) and (A8-10b), it holds that

$$\frac{da_{11}}{dz_4} = \frac{da_{12}}{dz_4} = \frac{da_{22}}{dz_4} = \frac{db_{22}}{dz_4} = |\, a_j \;\; b_j \;\; c_j' \;\; d_j' \;\; e_j' \;\; f_j' \,| = a_{111},$$

so that

$$\frac{d^4|A|}{dz_4^{\,4}} = 2 \cdot 3 \cdot 4\, a_{111};$$

thus

$$\frac{1}{4!}\left[\frac{d_4|A|}{dz_4^{\,4}}\right]_{z_4=0} = \overline{a}_{111} \tag{A8-17}$$

where

$$\overline{a}_{111} = Z_1 Z_2 \ldots Z_6 \left| X_j - x_4,\; Y_j - y_4,\; -1,\; \frac{Y_j}{Z_j}(X_j - x_3),\; \frac{Y_j}{Z_j}(Y_j - y_3),\; \left(\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3}\right)(X_j - x_2) \right|. \tag{A8-17a}$$

Finally, using (A8-7), (A8-9), (A8-13), (A8-16), and (A8-17), $|A|$ is obtained from (A8-6) as follows:

$$|A| = (\overline{b}_2 + \overline{b}_3 + \overline{c}_3)\, z_4^{\,2} + \overline{b}_{22}\, z_4^{\,3} + \overline{a}_{111}\, z_4^{\,4},$$

or

$$|A| = z_4^{\,2}(a_2 + a_3 z_4 + a_4 z_4^{\,2}) \tag{A8-18}$$

where

$$a_2 = \overline{b}_2 + \overline{b}_3 + \overline{c}_3, \tag{A8-18a}$$

$$a_3 = \overline{b}_{22}, \tag{A8-18b}$$

and

$$a_4 = \overline{a}_{111}, \tag{A8-18c}$$

and where $\overline{b}_2$, $\overline{b}_3$, $\overline{c}_3$, $\overline{b}_{22}$, and $\overline{a}_{111}$ are given in (A8-13a), (A8-13b), (A8-13c), (A8-16a), and (A8-17a) respectively. From any of these relations it is clear that

$$Z_J \neq 0 \tag{A8-19}$$

should hold, saying that no satellite point can be in plane with the stations 1, 2, 3, should this method be applicable.

The condition $|A| = 0$ represents a fourth order surface in x_4, y_4, z_4, as it was stated earlier, since $|A|$ was shown to be of order four in x_4, y_4, z_4. Due to (A8-18) one can also write this condition as

$$|A| = z_4^2 \, G(x_4, y_4, z_4) = 0 \tag{A8-20}$$

where

$$G(x_4, y_4, z_4) = a_2 + a_3 z_4 + a_4 z_4^2 \tag{A8-20a}$$

and G is of order at most two in the variables x_4, y_4, z_4; it means that a_2 is of order at most two, a_3 at most one, and a_4 a constant with respect to these variables (this will also be verified later). Since $z_4 \neq 0$, the expression (A8-20) implies that

$$G(x_4, y_4, z_4) = 0, \tag{A8-21}$$

which represents a second order surface in x_4, y_4, z_4. It will be proved now that this surface passes through the six satellite points, 1, 2, ..., 6 and through the three ground stations, 1, 2, 3, (since a second order surface is determined by nine points in general, the above points could be used to determine the surface defined by (A8-21) in the way presented in Appendix 6). It is easy to show that the second order surface (A8-21) passes through the six satellite points associated with matrix A. In (A8-5a) it was shown that $|A| = 0$ whenever any X_J, Y_J, Z_J were plugged for x_4, y_4, z_4. But this means, according to (A8-20), that

$$Z_J^2 \, G(X_J, Y_J, Z_J) = 0,$$

and since $Z_J \neq 0$ as stated in (A8-19), this yields

$$G(X_J, Y_J, Z_J) = 0,$$

which proves the asserted statement. A different approach will have to be taken to show that this surface also passes through the stations 1, 2, 3. It also held that $|A| = 0$ whenever any of the three ground station's coordinates were plugged for x_4, y_4, z_4, but this was true, as it can be seen from (A8-20), because z_4 was replaced by zero, while nothing can be said about the second order surface $G = 0$ so far. From the equation (A8-20a) it is now seen that

