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INVESTIGATIONS RELATED TO THE ESTABLISHMENT OF  
A WORLD GEODETIC SYSTEM

by

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## SUMMARY

A world-wide geodetic system which will best serve the present needs of geodesy is defined by four groups of components. These are geocentric coordinates and associated variances of physical points, spherical harmonic coefficients of the geopotential, a mean earth ellipsoid, and a group of seven transformation parameters for each datum which connect the geocentric coordinates with the geodetic coordinates of that datum. The method of determination of that system is given, based on all types of presently available geodetic observations, namely, terrestrial triangulation, astronomic observations, terrestrial gravity observations, and geometric and dynamic satellite observations. For all these observations, proper mathematical models are given. In addition, all the constraints which will guarantee a unique mean earth ellipsoid are introduced. Using the specified geodetic observations in the mathematical models developed, groups of observation equations and then normal equations may be formed. The solution for the components of the world geodetic system will be the solution of the summation of these normals, properly scaled to account for different model or weighting errors in each group. Finally, the contribution of each type of data to the determination of the components of the world geodetic system as well as the overall accuracy achieved is investigated.

## PREFACE

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## 1. INTRODUCTION

The recent development of civilization brought together different local activities and created the present international community. The needs and the aims of the new community have grown rapidly in various aspects. In geodesy it has been realized for some years that the long-established different datums, which served the local needs very well and were consistent with the means of their times, could not satisfy the geodetic needs of the new community.

First an attempt was made to tie together some of the different datums using terrestrial techniques. But then it became evident that even if the local datums would be brought together this would not have solved the whole problem as the need for geocentric coordinates and knowledge of the gravity field came into the picture. The need for the establishment of a world-wide geocentric coordinate system became evident; it is the purpose of this project as it is spelled out in its objectives to develop an analytical procedure for the establishment of a world geodetic system and to investigate the existing and the required data to achieve a satisfactory solution.

Any world-wide or universal representation is to be constructed by piecing together, so to speak, the local views defined by the different groups of data. We cannot make a universal representation from all data of all groups, so a selection or an estimation must be made; this estimation will be done according to the principle of least squares for correlated observations [Brown, 1955].

The way, however, in which the data will be treated and the selection of the mathematical models which will be used take different interpretations. It is to this subject that the word "ideal" refers. The treatment of the data and the selection of the mathematical models will be such that the minimum bias will be introduced in the determination of the parameters defining the

world geodetic system, while the maximum amount of information contained in the data will be used.

It is fair to ask in what respects this determination, when done, is going to serve better than the individual datums or the solutions from individual groups. When geocentric coordinates or coordinates referring to a single reference frame are needed, it is obviously much more desirable to use coordinates in a world geodetic system than to use the coordinates on the various datums. It is also intuitively obvious and may be proven that an estimate based on a combination of two or more groups of data will always have a smaller standard error, and consequently must be considered better than the estimates from the individual groups.

However, because the systematic errors dominate some of the present mathematical models and consequently the present solutions [Köhnlein, 1966b], we actually have a change in the definition of the involved parameters from one model to another. The combined solution then, although it is an improvement toward the true values of the parameters, may not be a better estimate of the same parameters as they are defined in each particular model. Consequently, functions similar to or the same as the ones expressed by a mathematical model are better served by values obtained through that model and the same kind of observations. We have indications of the truth of this idea first from the gravity model comparisons [Lerch et al., 1967]. There it is demonstrated that a set of harmonic coefficients describing the gravity field which has been derived by optical observations gives better predictions for an orbit established with optical observations than a gravity field derived from Doppler observations. Another example is given in [Mancini, 1968] in which the SAO gravity model of 121 coefficients, derived from satellite observations only, is reported to give better predictions for the orbit of the GEOS-A satellite than Mancini's model of 251 coefficients derived from a combination of satellite and terrestrial data. It seems then that a general combination solution can serve more general purposes, increase the knowledge of the true values of the parameters defining

the size and shape of the earth, and will be used in other fields besides geodesy, for example, in geophysics.

Another problem which arises after such a combination solution, in which all the existing data has been used, is that there is not any way to check it. This problem is complicated even more by the fact that the precisions obtained by least squares solutions of the individual groups are not reliable as accuracy figures, for example, satellite solutions [Gaposchkin, 1966b]. The precision is a measure of the consistency of the estimation of a parameter and describes the effect of the random errors on that estimation; this effect decreases with the number of observations. As the number of observations is very large, the precisions obtained are very small. The effect of the existing systematic errors, on the other hand, do not decrease with the number of observations; thus they cannot be described by the precision figures obtained, and these figures cannot be used as accuracy figures. Only by combining the data in different combinations of groups can we get an idea of the accuracy of the solution.

As this topic contains the main scope of geodesy, that is, the determination of the size and shape of the earth, to cover all problems completely and provide definite answers to all the questions involved would be impossible. The complexities of the problems involved in getting solutions from individual groups of data or from a combination of them is indicated by the amount of work that prominent scientists have devoted to them. The study of such an extended and divergent bibliography and the attempt to make a contribution on the subject appear to be a more suitable task for a group or an organization than for the efforts of an individual who is limited by means and time.

This study then is mainly concerned with the formulation of a consistent and exhaustive group of mathematical models which will constitute the basis of the determination of the world geodetic system. No attempt was made to bring the system into a computational working stage or to collect and use in that system the existing data. Whenever some preliminary computations were necessary, existing programs were used, after limited modifications.

Although no attempt was made for the determination of the values of the parameters of the world geodetic system, we did make an attempt to estimate the accuracy with which these parameters can be determined with the present data. For this estimation approximate procedures were used for obtaining estimates.

## 2. DEFINITION OF THE WORLD GEODETIC SYSTEM

For some years now it has been realized that the use of an ellipsoid as a geometric reference surface and a level ellipsoid as a dynamic reference model could not serve the more complicated functions of modern geodesy. The need for the introduction of a new kind of reference frame has become apparent with the development of satellite geodesy. Already individuals and organizations working in satellite geodesy use more sophisticated models to define a reference coordinate system or to describe the gravity field of the earth. To preserve the notion of the ellipsoid used for the different datums and in astronomic computations and to provide the details of the gravity field needed in the computations of trajectories, we formulated another reference system to be used as a world geodetic system which is composed of the following four components.

(1) The geocentric Cartesian coordinates of a set of physical points, with their variances and covariances. As these are points where geocentric coordinates will be computed and as today the most accurate method for geocentric position determinations is the observation of satellite orbits, these must be points from which orbits have been observed. The geocentric coordinate system that will constitute the base of the world geodetic system, and to which these points will be referred, is fixed with respect to the earth and is often called the average terrestrial coordinate system. Its origin is at the geocenter with its Z axis through the average terrestrial pole of 1900-05 as adopted by the IUGG [Garland, 1967, Resolution No. 19] and was designated Conventional International Origin (CIO). The X axis is parallel to the plane of the mean Greenwich astronomic meridian as defined by the Bureau International de l'Heure [BIH, 1968]. This coordinate system is also called System BIH 1968.

(2) A set of constants with their variances and covariances describing the gravity field of the earth, consistent with the above coordinates. This set could be a set of mean values of gravity, a set of mean gravity anomalies, or a set of spherical harmonic coefficients. As the most efficient way of orbit computations today is through spherical harmonic coefficients, we decided to accept such a set of constants to describe the gravity field of the earth. The spherical harmonic coefficients will be used in the fully normalized form  $\bar{C}$  and  $\bar{S}$  [Mueller, 1964]. The spherical harmonics will refer to the average terrestrial coordinate system, which is a geocentric coordinate system. Therefore, the first-degree, and the second-degree, first-order harmonics will not be present [Heiskanen and Moritz, 1967]. The zero-degree harmonic will not be used, but it will be determined from the rest of the parameters. The spherical harmonic coefficients are determined from satellites together with station coordinates and are correlated with them. Thus the adoption of the set of harmonics is not independent of the set of adopted station coordinates. These two sets then must be consistent and they must be determined simultaneously through the same adjustment.

(3) A mean earth ellipsoid consistent with the components of the above gravity field and the rate of rotation of the earth, with its center at the geocenter. The mean earth ellipsoid is defined by four parameters. In our system these parameters will be

$\omega$	the rate of rotation of the earth
$\bar{C}_{20}$	the second-degree zonal harmonic
$a$	the semidiameter
$GM$	the gravitation constant times the mass of the earth and its atmosphere

This surface, which best approximates the geoid in a world-wide sense, can be used as a simple reference surface and can serve in astronomic computations.

(4) The parameters defining the positions in the geocentric system of the existing major geodetic datums, established through the geodetic coordinates of the points of group (1). If the existing geodetic datums are to be preserved, either because they may better approximate the geoid in their domain or because of the amount of work related to them, then the determination of a world geodetic system must contain the determination of the positions of the existing datums relative to the geocentric system. As every geodetic datum defines a three-dimensional coordinate system, the position of this coordinate system relative to the geocentric system can be expressed by six parameters; these parameters are three shifts and three rotations. Because there is also a possible difference in scale between a geodetic and the geocentric system, one more parameter has been introduced to account for such differences. Thus finally the relative position of a geodetic system defined by a datum with respect to the geocentric system is expressed by seven parameters. The zero-degree harmonic of the gravity anomalies  $\Delta g_0$  is not a parameter of the world geodetic system, but a quantity which may be derived from them. We will carry it, however, through the formation of the whole system for mathematical convenience. After the normal equations have been formed,  $\Delta g_0$  may be eliminated from the system.

There are many underlying assumptions in that definition: as the average terrestrial coordinate system is materialized through the observations of stars; therefore, a certain star catalog, a certain theory of the motion of the earth (including time), and a certain ephemeris of polar motion are implied in the definition of the world geodetic system. The polar motion used in our system is that determined by the International Polar Motion Service; the value for the rate of rotation of the earth ( $\omega$ ) in our system is as defined in the "Explanatory Supplement to the Ephemeris."

### 3. THE ADJUSTMENT SYSTEM

#### 3.1 Observations for the World Geodetic System

For the establishment of an analytical adjustment system to determine the parameters of the world geodetic system, the basic idea was to use every kind of observation which could contribute to such a determination, together with all the constraints on the parameters implied by the definition of the world geodetic system. The selection of the observed quantities is a basic step; it must be done in such a way that a maximum of information concerning the parameters is extracted, with minimum correlations introduced, especially between different groups. The different groups of observations and the observed quantities for each group are shown in Fig. 3.1-1.

Observation equations for all these groups will be formulated in the following sections together with estimates of covariance matrices  $\hat{\Sigma}$  associated with the different types of observations. These variance-covariance matrices will be used to form the weight matrices  $P = \hat{\Sigma}^{-1}$ . The weights in this adjustment will necessarily have dimensions since in the quadratic form  $V'PV$ , which is minimized,  $V$ 's of different dimensions have been included [Kaula, 1959b].

The triangulation and leveling which constitute the first group of observations refer to the necessary operations for the determination of the geodetic coordinates  $x, y, z$  of the points where geocentric coordinates will be determined. To achieve this, classical triangulation can be supported by satellite observations used in geometric or short-arc mode, accurate traverses, and some first-order astronomic observations; all of the above and especially the satellite triangulation will be excluded from the same observations of the next groups, in order to avoid correlation between observations of different groups.



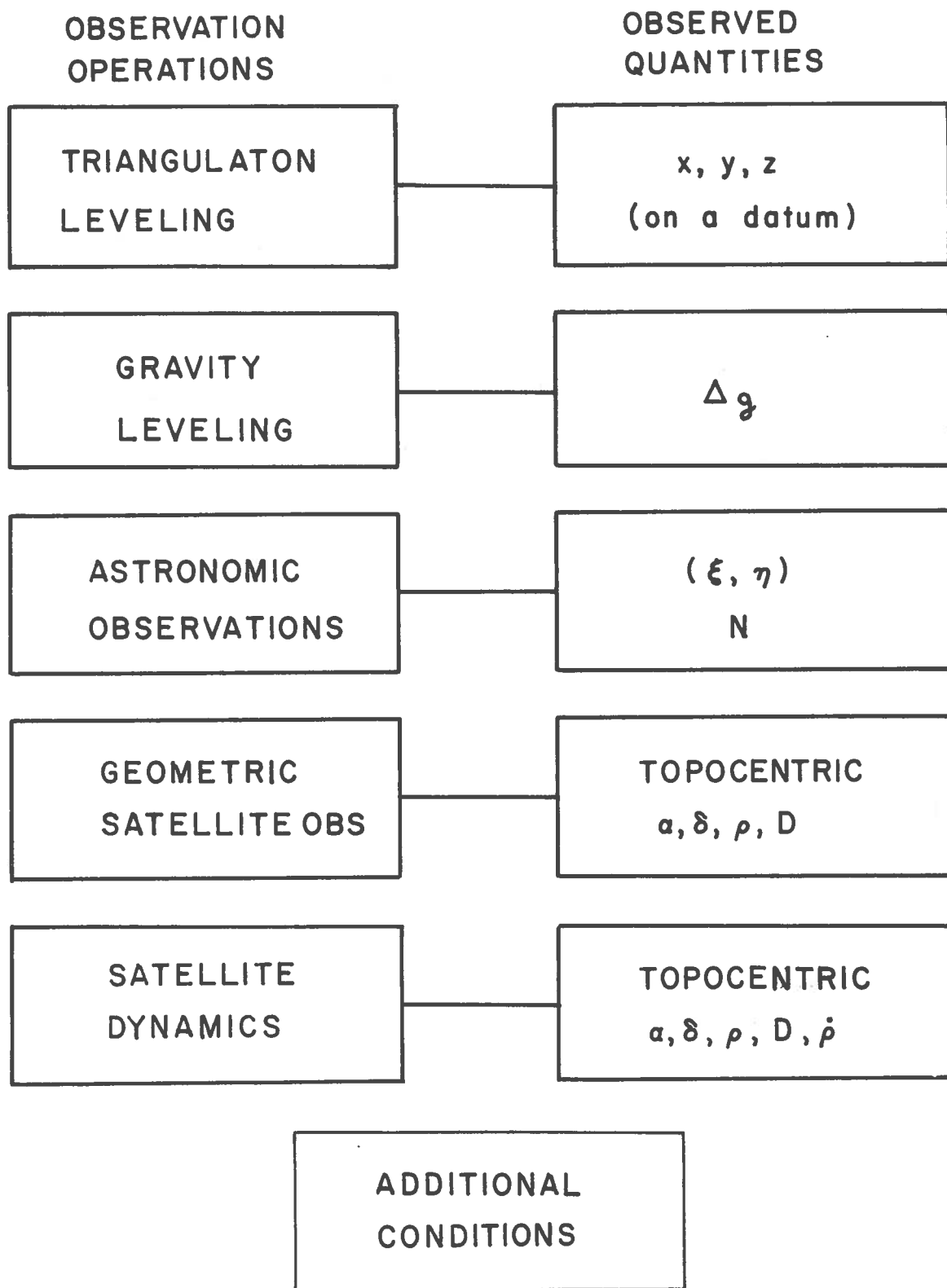


Fig. 3.1-1 Observations for the World Geodetic System

The gravity observations are used as free-air gravity anomalies; thus, besides the gravity observations, leveling is required.

The next group is composed basically of the astronomic observations, spaced approximately every 50 km and at triangulation points. From these, the astrogeodetic deflections of the vertical can be computed. Finally, the astrogeodetic undulations, which will be observed quantities for that group, may be computed.

Satellite observations treated in a geometric mode are simultaneous or quasi-simultaneous topocentric right ascension  $\alpha$ , declination  $\delta$ , topocentric range  $\rho$ , and range difference  $D$ . The topocentric ranges and range differences could come from SECOR, laser, C-Band radar, or Goddard Space Flight Center range and range-rate observation systems. The justification for the use of the simultaneous observations in a geometric mode, in addition to the dynamic mode (where they have also been used), is that there are many more modeling errors in the dynamic mode than in the geometric mode, which may be viewed as giving more weight to the geometric mode against the dynamic. The types of satellite observations treated in a dynamic mode are those used in geometric mode, as well as Minitrack and range-rate observations  $\dot{\rho}$ .

There are certainly types of observations which could be used in the determination of the parameters of the world geodetic system, but which have been omitted from the above list and consequently from the following discussion. Some of these types, such as solar eclipses, are omitted because their contribution is considered very poor today. Others, such as the very high accuracy absolute measurements of gravity [Sakuma, 1968] or observations with the radio interferometer [Gold, 1967] are omitted because they are still in an experimental stage. There can also be some special cases; e.g., if a set of astronomic observations has been made in an extended area of a datum, but has not been connected with the main network of astrogeodetic observations on the same datum, then undulations computed from this set cannot refer to the same origin. This case can be handled by introducing an additional parameter  $N$

which will bring the independent astrogeodetic undulations onto the same origin and which can be eliminated before the solution. Thus no effort was made to cover all possible special cases.

Besides the observation equations, the parameters must satisfy the definition of the world geodetic system, that is, the ellipsoid provided by the solution must be the mean earth ellipsoid; thus some constraints are introduced into the system.

The observation equations are often of the mixed type, that is, they contain parameters and observed quantities on the same side of the equation. As we are seeking a solution for the parameters of the world geodetic system which are never observed directly, the adjustment system is finally reduced to a simple adjustment of estimation of parameters only. This is done by eliminating the correlates in the cases of mixed condition equations. The additional constraints, when they are simple mathematical expressions, are used to eliminate some of the unknowns; otherwise they are treated either as observation equations by heavily weighting the misclosures, or according to the method for constraints described in [Uotila, 1967b].

Kalman-Schmidt filtering [Schwarz, 1967], although it has been considered, has not been found necessary, as the solution, once found, is not likely to change with a small number of observations. Kalman's approach is advantageous for a small number of observations and a large number of unknowns. The solution for the world geodetic system is not likely to change with a small number of observations, and it will be repeated after a considerable number of observations are collected for which an ordinary least squares technique is adequate. Thus the adjustment system will be solved by standard least squares techniques.

### 3.2 Triangulation

#### 3.21 Mathematical Model for Terrestrial Triangulation

By triangulation we mean the whole groups of observations and processes which are necessary to provide geodetic coordinates  $\varphi, \lambda$  and the geometrical height above the ellipsoid  $h$ . The coordinates  $\varphi$  and  $\lambda$  are the adjusted geodetic coordinates of a point of a triangulation which has been computed according to the projective method, and the ellipsoid height has been computed as

$$h = H + N_{AG}$$

where

$H$  is the orthometric height

$N_{AG}$  is the astrogeodetic undulation

The above equation implies that a systematic error in the order of a fraction of a millimeter has been neglected. In establishing a geodetic datum, a three-dimensional rectangular coordinate system ( $x$ ) is also defined. The geodetic coordinates in this rectangular system are related to  $\varphi, \lambda$  and  $h = H + N_{AG}$  by

$$\begin{aligned}x &= (N + H + N_{AG}) \cos\varphi \cos\lambda \\y &= (N + H + N_{AG}) \cos\varphi \sin\lambda \\z &= [N(1 - e^2) + H + N_{AG}] \sin\varphi\end{aligned}\tag{3.2-1}$$

where

$N$  is the radius of curvature of the datum ellipsoid in the prime vertical at the point

$e$  is the first eccentricity of that ellipsoid

This coordinate system differs from the average terrestrial ( $X$ ) (Fig. 3.2-1). The adoption of erroneous deflections of the vertical at the datum origin, errors in the astronomic latitude and longitude, and the adoption of the nonproper parameters for the reference ellipsoid introduce a shift of the origin of the geodetic system away from the geocenter. Errors in the astronomic azimuth

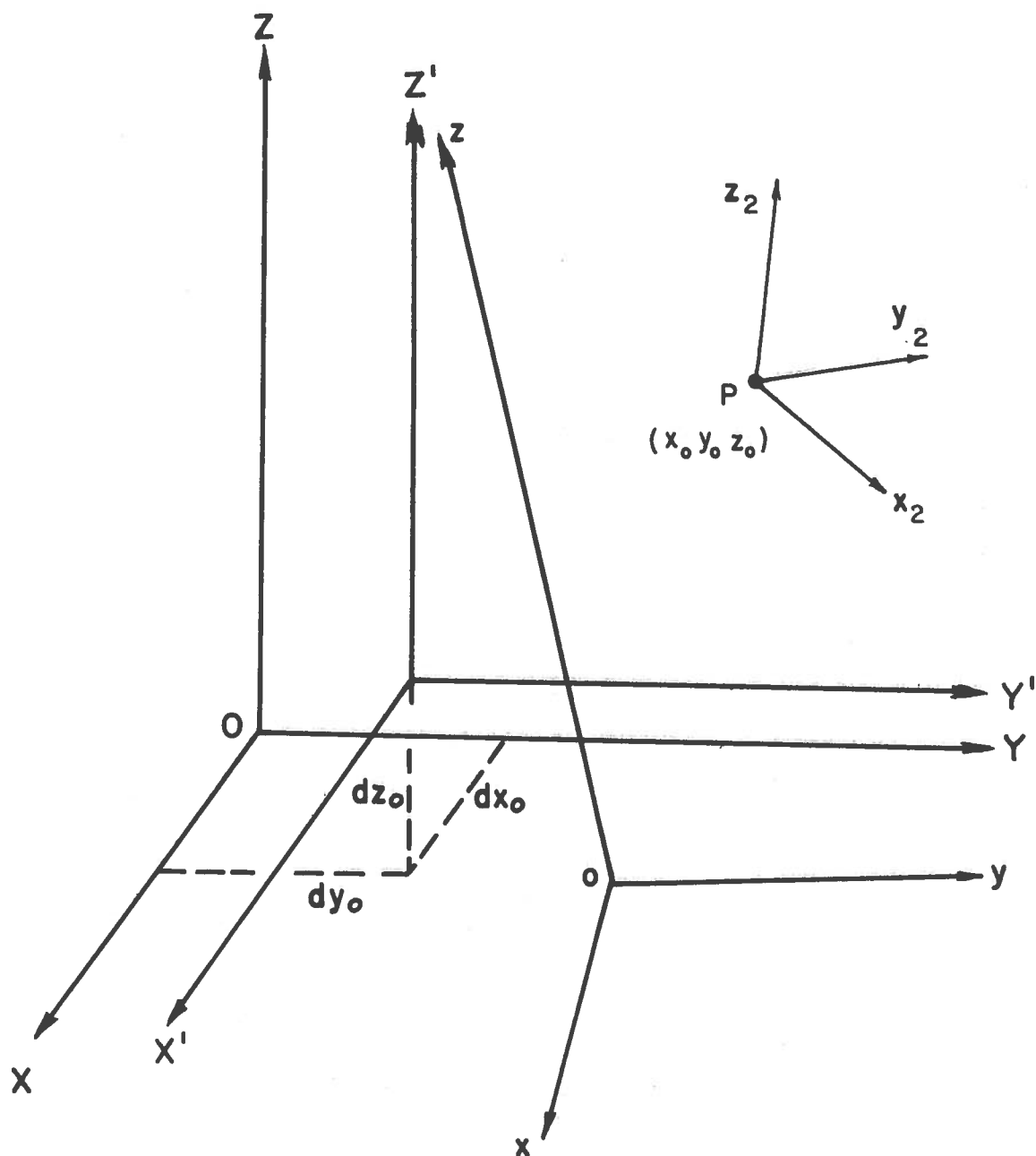


Fig. 3.2 - 1 Geocentric and geodetic coordinate systems

used for the definition of the datum and improper application of the Laplace condition introduce a rotation between the two systems. Furthermore, as every datum is scaled by geodetic baselines, errors in measurements or in the definition of scale introduce a discrepancy in scale between the average terrestrial and the geodetic systems. Though the geodetic coordinate system implied by a datum is clearly defined by the definition of the datum, it is materialized and its position relative to the average terrestrial system is determined by the coordinates of the points that lie on it. Because these coordinates may have systematic errors the determined transformation parameters reflect not only the errors made in the definition of the datum, but also all the systematic errors committed over the datum.