$$G(x_i, y_i, z_i) = 0, \quad i = 1, 2, 3, \tag{A8-22}$$

whenever $a_2 = 0$. It will be shown that $a_2 = 0$ holds if any of these x_i, y_i replace x_4, y_4 (z_4 does not appear in any of (A8-13a), (A8-13b), (A8-13c) which form a_2). From (A8-13a) one can see that

$$\bar{b}_2 = 0$$

whenever $x_1 = 0$ and $y_1 = 0$, or $x_2 \neq 0$ and $y_2 = 0$, or x_3 and $y_3 \neq 0$, replace x_4 and y_4. Similar considerations yield

$$\bar{b}_3 = 0$$

using (A8-13b) and

$$\bar{c}_3 = 0$$

using (A8-13c). Thus the asserted relation for the three ground stations follows and the second order surface $G = 0$ passes through the six satellite points and three ground stations. The considerations and derivations in this section have been based on a similar presentation in [1], Annex A, or in [10].

A8.2 <u>Explicit Expression for Second Order Surface $G(x_4, y_4, z_4) = 0$</u>.

In order to find a_2, a_3, a_4, necessary for computation of $G(x_4, y_4, z_4)$ from (A8-20a), suitable expressions have to be found for $\bar{b}_2$, $\bar{b}_3$, $\bar{c}_3$, $\bar{b}_{22}$, and $\bar{a}_{111}$, as seen from (A8-18a) - (A8-18c). If in the determinant on the right side of equation (A8-13a) the first row is subtracted from all the other rows and the determinant developed by the first column, a new determinant of (5×5) matrix is obtained and $\bar{b}_2$ can be given as

$$\bar{b}_2 = Z_1 Z_2 \ldots Z_5 (x_4 - y_4 \frac{x_3}{y_3})(x_4 - x_2) \left| Y_j - Y_1,\ Z_j - Z_1,\ \frac{Y_j}{Z_j}(X_j - x_3) - \frac{Y_1}{Z_1}(X_1 - x_3), \right.$$

$$\frac{Y_j}{Z_j}(Y_j - y_3) - \frac{Y_1}{Z_1}(Y_1 - y_3),\ X_j - X_1 \,\Big|, \tag{A8-23}$$

which is an expression of degree one and two in x_4, y_4. In this and the following determinants of (5×5) matrices, it is assumed that $j = 2, 3, \ldots, 6$. If the same procedure as above is carried out in (A8-13b) with the exception that the determinant is developed by the second column, it is obtained:

$$\bar{b}_3 = -Z_1 Z_2 \ldots Z_6 y_4(y_4 - y_3)\,\Big|\, X_j - X_1,\ Z_j - Z_1,\ \frac{Y_j}{Z_j}(X_j - x_3) - \frac{Y_1}{Z_1}(X_1 - x_3),\ Y_j - Y_1,$$

$$\left(\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3}\right)(X_j - x_2) - \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2)\,\Big|, \tag{A8-24}$$

which is of degree one and two in y_4. Similarly, for (A8-13c) it follows (determinant developed by the first column):

$$\bar{c}_3 = Z_1 Z_2 \ldots Z_6 y_4(x_4 - x_3)\,\Big|\, Y_j - Y_1,\ Z_j - Z_1,\ X_j - X_1,\ \frac{Y_j}{Z_j}(Y_j - y_3) - \frac{Y_1}{Z_1}(Y_1 - y_3),$$

$$\left(\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3}\right)(X_j - x_2) - \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2)\,\Big|, \tag{A8-25}$$

which is of degree one and two in x_4, y_4. Thus, considering (A8-18a), a_2 is found to be of degree one and two in x_4, y_4. Next, in the determinant of (A8-16a), the first row will be subtracted from all the other rows, giving thus