The average terrestrial and geodetic coordinate systems can be related by the following equation:

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} + \begin{vmatrix} x_0 \\ y_0 \\ z_0 \end{vmatrix} + M \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} + \epsilon \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} \quad (3.2 - 2)$$

where

$X, Y, Z$	are the coordinates of any point in the average terrestrial coordinate system
$x, y, z$	are the geodetic coordinates of the same point
$M$	is the matrix of three rotations necessary to make the geodetic system parallel to the average terrestrial
$\epsilon$	is a scale correction
$dx_0, dy_0, dz_0$	are the coordinates of the origin of the geodetic system ( $x_1$ ) after it has been rotated and has become parallel to the average terrestrial system
$x_0, y_0, z_0$	are the geodetic coordinates of the point P which is kept fixed during rotations

Now at point P we select arbitrarily a Cartesian coordinate system ( $x_2$ ). We want to apply three rotations around the axes of this system to make the geodetic system ( $x$ ) parallel to the average terrestrial system ( $X$ ). To express the relation between the two systems elementary rotation matrices  $R_i(\theta)$  will be used. The matrix  $R_i(\theta)$  rotates a Cartesian coordinate system around its axis  $i$  through an angle  $\theta$ . Expressions for the rotation matrices are given in [Mueller, 1969, p. 43].

In Fig. 3.2-1 we suppose first that the origin  $o$  of the geodetic system ( $x$ ) after a parallel translation has been brought to point P. Second, we suppose that the three rotations which rotate the geodetic coordinate system ( $x$ ) and bring it into coincidence with the coordinate system ( $x_2$ ) are  $R_1(\theta_1)$ ,  $R_3(\theta_3)$ , and  $R_2(\theta_2)$ . The first rotation ( $\theta_1$ ) makes the axis  $Py_2$  lie on the plane  $x_2Py$ . The second rotation ( $\theta_3$ ) makes the rotated  $y_2$  axis coincide with axis  $y$ . The third rotation ( $\theta_2$ ) makes the rotated  $z_2$  axis coincide with  $z$  and thus the coordinate system ( $x$ ) coincides with ( $x_2$ ).

Third, we suppose that the rotations that must be applied to the coordinate system ( $x$ ) to make it parallel to the average terrestrial are very small, so that we can substitute their cosines by 1 and their sines by the angles without appreciable error. We call the rotation angles around the respective axes  $da_1$ ,  $da_2$ ,  $da_3$  and the matrix of the three rotations, whose sequence is now irrelevant, is

$$R_1(da_1) R_2(da_2) R_3(da_3) = \begin{vmatrix} 1 & da_3 & -da_2 \\ -da_3 & 1 & da_1 \\ da_2 & -da_1 & 1 \end{vmatrix}$$

After the above, the matrix  $M$  of equation (3.2 - 2) is

$$M = R'_1(\theta_1) R'_3(\theta_3) R'_2(\theta_2) R_1(da_1) R_2(da_2) R_3(da_3) R_2(\theta_2) R_3(\theta_3) R_1(\theta_1)$$

where  $R'_i$  is the transpose of the matrix  $R_i$ . Equation (3.2 - 2) becomes

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} + \begin{vmatrix} x_0 \\ y_0 \\ z_0 \end{vmatrix} +$$

(3.2 - 2a)

$$+ R_1'(\theta_1) R_3'(\theta_3) R_2'(\theta_2) R_1(da_1) R_2(da_2) R_3(da_3) R_2(\theta_2) R_3(\theta_3) R_1(\theta_1) \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} + \epsilon \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix}$$

The above is a general equation relating two coordinate systems with seven parameters. By keeping different points fixed during rotations and by selecting different directions for the axes of the coordinate system ( $x_2$ ), around which the rotations are made, different investigators have given different forms to the equation (3.2 - 2a).

First is the form given by Wolf [1963] and Bursa [1965]. They have accepted the geodetic coordinate system as coordinate system ( $x_2$ ). Thus point P is now the origin o and

$$\theta_1 = \theta_2 = \theta_3 = 0$$

$$x_0 = y_0 = z_0 = 0$$

Equation (3.2 - 2a) then gives

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} + \begin{vmatrix} 1 & da_3 & -da_2 \\ -da_3 & 1 & da_1 \\ da_2 & -da_1 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} + \epsilon \begin{vmatrix} x \\ y \\ z \end{vmatrix} \quad (3.2 - 2b)$$

The second form is the one given by Molodensky et al. [1962]. He defines the point P around which the rotation will be made as the datum origin with geodetic coordinates ( $x_0, y_0, z_0$ ). The coordinate system ( $x_2$ ) is defined as one with axes parallel to the axes of the geodetic system ( $x$ ) (Fig. 3.2-2). Thus

$$\theta_1 = \theta_2 = \theta_3 = 0$$

and equation (3.2 - 2a) gives

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} + \begin{vmatrix} x_0 \\ y_0 \\ z_0 \end{vmatrix} + \begin{vmatrix} 1 & da_3 & -da_2 \\ -da_3 & 1 & da_1 \\ da_2 & -da_1 & 1 \end{vmatrix} \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} + \epsilon \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix}$$



This can be further simplified to

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} + \begin{vmatrix} x \\ y \\ z \end{vmatrix} + \begin{vmatrix} 0 & da_3 & -da_2 \\ -da_3 & 0 & da_1 \\ da_2 & -da_1 & 0 \end{vmatrix} \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} + \epsilon \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} \quad (3.2-2c)$$

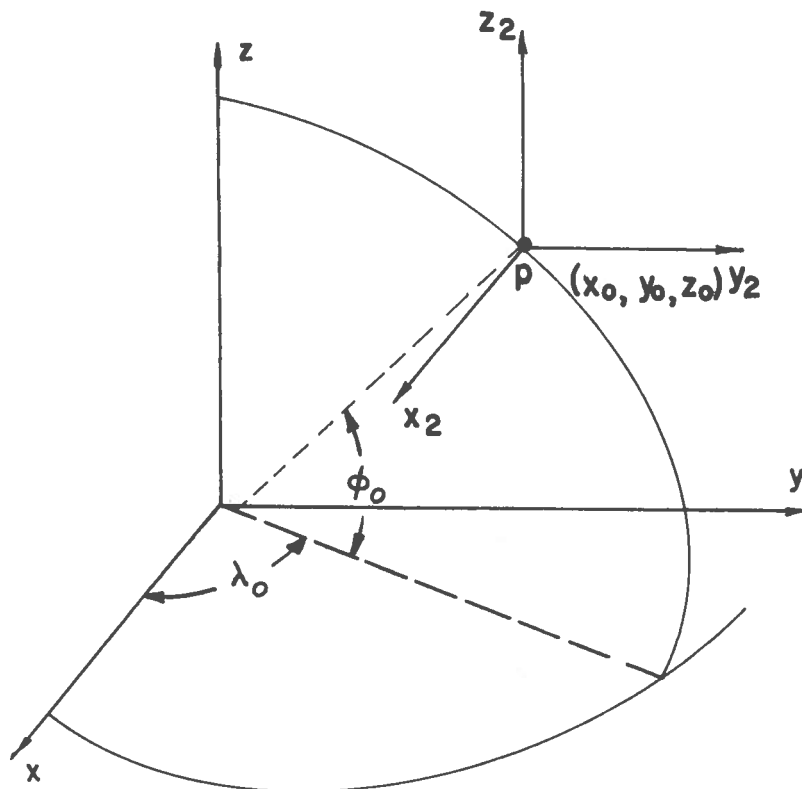


Fig. 3.2-2 Fixed point and axes of rotations in Molodensky system

Third, we have the form given by Veis [1960]. He retains the datum origin as point  $P$ , but his  $(x_2)$  coordinate system has the  $x_2$  axis tangent to geodetic meridian with positive direction toward the south; the  $y_2$  axis is perpendicular to the meridian plane and it is positive eastward; finally the  $z_2$  axis is along the geodetic normal with its positive direction upward, forming a right-handed system with  $x_2$  and  $y_2$  (Fig. 3.2-3). For this  $(x_2)$  coordinate system we have

$$\begin{aligned}\theta_1 &= 0 \\ \theta_3 &= \lambda_0 \\ \theta_2 &= 90 - \phi_0\end{aligned}$$

where

$\phi_0, \lambda_0$  are the geodetic coordinates of the point P, now being the datum origin

Equation (3.2 - 2a) now gives

$$\begin{aligned}\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} &= \begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} + \begin{vmatrix} x_0 \\ y_0 \\ z_0 \end{vmatrix} + \\ &+ R'_3(\lambda_0) R'_2(90 - \phi_0) R_1(da_1) R_2(da_2) R_3(da_3) R_2(90 - \phi_0) R_3(\lambda_0) \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} + \epsilon \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix}\end{aligned}\quad (3.2 - 2d)$$

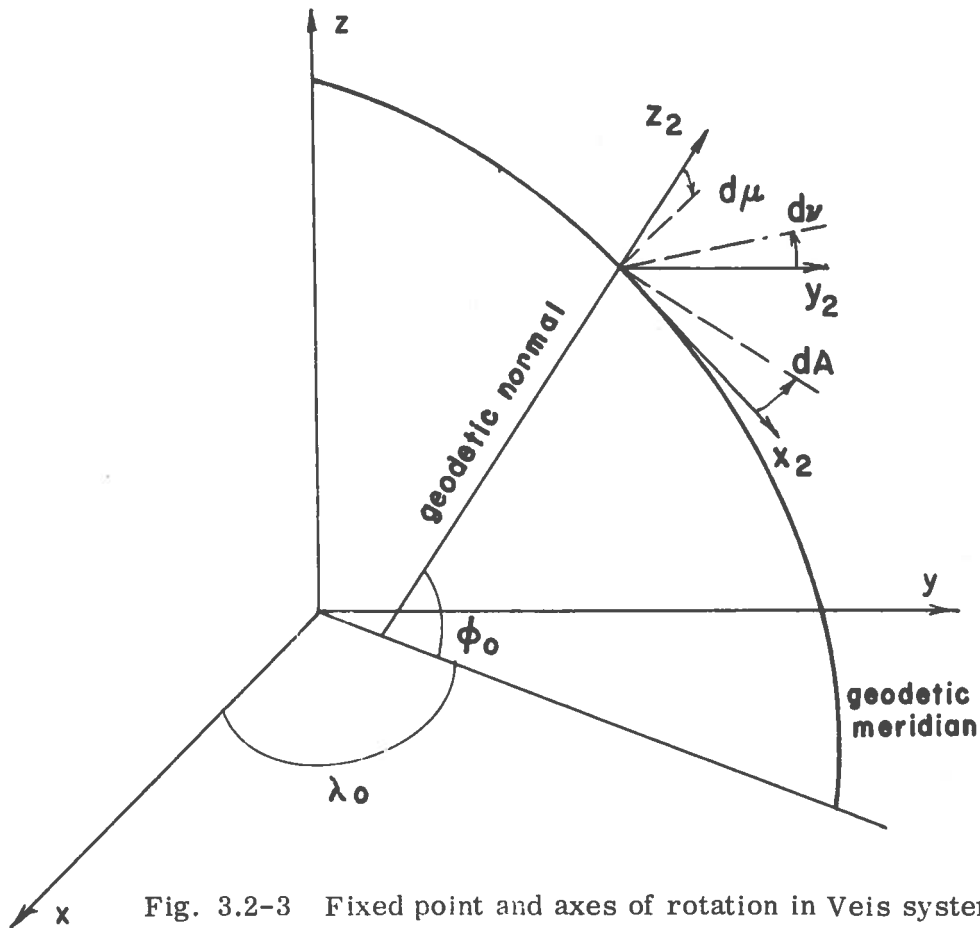


Fig. 3.2-3 Fixed point and axes of rotation in Veis system

All three forms of transformations are mathematically equivalent and a selection among them must be based on other considerations. First we notice that although all forms are mathematically equivalent it is not clear whether they are completely equivalent when used as mathematical models in an adjustment for the determination of the transformation parameters. As a shift is a rotation around an origin at infinity, the more the rotation axes move away from the area which is rotated, the more these rotations become similar to shifts, and therefore more difficult to separate in an adjustment. It appears then that Molodensky's or Veis' form may serve better than Bursa's form as a mathematical model in an adjustment for the determination of shifts and rotations. The selection between Molodensky's and Veis' form is completely irrelevant and depends only on what kind of physical quantities we prefer to express by the parameters used and to have at our immediate disposal.

We decided to use in our formulation and for our computer program Veis' parameters. This was because Veis' relations are closer to the general form and therefore we can shift to Bursa's or Molodensky's parameters by simply substituting zero for some of the parameters involved.

We denote

$$da_1 = d\nu$$

$$da_2 = d\mu$$

$$da_3 = dA$$

Then the M matrix is written

$$M = R'_3(\lambda_0) R'_2(90 - \varphi_0) R_1(d\nu) R_2(d\mu) R_3(dA) R_2(90 - \varphi_0) R_3(\lambda_0) \quad (3.2 - 3)$$

Making the substitutions and multiplications we have

$$M = \begin{vmatrix} 1 & \sin\varphi_0 dA - \cos\varphi_0 d\nu \\ -\sin\varphi_0 dA + \cos\varphi_0 d\nu & 1 \\ \cos\varphi_0 \sin\lambda_0 dA + \cos\lambda_0 d\mu + \sin\varphi_0 \sin\lambda_0 d\nu & -\cos\varphi_0 \cos\lambda_0 dA + \sin\lambda_0 d\mu - \sin\varphi_0 \cos\lambda_0 d\nu \\ -\cos\varphi_0 \sin\lambda_0 dA - \cos\lambda_0 d\mu - \sin\varphi_0 \sin\lambda_0 d\nu & \\ \cos\varphi_0 \cos\lambda_0 dA - \sin\lambda_0 d\mu + \sin\varphi_0 \cos\lambda_0 d\nu & \end{vmatrix} \quad (3.2 - 4)$$

We put

$$M = M_1 + I$$

which gives

$$M_1 = \begin{vmatrix} 0 & \sin\varphi_0 dA - \cos\varphi_0 d\nu \\ -\sin\varphi_0 dA + \cos\varphi_0 d\nu & 0 \\ \cos\varphi_0 \sin\lambda_0 dA + \cos\lambda_0 d\mu + \sin\varphi_0 \sin\lambda_0 d\nu & -\cos\varphi_0 \cos\lambda_0 dA + \sin\lambda_0 d\mu - \sin\varphi_0 \cos\lambda_0 d\nu \\ -\cos\varphi_0 \sin\lambda_0 dA - \cos\lambda_0 d\mu - \sin\varphi_0 \sin\lambda_0 d\nu & \\ \cos\varphi_0 \cos\lambda_0 dA - \sin\lambda_0 d\mu + \sin\varphi_0 \cos\lambda_0 d\nu & \end{vmatrix} \quad (3.2 - 5)$$

and equation (3.2 - 2) becomes

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} + \begin{vmatrix} x \\ y \\ z \end{vmatrix} + M_1 \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} + \epsilon \begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} \quad (3.2 - 6)$$

We also denote

$$\begin{vmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{vmatrix} = \begin{vmatrix} \Delta x \\ \Delta y \\ \Delta z \end{vmatrix}$$

After substitution, equation (3.2 - 6) yields the following three equations:

$$X = x + dx_0 + \epsilon \Delta x \quad (3.2 - 7)$$

$$+ (\sin \varphi_0 \Delta y - \cos \varphi_0 \sin \lambda_0 \Delta z) dA - (\cos \lambda_0 \Delta z) d\mu \\ - (\cos \varphi_0 \Delta y + \sin \varphi_0 \sin \lambda_0 \Delta z) d\nu$$

$$Y = y + dy_0 + \epsilon \Delta y$$

$$+ (-\sin \varphi_0 \Delta x + \cos \varphi_0 \cos \lambda_0 \Delta z) dA - (\sin \lambda_0 \Delta z) d\mu \\ + (\cos \varphi_0 \Delta x + \sin \varphi_0 \cos \lambda_0 \Delta z) d\nu$$

$$Z = z + dz_0 + \epsilon \Delta z$$

$$+ (\cos \varphi_0 \sin \lambda_0 \Delta x - \cos \varphi_0 \cos \lambda_0 \Delta y) dA \\ + (\cos \lambda_0 \Delta x + \sin \lambda_0 \Delta y) d\mu \\ + (\sin \varphi_0 \sin \lambda_0 \Delta x - \sin \varphi_0 \cos \lambda_0 \Delta y) d\nu$$

We want to separate the shifts in two terms. The first  $dx_{01}$  is due to the adopted geodetic coordinates and undulation at the initial point; the second  $dx_{02}$  is due to the change in the parameters of the adopted ellipsoid when the adopted coordinates and undulation at the initial point are kept fixed. Thus

$$\begin{vmatrix} dx_0 \\ dy_0 \\ dz_0 \end{vmatrix} = \begin{vmatrix} dx_{01} \\ dy_{01} \\ dz_{01} \end{vmatrix} + \begin{vmatrix} dx_{02} \\ dy_{02} \\ dz_{02} \end{vmatrix} \quad (3.2 - 8)$$

Since  $dx_{02}$  is the change of the geodetic coordinates due to a change in  $a$  and  $e^2$ , it is given by

$$\begin{vmatrix} dx_{02} \\ dy_{02} \\ dz_{02} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial a} \\ \frac{\partial z}{\partial a} \end{vmatrix} da + \begin{vmatrix} \frac{\partial x}{\partial e^2} \\ \frac{\partial y}{\partial e^2} \\ \frac{\partial z}{\partial e^2} \end{vmatrix} de^2 \quad (3.2 - 9)$$

Evaluating the partial derivatives of equation (3.2 - 1) at the datum origin we have

$$\begin{pmatrix} \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial a} \\ \frac{\partial z}{\partial a} \end{pmatrix} = \begin{pmatrix} \frac{\cos \varphi_0 \cos \lambda_0}{W_0} \\ \frac{\cos \varphi_0 \sin \lambda_0}{W_0} \\ \frac{(1-e^2) \sin \varphi_0}{W_0} \end{pmatrix} \quad (3.2-10)$$

and

$$\begin{pmatrix} \frac{\partial x}{\partial e^2} \\ \frac{\partial y}{\partial e^2} \\ \frac{\partial z}{\partial e^2} \end{pmatrix} = \begin{pmatrix} \frac{a \sin^2 \varphi_0 \cos \varphi_0 \cos \lambda_0}{2W_0^3} \\ \frac{a \sin^2 \varphi_0 \cos \varphi_0 \sin \lambda_0}{2W_0^3} \\ \left( \frac{M_0}{2} \sin^2 \varphi_0 - N_0 \right) \sin \varphi_0 \end{pmatrix} \quad (3.2-11)$$

where

$$W_0 = \sqrt{1 - e^2 \sin^2 \varphi}$$

$M_0, N_0$  are the radii of curvature in the meridian and the prime vertical of the datum ellipsoid at the initial point

Using

$$e^2 = 2f - f^2 \quad (3.2-12)$$

where

$f$  is the flattening of the ellipsoid

we have

$$\frac{\partial e^2}{\partial f} = 2(1 - f) \quad (3.2-13)$$

From [Heiskanen and Moritz, p. 78], we have

$$J_2 = \frac{2}{3} f - \frac{1}{3} m - \frac{1}{3} f^2 + \frac{2}{21} fm \quad (3.2-14)$$

where

$J_2$  is the second-degree zonal harmonic

By differentiation we get

$$dJ_2 = \frac{2}{3} df - \frac{2}{3} fdf + \frac{2}{21} mdf$$

from which

$$df = \frac{3}{2} \left( 1 + f - \frac{m}{7} \right) dJ_2 \quad (3.2-14a)$$

The quantity  $f - \frac{m}{7}$  is less than the flattening and by neglecting it we make an error of the order of the flattening in the correction for the second-degree harmonic. As the correction to the second-degree harmonic is small and as we make an error of the order of the flattening in subsequent models anyway, we can safely neglect this term and use

$$df = \frac{3}{2} dJ_2$$

Substituting  $-C_{20}$  for  $J_2$  in equation (3.2-14), we get

$$df = -\frac{3}{2} dC_{20} \quad (3.2-15)$$

For a fully normalized coefficient the above will be

$$df = -\frac{3}{2} \sqrt{5} d\bar{C}_{20} \quad (3.2-16)$$

We write equation (3.2-9) as

$$dx_{02} = \frac{\partial x}{\partial a} da + \frac{\partial x}{\partial e^2} \frac{\partial e^2}{\partial f} \frac{\partial f}{\partial \bar{C}_{20}} d\bar{C}_{20} \quad (3.2-17)$$

which, after substitution, gives

$$\begin{aligned} dx_{02} &= \frac{\cos \varphi_0 \cos \lambda_0}{W_0} da - \frac{3\sqrt{5} a (1-f) \sin^2 \varphi_0 \cos \varphi_0 \cos \lambda_0}{2W_0^3} d\bar{C}_{20} \\ dy_{02} &= \frac{\cos \varphi_0 \sin \lambda_0}{W_0} da - \frac{3\sqrt{5} a (1-f) \sin^2 \varphi_0 \cos \varphi_0 \cos \lambda_0}{2W_0^3} d\bar{C}_{20} \\ dz_{02} &= \frac{(1-e^2) \sin \varphi_0}{W_0} da - 3\sqrt{5} (1-f) \frac{M_0}{2} \sin^2 \varphi_0 - N_0 \sin \varphi_0 d\bar{C}_{20} \end{aligned} \quad (3.2-17b)$$

Substituting (3.2-17b) into (3.2-7), we have

$$\begin{aligned}
X = x + dx_{01} + (\sin \varphi_0 \Delta y - \cos \varphi_0 \sin \lambda_0 \Delta z) dA & \quad (3.2-18) \\
- (\cos \lambda_0 \Delta z) d\mu - (\cos \varphi_0 \Delta y + \sin \varphi_0 \sin \lambda_0 \Delta z) d\nu \\
+ \frac{\cos \varphi_0 \cos \lambda_0}{w_0} da - \frac{3\sqrt{5} a (1-f) \sin^2 \varphi_0 \cos \varphi_0 \cos \lambda_0}{2W_0^3} d\bar{C}_{20} + \epsilon \Delta x
\end{aligned}$$

$$\begin{aligned}
Y = y + dy_{01} + (-\sin \varphi_0 \Delta x + \cos \varphi_0 \cos \lambda_0 \Delta z) dA - (\sin \lambda_0 \Delta z) d\mu \\
+ (\cos \varphi_0 \Delta x + \sin \varphi_0 \cos \lambda_0 \Delta z) d\nu \\
+ \frac{\cos \varphi_0 \sin \lambda_0}{w_0} da - \frac{3\sqrt{5} a (1-f) \sin^2 \varphi_0 \cos \varphi_0 \cos \lambda_0}{2W_0^3} d\bar{C}_{20} + \epsilon \Delta y
\end{aligned}$$

$$\begin{aligned}
Z = z + dz_{01} + (\cos \varphi_0 \sin \lambda_0 \Delta x - \cos \varphi_0 \cos \lambda_0 \Delta y) dA \\
+ (\cos \lambda_0 \Delta x + \sin \lambda_0 \Delta y) d\mu + (\sin \varphi_0 \sin \lambda_0 \Delta x - \sin \varphi_0 \cos \lambda_0 \Delta y) d\nu \\
+ \frac{(1-e^2) \sin \varphi_0}{w_0} da - 3\sqrt{5} (1-f) \left( \frac{M_0}{2} \sin^2 \varphi_0 - N_0 \right) \sin \varphi_0 d\bar{C}_{20} + \epsilon \Delta z
\end{aligned}$$

Equations (3.2-18) will be used as the mathematical model relating the geocentric and the geodetic coordinates. This mathematical model will be used to form observation equations in which the observed quantities are the rectangular geodetic coordinates given by equation (3.2-1). Although equations (3.2-18) are linear with respect to the parameters involved, it is still necessary to introduce approximate values,  $X^0, Y^0, Z^0$ , of the geocentric coordinates  $X, Y, Z$  and to solve for corrections to the approximations. This will allow the unknown parameters in the triangulation observation equations to be consistent with the unknowns in the other models to be derived.

The corrections  $da$  and  $df$  (or  $d\bar{C}_{20}$ ) are different for each datum because they are the differences between the elements of the average terrestrial ellipsoid and the ellipsoid of the datum in question. In order to make these corrections always to refer to the same approximate values, we select an approximate reference ellipsoid with semidiameter  $a$  and flattening  $f$ , and we change the ellipsoids of all datums to that one. We effect this change by introducing the approximate values



$$da_0 = a - a_1$$

$$df_0 = f - f_1$$

where

$a_1, f_1$  are the semidiameter and flattening of the  $i$ th datum

The unknowns  $da$  and  $df$  are then interpreted as the differences between the average terrestrial ellipsoid and the ellipsoid selected to approximate it.