$$\bar{b}_{22} = Z_1 Z_2 \ldots Z_6 \times$$

$$\times \begin{vmatrix} X_1 - x_4, & Y_1 - y_4, & Z_1, & \frac{Y_1}{Z_1}(X_1 - x_3), & \frac{Y_1}{Z_1}(Y_1 - y_3), & \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_j - X_1, & Y_j - Y_1, & Z_j - Z_1, & \frac{Y_j}{Z_j}(X_j - x_3) - \frac{Y_1}{Z_1}(X_1 - x_3), & \frac{Y_j}{Z_j}(Y_j - y_3) - \frac{Y_1}{Z_1}(Y_1 - y_3), & \left(\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3}\right)(X_j - x_2) - \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix},$$

(A8-26)

which is of order at most one in x_4, y_4, and thus a_3 is of order at most one in x_4, y_4 ($a_3 = \bar{b}_{22}$ by (A8-18b)). Later the determinant of (6 × 6) matrix in (A8-26) will be developed by the first row, giving thus six determinants of (5 × 5) matrices with patterns similar to those in the previously mentioned determinants. Finally, considering (A8-17a) with the first row of the determinant subtracted from the other rows and the determinant developed by the third column, it follows that

$$\bar{a}_{111} = -Z_1 Z_2 \ldots Z_6 \left| X_j - X_1,\ Y_j - Y_1,\ \frac{Y_j}{Z_j}(X_j - x_3) - \frac{Y_1}{Z_1}(X_1 - x_3),\ \frac{Y_j}{Z_j}(Y_j - y_3) - \frac{Y_1}{Z_1}(Y_1 - y_3),\right.$$
$$\left.\left(\frac{X_j}{Z_j} - \frac{Y_j}{Z_j}\frac{x_3}{y_3}\right)(X_j - x_2) - \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2) \right| \qquad \text{(A8-27)}$$

where $\bar{a}_{111}$ and thus a_4 ($a_4 = \bar{a}_{111}$ by (A8-18c)) is a constant with respect to x_4, y_4, z_4. From (A8-20a) and the above results it is now verified that $G(x_4, y_4, z_4)$ is of order one and two and therefore (A8-21) does indeed represent a second order surface. As a matter of fact, since

$$G(x_4,\ y_4,\ z_4) = (\bar{b}_2 + \bar{b}_3 + \bar{c}_3) + \bar{b}_{22} z_4 + \bar{a}_{111} z_4^2 \qquad \text{(A8-28)}$$

(following from (A8-20a) and (A8-18a) - (A8-18c)), one can readily find that

$$G(x_4,\ y_4,\ z_4) = c_1 x_4^2 + c_2 x_4 y_4 + c_3 x_4 z_4 + c_4 y_4^2 + c_5 y_4 z_4 + c_6 z_4^2 + c_7 x_4 + c_8 y_4 + c_9 z_4.$$

This form contains all the terms of a second degree equation with exception of the constant term (this indicates that the second order surface $G = 0$ passes through the origin, i.e., ground station 1, which is clearly true since it was shown to pass through all the six satellite points and three ground stations).

For practical computations the expressions for $\bar{b}_2$, $\bar{b}_3$, $\bar{c}_3$, $\bar{b}_{22}$ and $\bar{a}_{111}$ can be simplified by using several determinants of (5 × 5) matrices which can be easily obtained one from another. For all these determinants, $j = 2, 3, \ldots, 6$ will be used. First, a determinant of (5 × 5) matrix, called D, will be defined as

$$= |X_J - X_1,\ Y_J - Y_1,\ \frac{Y_J}{Z_J}(X_J - x_3) - \frac{Y_1}{Z_1}(X_1 - x_3),\ \frac{Y_J}{Z_J}(Y_J - y_3) - \frac{Y_1}{Z_1}(Y_1 - y_3),\ (\frac{X_J}{Z_J} - \frac{Y_J}{Z_J}\frac{x_3}{y_3})(X_J - x_2) -$$

$$(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3})(X_1 - x_2)\ | .$$

xt, six determinants will be defined as follows:

D_1 ... replace the first column of D by $(Z_J - Z_1)$ and change the sign.