This allows us to enforce the condition that the semidiameter and flattening of the mean earth ellipsoid computed from data of different datums will be unique.

The observation equations now have the form

$$dX - dx_{01} - (\sin\varphi_0 \Delta y - \cos\varphi_0 \sin\lambda_0 \Delta z) dA + (\cos\lambda_0 \Delta z) d\mu \quad (3.2-19)$$

$$+ (\cos\varphi_0 \Delta y + \sin\varphi_0 \sin\lambda_0 \Delta z) d\nu - \epsilon \Delta x$$

$$- \frac{\cos\varphi_0 \cos\lambda_0}{W_0} da + \frac{3\sqrt{5}a(1-f)\sin^2\varphi_0 \cos\varphi_0 \cos\lambda_0}{2W_0^3} d\bar{C}_{20}$$

$$= x - X^0 + \frac{\cos\varphi_0 \cos\lambda_0}{W_0} da_0 + \frac{a(1-f) \sin^2\varphi_0 \cos\varphi_0 \cos\lambda_0}{W_0^3} df_0 + V_x$$

$$dY - dy_{01} - (-\sin\varphi_0 \Delta x + \cos\varphi_0 \cos\lambda_0 \Delta z) dA + (\sin\lambda_0 \Delta z) d\mu$$

$$- (\cos\varphi_0 \Delta x + \sin\varphi_0 \cos\lambda_0 \Delta z) d\nu - \epsilon \Delta y$$

$$- \frac{\cos\varphi_0 \sin\lambda_0}{W} da + \frac{3\sqrt{5}a(1-f)\sin^2\varphi_0 \cos\varphi_0 \sin\lambda_0}{2W_0^3} d\bar{C}_{20}$$

$$= y - Y^0 + \frac{\cos\varphi_0 \sin\lambda_0}{W_0} da_0 + \frac{a(1-f) \sin^2\varphi_0 \cos\varphi_0 \sin\lambda_0}{W_0^3} df_0 + V_y$$

$$dZ - dz_0 - (\cos\varphi_0 \sin\lambda_0 \Delta x - \cos\varphi_0 \cos\lambda_0 \Delta y) dA$$

$$- (\cos\lambda_0 \Delta x + \sin\lambda_0 \Delta y) d\mu - (\sin\varphi_0 \sin\lambda_0 \Delta x - \sin\varphi_0 \cos\lambda_0 \Delta y) d\nu$$

$$- \epsilon \Delta z - \frac{(1-e^2) \sin\varphi_0}{W_0} da + 3\sqrt{5}(1-f) \frac{M_0}{2} \sin^2\varphi_0 - N_0 \sin\varphi_0 d\bar{C}_{20}$$

$$= z - Z^0 + \frac{(1-e^2) \sin\varphi_0}{W_0} da_0 + 2(1-f) \frac{M_0}{2} \sin^2\varphi_0 - N_0 \sin\varphi_0 df_0 + V_z$$

where

$V_x, V_y, V_z$  are the errors assigned to geodetic coordinates  $x, y, z$

The above observation equations can be easily modified to include the case in which we have known triangulation points on two different datums. The mathematical model for this case will be derived by subtracting equation (3.2 - 6) formed for one point on one datum from the same equation formed for the same point on the second datum. The geocentric coordinates are eliminated and the observation equation is

$$\begin{vmatrix} dx_{O_1} \\ dy_{O_1} \\ dz_{O_1} \end{vmatrix} + M_{11} \begin{vmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \end{vmatrix} + \epsilon_1 \begin{vmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \end{vmatrix} - \begin{vmatrix} dx_{O_j} \\ dy_{O_j} \\ dz_{O_j} \end{vmatrix} - M_{1j} \begin{vmatrix} \Delta x_j \\ \Delta y_j \\ \Delta z_j \end{vmatrix} - \epsilon_j \begin{vmatrix} \Delta x_j \\ \Delta y_j \\ \Delta z_j \end{vmatrix} =$$

$$= \begin{vmatrix} x_j \\ y_j \\ z_j \end{vmatrix} - \begin{vmatrix} x_1 \\ y_1 \\ z_1 \end{vmatrix} + \begin{vmatrix} V_x \\ V_y \\ V_z \end{vmatrix}$$

where the subscripts i and j are denoted the first and second datum respectively.

After the formation of the observation equations, normal equations can be formed. They will be of the form

$$N_1 \begin{vmatrix} dX \\ dx \\ da \\ dC \end{vmatrix} = U_1 \quad (3.2 - 20)$$

where

- dX are corrections to the geocentric coordinates
- dx are datum transformation parameters, namely, the three shifts of the origins, three rotations, and the correction of the scale
- da are corrections to the elements of the reference ellipsoid. In this group we will consider two parameters, the correction to the semidiameter da and to the constant GM. Only the correction to the semidiameter is involved in this particular set of normal equations.
- dC are corrections to the harmonic coefficients. In this case only  $d\bar{C}_{20}$  is involved.

It is immediately seen that these normal equations cannot be solved. The number of unknowns is  $3n + 7d + 2$  where  $n$  is the number of stations and  $d$  is the number of datums. However, the number of equations is  $3n$ . As the number of unknowns is always greater than the number of equations, the matrix of the coefficients of the normal equations will be singular. This should be expected as it is well known that triangulation alone cannot provide geocentric positions. The normal equations are also singular for another reason. Even if we treat the geocentric coordinates as known quantities, thus eliminating the corrections  $dX$  to the geocentric coordinates, and we limit the unknowns to the  $7d$  datum transformation parameters plus  $da$  and  $dC_{20}$  so that there will be more equations than unknowns, the system cannot be solved. This is because we make the adjustment between three-dimensional coordinates where changes of the reference ellipsoid have the same effect as datum shifts and thus cannot be separated by the adjustment. We can see this by examining the coefficients of the shifts and of the changes of the ellipsoid in equations (3.2-18). These coefficients are independent of the station coordinates and thus constant throughout a datum so that the normal equations generated by such observation equations are again singular. From such an adjustment only the total shifts can be recovered, not corrections for the semidiameter and the flattening. Only by combining the above normals with those of another type of information can we have a solution in which the triangulation has its own contribution.

### 3.22 Weighting of the Geodetic Coordinates

As has been already stated, the weights which are going to be applied through all of the system will basically be the inverse of the variance-covariance matrix of the observed quantities. For this case the weight matrix  $P_x$  will be

$$P_x = \Sigma_x^{-1} \quad (3.2-21)$$

where

$\Sigma_x$  is the variance-covariance matrix of the rectangular geodetic coordinates

Our problem then is to determine the variance-covariance matrix  $\Sigma_x$ .

When the geodetic coordinates  $\phi$  and  $\lambda$  are determined by rigorous unified adjustment for every datum, the variance-covariance matrix for  $\phi$  and  $\lambda$  is available. To these the uncertainty of the height above the ellipsoid must be added. The latter depends mainly on the uncertainty of the astrogeodetic undulations and the estimation of that uncertainty will be investigated in the next chapter. It depends also on the uncertainty of the orthometric height. This uncertainty is usually very small and is provided by the adjustment of leveling. If  $\Sigma\phi$  is the variance-covariance matrix of the geodetic coordinates, properly modified to include the covariance of astrogeodetic undulations and orthometric heights,  $\Sigma_x$  will be

$$\Sigma_x = G\Sigma\phi G' \quad (3.2 - 22)$$

where

$G$  is the matrix which transforms the variables involved to the rectangular geodetic system ( $x$ )

In practice we do not have a variance-covariance matrix for  $\phi$  and  $\lambda$  as the triangulation is adjusted in parts. Instead, variances for these are computed by methods such as those described by [Bomford, 1962]. They may also be estimated, as in [Kaula, 1959a], which is based on discussions such as those by [Simmons, 1951], [Whitten, 1952], and [Ross, 1957]. In all these cases the covariance between  $\phi$  and  $\lambda$  of the same point as well as the covariances between different points were considered negligible. Thus the matrix  $\Sigma\phi$  becomes a diagonal matrix and for every point the variance of the horizontal position, in linear units, in the meridian  $V(\phi)$  and the prime vertical  $V(\lambda)$  are taken equal. Thus the  $\Sigma\phi$  for every point is

$$\Sigma\phi = \begin{vmatrix} V(\phi) & 0 & 0 \\ 0 & V(\lambda) & 0 \\ 0 & 0 & V(h) \end{vmatrix} \quad (3.2 - 23)$$

and the corresponding error ellipsoid is an ellipsoid of revolution with the one

axis in the direction of the normal. If the geodetic coordinates of a point are  $\varphi$  and  $\lambda$  (Fig. 3.2-4), then the G matrix with a spherical approximation will be

$$G = R_3(-\lambda) R_2[-(90 - \varphi)] \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad (3.2 - 24)$$

or

$$G = \begin{vmatrix} \cos \lambda & -\sin \lambda & 0 \\ \sin \lambda & \cos \lambda & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \sin \varphi & 0 & \cos \varphi \\ 0 & 1 & 0 \\ -\cos \varphi & 0 & \sin \varphi \end{vmatrix} \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

or

$$G = \begin{vmatrix} -\sin \varphi \cos \lambda & -\sin \lambda & \cos \varphi \cos \lambda \\ -\sin \varphi \sin \lambda & \cos \lambda & \cos \varphi \sin \lambda \\ \cos \varphi & 0 & \sin \varphi \end{vmatrix} \quad (3.2 - 25)$$

Finally, the terms of the  $\Sigma_x$  of one point are

$$\begin{aligned} S_{11} &= \sin^2 \varphi \cos^2 \lambda V(\varphi) + \sin^2 \lambda V(\lambda) + \cos^2 \varphi \cos^2 \lambda V(h) \\ S_{12} &= \sin^2 \varphi \cos \lambda \sin \lambda V(\varphi) - \sin \lambda \cos \lambda V(\lambda) + \cos^2 \varphi \cos \lambda \sin \lambda V(h) \\ S_{13} &= -\sin \varphi \cos \varphi \cos \lambda V(\varphi) + \sin \varphi \cos \varphi \cos \lambda V(h) \\ S_{22} &= \sin^2 \varphi \sin^2 \lambda V(\varphi) + \cos^2 \lambda V(\lambda) + \sin^2 \lambda \cos^2 \varphi V(h) \\ S_{23} &= -\sin \varphi \cos \varphi \sin \lambda V(\varphi) + \cos \varphi \sin \varphi \sin \lambda V(h) \\ S_{33} &= \cos^2 \varphi V(\varphi) + \sin^2 \varphi V(h) \end{aligned} \quad (3.2 - 26)$$

Thus we see that even if we neglect the correlation between  $\varphi, \lambda, h$  of a point, of between different points, there is still correlation between the rectangular geodetic coordinates, and the whole  $\Sigma_x$  matrix is composed of  $3 \times 3$  submatrices along the diagonal.

For an approximate estimation of the horizontal uncertainty of a triangulation point, Simmons' equations has been used [Brown, 1968, p. 22]. This equation [Simmons, 1951] can be written

$$\sigma = \frac{k^{2/3}}{23.4} \quad (3.2 - 27)$$

where

$\sigma$  is the standard deviation of a geodetic position on the horizontal plane in meters. It is the same in all directions.

$k$  is the distance from the datum origin in kilometers

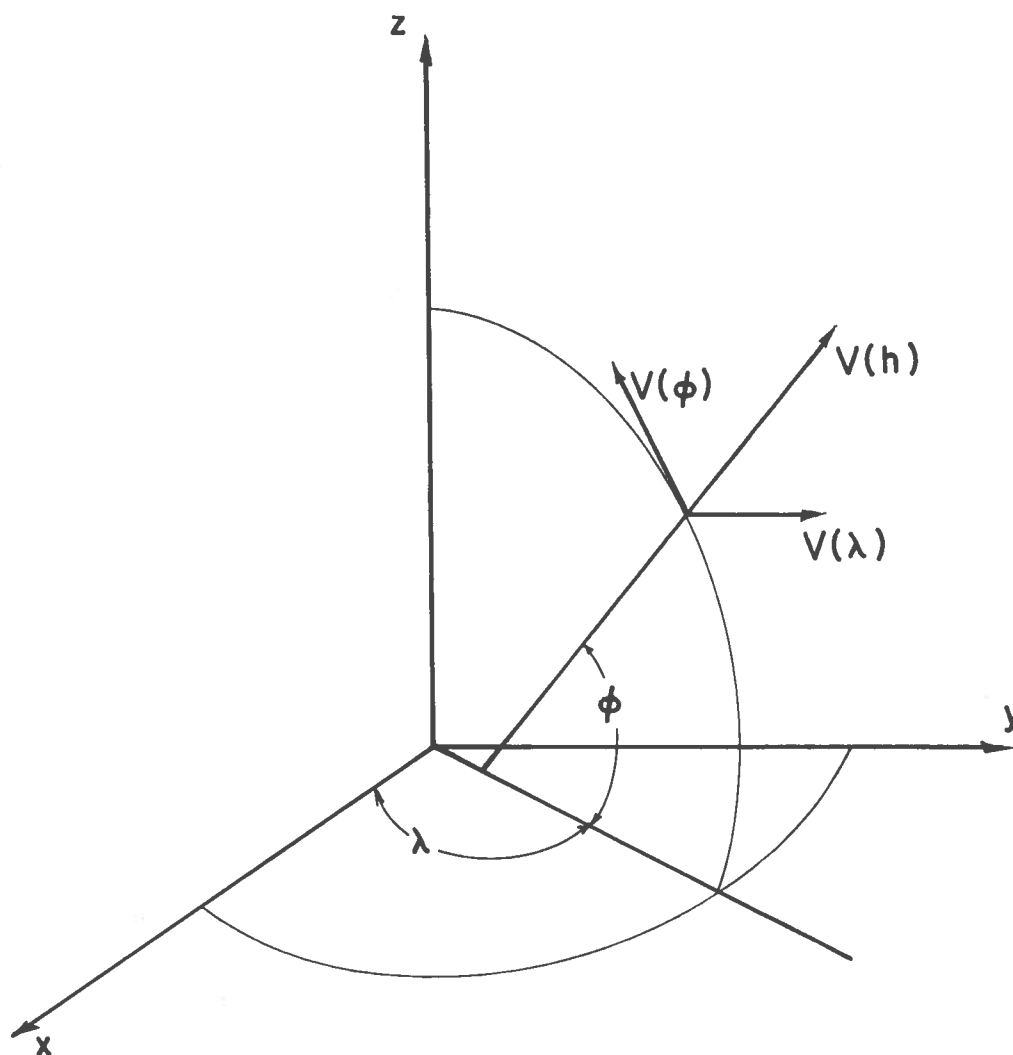


Fig. 3.2-4      Uncertainties of the triangulation and geodetic coordinate system

### 3.3 Gravity Observations

The terrestrial gravity observations will be used in the form of gravity anomalies. This information will be used for determination of spherical harmonic coefficients and geocentric coordinates. The spherical harmonic coefficients will be determined by least squares from the mean gravity anomalies and the geocentric positions will be determined from gravimetric deflections of the vertical, computed by Vening-Meinesz's equation [Heiskanen and Moritz, 1967] from the gravity anomalies.

#### 3.31 Gravity Anomalies

##### 3.311 Modeling the gravity anomalies

The gravimetric observations in the form of gravity anomalies  $\Delta g$  will provide valuable information in the determination of the harmonic coefficients of the gravity field. For such a determination the conventional method, which involves summation of gravity anomalies over the earth's surface which is considered to be a sphere, has been applied by various investigators. The method of estimation by least squares has also been used. According to the first method, the harmonic coefficients are given in general by

$$\bar{a}_{nm} = \frac{1}{4\pi} \iint_{\sigma} f(\theta, \lambda) \bar{R}_{nm}(\theta, \lambda) d\sigma \quad (3.3-1)$$

$$\bar{b}_{nm} = \frac{1}{4\pi} \iint_{\sigma} f(\theta, \lambda) \bar{S}_{nm}(\theta, \lambda) d\sigma$$

where

$f(\theta, \lambda)$  is an arbitrary function on the sphere  $\sigma$  which is to be expanded in spherical harmonics

$\bar{a}_{nm}, \bar{b}_{nm}$  are fully normalized harmonic coefficients of  $n$  degree and  $m$  order

$$\bar{R}_{nm}(\theta, \lambda) = \bar{P}_{nm}(\cos \theta) \cos m \lambda$$

$$\bar{S}_{nm}(\theta, \lambda) = \bar{P}_{nm}(\cos \theta) \sin m \lambda$$

and

$\bar{P}_{nm}$  is the fully normalized Legendre function of degree  $n$  and order  $m$

$$\theta = 90^\circ - \varphi$$

Thus the coefficients are the average products of the values of the function  $f(\theta, \lambda)$  and the corresponding harmonics  $\bar{R}_{nm}$  or  $\bar{S}_{nm}$ . To perform this integration we need to have the function, or values of it, all over the sphere; this means complete coverage of the earth with gravity observations or with estimates of gravity anomalies. In addition we can see that equation (3.3-1) can be derived from a least squares fit of the harmonic series to the function  $f(\theta, \lambda)$  [Parzen, 1967]. To understand this, consider uncorrelated values of  $f(\theta, \lambda)$  with variances  $v(\theta, \lambda)$  all over the sphere  $\sigma$ . The least squares minimization will be

$$\Phi = \iint_{\sigma} v^{-1}(\theta, \lambda) \left[ f(\theta, \lambda) - \sum_{n=0}^{\infty} \sum_{m=0}^n (\bar{a}_{nm} \bar{R}_{nm}(\theta, \lambda) + \bar{b}_{nm} \bar{S}_{nm}(\theta, \lambda)) \right]^2 d\sigma = \min \quad (3.3-2)$$

The normal equations for the harmonic coefficients will be found by equating to zero the partial derivatives of the function  $\Phi$  with respect to the harmonic coefficients. Thus we have

$$-\frac{\partial \Phi}{2 \partial \bar{a}_{nm_1}} = \iint_{\sigma} v^{-1}(\theta, \lambda) \left[ f(\theta, \lambda) - \sum_{n=0}^{\infty} \sum_{m=0}^n (\bar{a}_{nm} \bar{R}_{nm}(\theta, \lambda) + \bar{b}_{nm} \bar{S}_{nm}(\theta, \lambda)) \right] \bar{R}_{nm_1}(\theta, \lambda) d\sigma = 0 \quad (3.3-3)$$

$$-\frac{\partial \Phi}{2 \partial \bar{b}_{nm_1}} = \iint_{\sigma} v^{-1}(\theta, \lambda) \left[ f(\theta, \lambda) - \sum_{n=0}^{\infty} \sum_{m=0}^n (\bar{a}_{nm} \bar{R}_{nm}(\theta, \lambda) + \bar{b}_{nm} \bar{S}_{nm}(\theta, \lambda)) \right] \bar{S}_{nm_1}(\theta, \lambda) d\sigma = 0$$

where the solution is

$$\bar{a}_{nm_1} = \frac{1}{\iint_{\sigma} v^{-1}(\varphi, \lambda) \bar{R}_{nm_1}^2(\varphi, \lambda) d\sigma} \iint_{\sigma} v^{-1}(\theta, \lambda) f(\theta, \lambda) \bar{R}_{nm_1}(\theta, \lambda) d\sigma \quad (3.3-4)$$



$$\begin{aligned}
& - \sum_{n=0}^{\infty} \sum_{m=0}^n \bar{a}_{nm} \int \int_{\sigma} v^{-1}(\theta, \lambda) \bar{R}_{nm}(\theta, \lambda) \bar{R}_{nm1}(\theta, \lambda) d\sigma + \bar{a}_{nm1} \int \int_{\sigma} v^{-1}(\theta, \lambda) \bar{R}_{nm1}^2 d\sigma \\
& - \sum_{n=0}^{\infty} \sum_{m=0}^n \bar{b}_{nm} \int \int_{\sigma} v^{-1}(\theta, \lambda) \bar{S}_{nm}(\theta, \lambda) \bar{R}_{nm1}(\theta, \lambda) d\sigma \Big] \\
\bar{b}_{nm1} = & \frac{1}{\int \int_{\sigma} v^{-1}(\theta, \lambda) \bar{S}_{nm1}^2(\theta, \lambda) d\sigma} \left[ \int \int_{\sigma} v^{-1}(\theta, \lambda) f(\theta, \lambda) \bar{S}_{nm1}(\theta, \lambda) d\sigma \right. \\
& - \sum_{n=0}^{\infty} \sum_{m=0}^n \bar{a}_{nm} \int \int_{\sigma} v^{-1}(\theta, \lambda) \bar{R}_{nm}(\theta, \lambda) \bar{S}_{nm1}(\theta, \lambda) d\sigma \\
& \left. - \sum_{n=0}^{\infty} \sum_{m=0}^n \bar{b}_{nm} \int \int_{\sigma} v^{-1}(\theta, \lambda) \bar{S}_{nm}(\theta, \lambda) \bar{S}_{nm1}(\theta, \lambda) d\sigma + \bar{b}_{nm1} \int \int_{\sigma} v^{-1}(\theta, \lambda) \bar{S}_{nm1}^2(\theta, \lambda) d\sigma \right]
\end{aligned}$$

When  $v(\theta, \lambda)$  is constant all over the sphere, equations (3.3-4) reduce to equations (3.3-1) with the help of the orthogonal relations [Heiskanen and Moritz, 1967, pp. 29-30]

$$\begin{aligned}
\int \int_{\sigma} \bar{R}_{nm}(\theta, \lambda) \bar{R}_{nm1}(\theta, \lambda) d\sigma &= 0 \\
\int \int_{\sigma} \bar{S}_{nm}(\theta, \lambda) \bar{S}_{nm1}(\theta, \lambda) d\sigma &= 0 \\
\int \int_{\sigma} \bar{R}_{nm}(\theta, \lambda) \bar{S}_{nm1}(\theta, \lambda) d\sigma &= 0
\end{aligned} \tag{3.3-5}$$

and the property

$$\int \int_{\sigma} \bar{R}_{nm}^2(\theta, \lambda) d\sigma = \int \int_{\sigma} \bar{S}_{nm}^2(\theta, \lambda) d\sigma = 4\pi$$

Since the existing gravity data does not cover the entire earth and since the gravity anomalies do not have the same variance, a least squares estimation is preferable as indicated by Rapp in an earlier report [Rapp, 1968].

In another paper the same investigator also gave the following equation relating gravity anomalies and harmonic coefficients [Rapp, 1967b].

$$\Delta g_T = - \frac{GM}{r^2} + \frac{2(W_0 - U_0)}{r} + \frac{GM}{r^2} \sum_{n=2}^{\infty} (n-1) \left[ \frac{a_0}{r} \right]^n \sum_{m=0}^n (\bar{C}_{nm}^* \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\mu) \quad (3.3-6)$$

where

$\Delta g_T$  is a terrestrial gravity anomaly

$\bar{C}_{nm}, \bar{S}_{nm}$  are the fully normalized harmonic coefficients of the gravity field

\* denotes the difference between the actual and the reference fields

$\bar{P}_{nm}(\mu)$  is the fully normalized Legendre polynomial of degree n and order m

GM is the gravitational constant times the mass of the earth

$\Delta GM$  is the difference between the constant GM of the geoid and the reference ellipsoid

$a_0$  is the equatorial radius of the reference ellipsoid

r is the radius vector from the origin of the coordinate system to the point in question

$W_0$  is the potential of the geoid

$U_0$  is the potential at the surface of the reference ellipsoid

By making the approximations

$$\frac{GM}{r^2} = \bar{\gamma}$$

$$\frac{a_0}{r} = 1$$

$$\Delta g_0 = \frac{2(W_0 - U_0)}{r} - \frac{\Delta GM}{r^2}$$

and by carrying the summation up to  $N_{max}$  degree, equation (3.3 - 6) becomes

$$\Delta g_T = \Delta g_0 + \bar{\gamma} \sum_{n=0}^{N_{max}} (n-1) \sum_{m=0}^n (\bar{C}_{nm}^* \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\mu) \quad (3.3 - 7)$$

The gravity anomalies considered in this equation are free-air gravity anomalies on a sphere of radius  $a_e$ . This is the extended mathematical formula given by Rapp for the determination of the harmonic coefficients and the zero-degree harmonic of the anomalies from terrestrial gravity anomalies. As the gravity anomalies are determined on the surface of the earth, some corrections and some assumptions must be made. If the gravity anomalies were referred to sea level, then the substitution of the reference ellipsoid, by a sphere, would have produced an error proportional to the flattening (spherical approximation) in the determination of the harmonic coefficients through equation (3.3 - 6) [Moritz, 1967a, Section 9]. Pellinen and Ostrach showed that this error actually increases with the degree of the determined coefficient. The proportional error is  $n \cdot f$ , where  $n$  is the degree and  $f$  is the flattening [Ostrach and Pellinen, 1966]. For  $n = 15$ , the above error amounts to 5 percent of the value of the coefficient.