D_2 ... replace the second column of D by $(Z_J - Z_1)$ and change the sign.

D_3 ... $D_3 = D$.

D_4 ... replace the third column of D by $(Z_J - Z_1)$ and change the sign.

D_5 ... replace the fourth column of D by $(Z_J - Z_1)$ and change the sign.

D_6 ... replace the fifth column of D by $(Z_J - Z_1)$ and change the sign.

rther, introduce the notation

$$K = Z_1 Z_2 \ldots Z_6.$$

w it follows from (A8-23), that

$$= K(x_4 - y_4\frac{x_3}{y_3})(x_4 - x_2)(-D_6) = -KD_6(x_4^2 - \frac{x_3}{y_3}x_4y_4 - x_2x_4 + x_2\frac{x_3}{y_3}y_4), \qquad (A8-29)$$

m (A8-24), that

$$\bar{b}_3 = Ky_4(y_4 - y_3)(-D_5) = -KD_6(\frac{D_5}{D_6}y_4^2 - \frac{D_5}{D_6}y_3y_4), \qquad (A8-30)$$

from (A8-25), that

$$\bar{c}_3 = Ky_4(x_4 - x_3)(-D_4) = -KD_6(\frac{D_4}{D_6}x_4y_4 - \frac{D_4}{D_6}x_3y_4). \qquad (A8-31)$$

order to find $\bar{b}_{22}$ from (A8-26), the determinant present there will be developed the first row as follows:

$$\bar{b}_{22} = K[(X_1 - x_4)D_1 + (Y_1 - y_4)D_2 + Z_1D_3 + \frac{Y_1}{Z_1}(X_1 - x_3)D_4 +$$

$$+ \frac{Y_1}{Z_1}(Y_1 - y_3)D_5 + (\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3})(X_1 - x_2)D_6],$$

$$\bar{b}_{22} = -KD_6\left[\frac{D_1}{D_6}x_4 + \frac{D_2}{D_6}y_4 - \frac{D_1}{D_6}X_1 - \frac{D_2}{D_6}Y_1 - \frac{D_3}{D_6}Z_1 - \right.$$

$$\left. -\frac{D_4}{D_6}\frac{Y_1}{Z_1}(X_1 - x_3) - \frac{D_5}{D_6}\frac{Y_1}{Z_1}(Y_1 - y_3) - \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2)\right] \qquad \text{(A8-31)}$$

Finally, from (A8-27) it is seen that

$$\bar{a}_{111} = -KD_3 = -KD_6\left(\frac{D_3}{D_6}\right). \qquad \text{(A8-32)}$$

Having all the terms necessary to obtain $G(x_4, y_4, z_4)$ from (A8-28), it follows that

$$G(x_4, y_4, z_4) = -KD_6\left[x_4^2 - \frac{x_3}{y_3}x_4y_4 - x_2x_4 + x_2\frac{x_3}{y_3}y_4 + \frac{D_5}{D_6}y_4^2 - \right.$$

$$-\frac{D_5}{D_6}y_3y_4 + \frac{D_4}{D_6}x_4y_4 - \frac{D_4}{D_6}x_3y_4 + \frac{D_1}{D_6}x_4z_4 + \frac{D_2}{D_6}y_4z_4 - \frac{D_1}{D_6}X_1z_4 -$$

$$-\frac{D_2}{D_6}Y_1z_4 - \frac{D_3}{D_6}Z_1z_4 - \frac{D_4}{D_6}\frac{Y_1}{Z_1}(X_1 - x_3)z_4 - \frac{D_5}{D_6}\frac{Y_1}{Z_1}(Y_1 - y_3)z_4 -$$

$$\left. -\left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2)z_4 + \frac{D_3}{D_6}z_4^2\right].$$

For the second degree surface as given by (A8-21), i.e.