The proportional error of a harmonic coefficient from a purely terrestrial solution with the present data is [Rapp, 1968a]

$$\frac{s}{\sigma} = 0.055 (n + 1) \quad (3.3 - 8)$$

where

- $s$  is the standard error of a harmonic coefficient of order  $n$
- $\sigma$  is the RMS coefficient variation

For  $n = 15$ , this equation gives an estimated error for a harmonic coefficient of 83 percent of its value. Thus with the present gravimetric data and the degree of expansion of the geopotential for which we are solving, the substitution of the ellipsoid by a sphere is permissible. In general the above error may

even be below the limit of round-off errors introduced by the computer and can safely be neglected. Accepting this approximation for the gravity anomalies we bypass the question of divergence of the harmonic series on the surface of the earth, about which contradictory opinions exist in the literature [Molodensky et al., 1962; Heiskanen and Moritz, 1967; Morrison, 1967; Brovar, 1961; Hirvonen, 1960; Moritz, 1968; Molodensky et al., 1962a; Köhnlein, 1966].

Although we can replace a sphere with the mean sea level, we cannot replace it with the physical surface because the inclination of the terrain is not negligible [Moritz, 1967a; Molodensky et al., 1962a]. This substitution would affect the estimate of the harmonic coefficient according to Pellinen by 15 to 20 percent [Pellinen, 1962]. Thus a reduction of the gravity anomaly to the mean sea level or a regularization must be made.

The free-air gravity anomaly reduced to sea level  $\Delta g^*$  is related to the surface free-air anomaly by [Moritz, 1967a]

$$\Delta g^* = \Delta g - \frac{\partial \Delta g}{\partial h} h \quad (3.3-9)$$

where

$\Delta g^*$  is the gravity anomaly at sea level

$h$  is the elevation of the point where  $\Delta g$  has been observed

In this equation higher-order terms have been neglected. The vertical gradient at a point P can be measured directly or computed by

$$\frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_p}{\ell_o^3} d\sigma \quad (3.3-10)$$

where  $\ell_o$  is the distance of the anomaly  $\Delta g$  from the surface element  $d\sigma$ .

Instead of an analytical continuation to some level surface, we may use the Molodensky correction  $G_1$ . In this case

$$\Delta g^* = \Delta g + G_1 \quad (3.3-11)$$

with

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{\ell_o^3} \Delta g d\sigma \quad (3.3-12)$$

This correction is a planar approximation of the actual correction and it holds only for low-degree harmonics. For a better approximation the above correction, as has been modified by Pellinen, must be applied [Pellinen, 1964]. The first term of this correction, which is adequate for most practical purposes, is

$$G' = \frac{R^2}{4\pi} \iint_{\sigma} \frac{(h - h_p)(\Delta g - \Delta g_p)}{l_0^3} d\sigma \quad (3.3-13)$$

This form also has another advantage; using the assumption that there is a linear relation between gravity anomalies and elevations, the above correction may be shown to be essentially identical with the conventional terrain correction for deviation from the Bouguer plate. Thus it can be computed by the topography alone as follows [Moritz, 1967a; Rapp, 1967a]

$$G' = \frac{1}{2} k \delta R^2 \iint_{\sigma} \frac{(h - h_p)^2}{l_0^3} d\sigma \quad (3.3-14)$$

where

$\delta$  is the density

$k$  is the gravitational constant

The other change in the correct equation (3.3-7), that is the truncation of the series, could become a serious error if not properly handled. It is known that because of the orthogonality between the spherical harmonics of different orders the truncation does not have any effect when there is a complete coverage of gravity anomalies with the same variance. However, when such complete coverage does not exist it introduces a systematic error which cannot be avoided will be examined later.

Because the coefficient  $\bar{C}_{20}$  is  $10^3$  times the size of most of the other coefficients and because in forming the gravity anomalies nominal values for  $\bar{C}_{20}$ ,  $\bar{C}_{40}$  have been used, it is both convenient and numerically more desirable to solve for corrections to these coefficients than for their whole values. This is not strictly required by the adjustment however, as the mathematical model

(3.3 - 7) is linear with respect to the coefficients. The gravity anomalies which will be used in equation (3.3 - 7) can be given with respect to any ellipsoid provided that a parameter  $\Delta g_0$  and an approximate value for  $d\bar{C}_{20}$  are introduced for each different ellipsoid.

As has been pointed out by Kaula, the observed gravity anomalies at discrete points cannot be directly used for the determination of spherical harmonic coefficients [Kaula, 1959]. Because of the nonuniform distribution and incomplete coverage, the effect on the higher-order terms is such that the results are so erratic as to be useless. Instead a smoothing of the observed anomalies has been applied. This smoothing consists of some kind of averaging of the anomalies, usually in squares of different sizes bordered by meridians and parallels, or sometimes in equal-area squares. The selection of the size of the block and, consequently, the selection of the method of smoothing depends on the maximum degree of the coefficients carried in the solution. Rapp, using two criteria, determined the size of the block for which the mean anomaly may be represented by a given set of coefficients of degree  $n$ . Inversely, this is the size of the block where mean anomalies must be computed for a better determination of a set of coefficients of degree  $n$ . Table 3.3 - 1 gives the values of Rapp based on the half-wave length criterion and the number of parameters to be solved criterion [Rapp, 1967b].

From Table 3.3 - 1 we see that for an extension of the gravity anomalies to the 14th degree, a smoothing in  $13^\circ \times 13^\circ$  blocks is recommended. However, the estimation of mean anomalies and their variances for larger blocks becomes more uncertain. Therefore, a comparison must be made to determine whether smaller errors will be provided by estimating mean values in larger areas, or by using smaller areas and accepting some effect from the higher-order terms. For an extension to the 14th degree, sizes of  $5^\circ \times 5^\circ$  and  $10^\circ \times 10^\circ$  have been tried. The degree up to which the present solution can go will be examined in Chapter 4. The effect of this smoothing on the estimated harmonic coefficients has been studied by Kaula [1959] and Pellinen [1966].

Table 3.3 - 1  
Block Size from an Expansion to Degree n

n	From Half-wave Length	From Number of Parameters	n	From Half-wave Length	From Number of Parameters
5	$36^\circ$	$33.8$	22	$8.2$	$8.8$
8	22.5	22.6	24	7.5	8.1
9	20	20.3	26	6.9	7.5
10	18	18.4	28	6.4	7.0
12	15	15.6	30	6.0	6.5
14	12.8	13.5	32	5.6	6.1
16	11.2	11.8	34	5.3	5.8
18	10.0	10.6	36	5.0	5.5
20	9.0	9.7			

Many methods have been tried or suggested for smoothing of the anomalies, e.g., simple averaging of the anomalies, detailed development in spherical harmonics within each square, etc. A system which provides sufficiently accurate mean gravity anomalies in  $1^\circ \times 1^\circ$  blocks and which is practical and possible has been suggested by Uotila [1967a]. In his method Uotila estimates mean Bouguer anomalies  $\Delta g_j^B$  using preselected b-values and the mean elevation of the observation stations in  $5' \times 5'$  sub-blocks. Then a plane which best fits the mean Bouguer anomalies of the  $5' \times 5'$  sub-blocks is determined by using the model

$$\Delta g_j^B = x_1 + \Delta \varphi_j x_2 + \Delta \lambda_j x_3 \quad (3.3-15)$$

where

$\Delta g_j^B$	is the mean Bouguer anomaly in a $5' \times 5'$ sub-block
$\Delta \varphi_j$	$= \varphi_m - \varphi_j$
$\Delta \lambda_j$	$= \lambda_m - \lambda_j$
$\varphi_m$	is the latitude of the central parallel of the $1^\circ \times 1^\circ$ block in question
$\lambda_m$	is the longitude of the central meridian of the $1^\circ \times 1^\circ$ block
$\varphi_j$	is the latitude of the central parallel of the sub-block j
$\lambda_j$	is the longitude of the central meridian of the sub-block j
$x_1, x_2, x_3$	are the parameters to be determined

Finally, the mean anomaly of a  $1^\circ \times 1^\circ$  block is computed by the equation

$$\Delta \bar{g} = \frac{1}{S} GX + \Delta g_m^B + bh_m \quad (3.3-16)$$

where

S	is the total number of $5' \times 5'$ sub-blocks
G	$= \left[ k; \sum_{j=1}^k \Delta \varphi_j; \sum_{j=1}^k \Delta \lambda_j \right]$
k	is the number of observed $5' \times 5'$ sub-blocks
X	$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$



$$\Delta g_m^B = \frac{1}{k} \sum_{j=1}^k \Delta g_j^B$$

$$\Delta g_j^B = \frac{1}{n} \sum_{i=1}^n \Delta g_i^B$$

$n$  is the number of observations in the  $j$  sub-block

$b$  is the Bouguer coefficient computed by the local data or constant

$h_m$  is the mean elevation of the  $1^\circ \times 1^\circ$  block

Mean anomalies in  $5^\circ \times 5^\circ$ , or any other size, block are estimated by methods developed by Kaula and Moritz [Kaula, 1959; Heiskanen and Moritz, 1967]. The gravity anomalies have been treated as a stochastic phenomenon on the earth's surface described by a stationary stochastic process. An estimated mean gravity anomaly is given in general by

$$\Delta \tilde{g} = \sum_{i=1}^n a_i \Delta \bar{g}_i$$

where

$\Delta \bar{g}$  are observed anomalies and, in this case, mean anomalies in  $1^\circ \times 1^\circ$  square

$a_i$  is a coefficient which depends on the relative position of the  $i$  square in the  $5^\circ \times 5^\circ$  block

Different methods of selection of the coefficients result in different interpolation methods. Although selection according to least squares principle seems more accurate, the amount of work involved, especially on a world-wide scale, led to the design of simpler expressions. When the mean values in the  $1^\circ \times 1^\circ$  squares are computed by simple averaging of the observed values, the mean anomaly of the  $5^\circ \times 5^\circ$  square is computed by simple averaging and

$$a_1 = a_2 = \dots a_n = \frac{1}{n}$$

where

$n$  is the number of the observed  $1^\circ \times 1^\circ$  squares.

When the mean values in the  $1^\circ \times 1^\circ$  blocks have been computed by Uotila's method, which also furnishes an estimate for the variance of the anomaly, the

mean anomaly for the  $5^\circ \times 5^\circ$  block will be the weighted mean of the anomalies in the smaller blocks. The weights will be the inverses of the estimated variances so that

$$\Delta \bar{g} = \frac{\sum_{i=1}^n p_i \Delta \bar{g}_i}{\sum_{i=1}^n p_i} \quad (3.3-17)$$

and

$$a_i = \frac{p_i}{\sum_{i=1}^n p_i} \quad (3.3-18)$$

After the computation of mean gravity anomalies, observation equations can be formed. The coefficients of the observation equations are

$$\begin{aligned} \bar{\gamma} (n-1) \cos m\lambda \bar{P}_{nm} & \text{ for } \bar{C}_{nm}^* \\ \bar{\gamma} (n-1) \sin m\lambda \bar{P}_{nm} & \text{ for } \bar{S}_{nm} \\ 1 & \text{ for } \Delta g_o \end{aligned}$$

The normal equations arising from the terrestrial gravity observations will have the form

$$N_2 \begin{vmatrix} \Delta g_o \\ dC \end{vmatrix} = U_2 \quad (3.3-19)$$

where

$\Delta g_o$  is the zero-degree harmonic of the gravity anomalies

$dC$  are corrections to the harmonic coefficients

By keeping the zero-degree harmonic in our system, the solution for the equatorial gravity will be consistent with the values of the coefficients which come out of the solution.