$$G(x_4, y_4, z_4) = 0,$$

the above equation can be divided by $(-KD_6)$, since $K \neq 0$ ($Z_j \neq 0$ for all j) and $D_6 \neq 0$ in general. Consequently, this second order surface can be expressed after rearranging the terms as

$$x_4^2 + \left(\frac{D_4}{D_6} - \frac{x_3}{y_3}\right)x_4y_4 + \frac{D_1}{D_6}x_4z_4 + \frac{D_5}{D_6}y_4^2 + \frac{D_2}{D_6}y_4z_4 + \frac{D_3}{D_6}z_4^2 - x_2x_4 +$$

$$+ \left(x_2\frac{x_3}{y_3} - \frac{D_5}{D_6}y_3 - \frac{D_4}{D_6}x_3\right)y_4 - \left[\frac{D_1}{D_6}X_1 + \frac{D_2}{D_6}Y_1 + \frac{D_3}{D_6}Z_1 + \frac{D_4}{D_6}\frac{Y_1}{Z_1}(X_1 - x_3) + \right.$$

$$\left. + \frac{D_5}{D_6}\frac{Y_1}{Z_1}(Y_1 - y_3) + \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2)\right]z_4 = 0. \qquad \text{(A8-33)}$$

From this last equation the second order surface will be expressed in a matrix form; writing $x \equiv x_4$, $y \equiv y_4$, $z \equiv z_4$, it follows that

$$x^T A x + x^T a + c = 0 \qquad \text{(A8-34)}$$

where A is a (3×3) symmetric matrix (not to be confused with A matrix of

(A8-2)) with a_{ij} as the element in its ith row and jth column, a is a (column) vector written as

$$a = [a_1\ a_2\ a_3]^T,$$

and c is a constant term; the variable (column) vector x is written as

$$x = [x\ y\ z]^T.$$

From the relation (A8-33) A-matrix and a-vector are found to be such that

$$a_{11} = 1,$$

$$a_{12} = a_{21} = \frac{1}{2}\left(\frac{D_4}{D_6} - \frac{x_3}{y_3}\right),$$

$$a_{13} = a_{31} = \frac{1}{2}\,\frac{D_1}{D_6},$$

$$a_{22} = \frac{D_5}{D_6},$$

$$a_{23} = a_{32} = \frac{1}{2}\,\frac{D_2}{D_6},$$

$$a_{33} = \frac{D_3}{D_6},$$

nd

$$a_1 = -x_2$$

$$a_2 = x_2\,\frac{x_3}{y_3} - \frac{D_5}{D_6}\,y_3 - \frac{D_4}{D_6}\,x_3,$$

$$a_3 = -\left[\frac{D_1}{D_6}X_1 + \frac{D_2}{D_6}Y_1 + \frac{D_3}{D_6}Z_1 + \frac{D_4}{D_6}\frac{Y_1}{Z_1}(X_1 - x_3) + \frac{D_5}{D_6}\frac{Y_1}{Z_1}(Y_1 - y_3) + \right.$$

$$\left. + \left(\frac{X_1}{Z_1} - \frac{Y_1}{Z_1}\frac{x_3}{y_3}\right)(X_1 - x_2)\right].$$

urthermore, here

$$c = 0.$$

hen A-matrix and a-vector of the second order surface given by (A8-34) were omputed numerically, they were the same (within round-off errors) as the nes computed from fitting of a second order surface to the nine points lying n it (six satellites points and three grounds stations), according to the description given in Appendix 6.

In conclusion it is emphasized that the problem with six satellite points will be singular if ground station 4 lies on a second order surface which passes through all six satellite points and the three remaining ground stations. If more than six satellite points are observed by the four ground stations, the same approach could take place using all possible combinations of six satellite points. The problem would then be singular if (A8-2) held for each such combination. This means that station 4 would have to lie simultaneously on all second order surfaces defined by stations 1, 2, 3 and any combination of six satellites. Since in general nine points define a second order surface, these surfaces would have to coincide in order to fulfill such a condition. Thus the general conclusion follows: the problem is singular whenever all the satellite points and all four ground stations are lying on one second order surface.

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