### 3.312 Various gravity estimates for the unsurveyed areas

For a reliable expansion of the gravity field in spherical harmonics, especially when based only on terrestrial gravity observations, a complete coverage of the earth's surface with gravity estimates is necessary. As the gravity observations cover only a small part of the earth's surface, estimates of the gravity anomalies for the unsurveyed areas have been attempted by various methods. This increase of the coverage has been effected by increasing the size of the squares where mean values were estimated [Jeffreys, 1943], by statistical predictions of the gravity anomalies in the unsurveyed areas [Hirvonen, 1956; Kaula, 1959], by predictions based on geological and geophysical evidence [Woollard, 1966], and by applying isostatic theories [Uotila, 1964; Kivioja, 1966].

The first method has only a phenomenological advantage and can be applied only for very low-degree harmonics, since the mean values for the new large blocks, where now the proportion of the observed area is much smaller than before, are estimated with large uncertainties. Consequently, the harmonic coefficients will have larger variances.

The statistical predictions are based on the study of the behavior of the available terrestrial gravity data. Although they are very valuable for the study of the gravity field itself, they present two difficulties in a world-wide application. First, in order to apply the method correctly, we need the covariance function  $C(s)$ . But to find this we must know the gravity field all over the world [Kaula, 1963]. Second, although the method gives very good results for short distances, it cannot be applied very successfully to fill in the unsurveyed areas which are far from observed regions.

The principal defect of predictions based on geological evidence is that in those areas where gravity observations do not exist geological maps are also inadequate.

The prediction of anomalies based on isostatic models [Prey, 1922; Jung, 1962] gives some information which is considered important for

higher-order harmonics. In general all these methods give predicted gravity anomalies with a standard error of around 20 mgals. Although there is some evidence that the inclusion of predicted anomalies for the unsurveyed areas contributes to the determination of the harmonic coefficients [Rapp, 1968c], there is no clear agreement on how much this contribution is for every type of prediction. The question of the contribution of the predictions for the unsurveyed areas will be investigated in a later chapter.

### 3.313 Weighting of gravity anomalies

To weight the gravity anomalies we need the variance of the mean anomalies in the  $5^\circ \times 5^\circ$  blocks.

We start from the variance of the  $1^\circ \times 1^\circ$  means which can be computed as follows [Uotila, 1967a]

$$\text{Var} (\Delta \bar{g}) = G \Sigma_x G' \quad (3.3 - 20)$$

where

$$\Sigma_x = m_0^{*2} N^{-1}$$

and

$m_0^{*2}$  is obtained through the adjustment if there are enough observations or it is computed from a large number of samples.

These variances are used in the computation of mean gravity anomalies and their variances in the  $5^\circ \times 5^\circ$  blocks. First, the variance  $V \Delta \tilde{g}$  of the  $5^\circ \times 5^\circ$  mean due to errors of the  $1^\circ \times 1^\circ$  blocks is computed by quadratic propagation of the errors of  $1^\circ \times 1^\circ$ . We have

$$V \Delta \tilde{g} = \sum_{i=1}^n a_{pi}^2 \text{Var} (\Delta \bar{g})_i$$

Next, the variance  $\text{Var} (\Delta \tilde{g})$  due to the determination of the mean from discrete gravity observations is computed. We have from [Heiskanen and Moritz, 1967]

$$\text{Var} (\Delta \tilde{g}) = C_0 - 2 \sum_{i=1}^n a_{pi} C_{pi} + \sum_{i=1}^n \sum_{k=1}^n a_{pi} a_{pk} C_{ik} \quad (3.3 - 21)$$

where

$$C_o = M \{ \Delta g_p^2 \}$$

$$C_{p1} = M \{ \Delta g_p \Delta g_1 \}$$

$$C_{1k} = M \{ \Delta g_1 \Delta g_k \}$$

$$a_{p1} = \frac{p_1}{[p]}$$

with

$M$  meaning average over the whole sphere

$p_1$  weights

Even if the gravity anomalies have the same variance, we still have to weight our observations because of the difference between the area of a square of  $5^\circ \times 5^\circ$  at the equator and at a latitude  $\varphi$ . Because the area of the  $5^\circ \times 5^\circ$ , or other, block bordered by meridians and parallels, changes with the latitude as  $\cos \varphi$ , the expression for the weight of a mean gravity anomaly will be

$$p = \frac{\cos \varphi}{\text{Var} (\Delta \tilde{g}) + V \Delta \tilde{g}} \quad (3.3-22)$$

We can also form estimates for the variances of the predicted gravity anomalies by comparing the predicted values with known values. From such a comparison Rapp estimates that the standard error of a mean model anomaly of a  $5^\circ \times 5^\circ$  block is of the order of 20 mgals. The accuracy of most of the statistical predictions depends usually on the distance of the block where mean anomaly is to be estimated from the known mean values on which the prediction is to be based; thus it cannot be described by one number.

### 3.32 Gravimetric Deflections

#### 3.321 Modeling the gravimetric deflections of the vertical

The observation equations for the deflections of the vertical at one point will be derived from the condition that the gravimetric and the astrogeodetic deflections, corrected for the terrain effect, must be equal. With a spherical approximation this is expressed as [Heiskanen and Moritz, 1967, pp. 213-214]

$$\begin{aligned}\xi^a - \xi^g &= \xi' - \xi - \frac{1}{R} (\sin\varphi \cos\lambda \, dx_0 + \sin\varphi \sin\lambda \, dy_0 - \cos\varphi \, dz_0) \\ \eta^a - \eta^g &= \eta' - \eta - \frac{1}{R} (\sin\lambda \, dx_0 - \cos\lambda \, dy_0)\end{aligned}\tag{3.3-23}$$

where

$\xi^a, \eta^a$  are the adjusted terrain-corrected astrogeodetic deflections of the vertical in the meridian and the prime vertical planes respectively

$\xi^g, \eta^g$  are the adjusted gravimetric deflections

$\xi', \eta'$  are the observed terrain-corrected astrogeodetic deflections

$\xi, \eta$  are the observed gravimetric deflections

$dx_0, dy_0, dz_0$  are the total shifts of the origin of the center of the ellipsoid used for the computation of the astrogeodetic deflections with respect to the center of mass

These equations in a more precise form as given by Molodensky [Molodensky et al., 1962, p. 15] are the following:

$$\begin{aligned}\xi^a - \xi^g &= \xi' - \xi - \frac{1}{M} (\sin\varphi \cos\lambda \, dx_0 + \sin\varphi \sin\lambda \, dy_0 - \cos\varphi \, dz_0) \\ \eta^a - \eta^g &= \eta' - \eta - \frac{1}{N} (\sin\lambda \, dx_0 - \cos\lambda \, dy_0)\end{aligned}\tag{3.3-24}$$

The total shift due to the orientation, rotations, size, and shape of the ellipsoid is derived from equation (3.2-18) as

$$\begin{aligned}
dx_o &= dx_{o1} + (\sin\varphi_o \Delta y - \cos\varphi_o \sin\lambda_o \Delta z) dA \\
&\quad - (\cos\lambda_o \Delta z) d\mu - (\cos\varphi_o \Delta y + \sin\varphi_o \sin\lambda_o \Delta z) d\nu \\
&\quad + \frac{\cos\varphi_o \cos\lambda_o}{w_o} da - \frac{3a(1-f)\sin^2\varphi_o \cos\varphi_o \cos\lambda_o}{2w_o^3} d\bar{C}_\infty \\
dy_o &= dy_{o1} + (-\sin\varphi_o \Delta x + \cos\varphi_o \cos\lambda_o \Delta z) dA - (\sin\lambda_o \Delta z) d\mu \\
&\quad + (\cos\varphi_o \Delta x + \sin\varphi_o \cos\lambda_o \Delta z) d\nu \\
&\quad + \frac{\cos\varphi_o \sin\lambda_o}{w_o} da - \frac{3a(1-f)\sin^2\varphi_o \cos\varphi_o \sin\lambda_o}{2w_o^3} d\bar{C}_\infty
\end{aligned} \tag{3.3-25}$$

$$\begin{aligned}
dz_o &= dz_{o1} + (\cos\varphi_o \sin\lambda_o \Delta x - \cos\varphi_o \cos\lambda_o \Delta y) dA \\
&\quad + (\cos\lambda_o \Delta x + \sin\lambda_o \Delta y) d\mu + (\sin\varphi_o \sin\lambda_o \Delta x - \sin\varphi_o \cos\lambda_o \Delta y) d\nu \\
&\quad + \frac{(1-e^2)\sin\varphi_o}{w_o} da - 3(1-f) \left[ \frac{M_o}{2} \sin^2\varphi_o - N_o \right] \sin\varphi_o d\bar{C}_\infty
\end{aligned}$$

Substituting (3.3-25) into (3.3-24), we have

$$\begin{aligned}
\xi^a - \xi^e &= \xi' - \xi - \frac{1}{M} \left[ \sin\varphi \cos\lambda dx_{o1} + \sin\varphi \sin\lambda dy_{o1} - \cos\varphi dz_{o1} \right. \\
&\quad + [\sin\varphi \cos\varphi_o \sin(\lambda - \lambda_o) \Delta z - \sin\varphi \sin\varphi_o (+\sin\lambda \Delta x - \cos\lambda \Delta y) \\
&\quad - \cos\varphi \cos\varphi_o (\sin\lambda_o \Delta x - \cos\lambda_o \Delta y)] dA \\
&\quad - [\sin\varphi \cos(\lambda - \lambda_o) \Delta z + \cos\varphi (\cos\lambda_o \Delta x + \sin\lambda_o \Delta y)] d\mu \\
&\quad + [\sin\varphi \sin\varphi_o \sin(\lambda - \lambda_o) \Delta z + \sin\varphi \cos\varphi_o (\sin\lambda \Delta x - \cos\lambda \Delta y) \\
&\quad \quad - \cos\varphi \sin\varphi_o (\sin\lambda_o \Delta x - \cos\lambda_o \Delta y)] d\nu \\
&\quad + \frac{1}{w_o} [\sin\varphi \cos\varphi_o \cos(\lambda - \lambda_o) - (1 - e^2) \sin\varphi_o \cos\varphi] da \\
&\quad - 3\sqrt{5}(1-f) \left[ \frac{a}{2w_o^3} \sin\varphi_o^2 \cos\varphi_o \sin\varphi \cos(\lambda - \lambda_o) \right. \\
&\quad \quad \left. - \frac{M_o}{2} \sin^2\varphi_o - N_o \right] \sin\varphi_o \cos\varphi d\bar{C}_\infty
\end{aligned} \tag{3.3-26}$$

$$\begin{aligned}
\eta^a - \eta^e &= \eta' - \eta - \frac{1}{N} \sin \lambda \, dx_{0i} - \cos \lambda \, dy_{0i} \\
&+ [\sin \varphi_0 (\sin \lambda \, \Delta y + \cos \lambda \, \Delta x) - \cos \varphi_0 \cos(\lambda - \lambda_0) \Delta z] dA \\
&- [\sin(\lambda - \lambda_0) \Delta z] d\mu \\
&- [\cos \varphi_0 (\sin \lambda \, \Delta y + \cos \lambda \, \Delta x) + \sin \varphi_0 \cos(\lambda - \lambda_0) \Delta z] d\nu \\
&+ \frac{\cos \varphi_0 \sin(\lambda - \lambda_0)}{w_0} da - \frac{3\sqrt{5} a(1-f)}{2w_0^3} \sin^2 \varphi_0 \cos \varphi_0 \sin(\lambda - \lambda_0) d\bar{C}_{20}
\end{aligned}$$

Equations (3.3-26) give the changes of the astrogeodetic deflections due to the orientation, the rotation of the ellipsoid, and the changes of the semidiameter and the flattening. The latter has been again introduced as a change in the fully normalized second-degree zonal harmonic of the geopotential.

The above equations contain corrections for parameters of the world geodetic system and can be used for the determination of them. The normal equations formed will be of the same form as those produced by the use of the terrestrial triangulation equations, except that they will not contain corrections for geocentric coordinates. They will then be of the form

$$N_3 \begin{vmatrix} dx \\ da \\ dc \end{vmatrix} = U_3$$

### 3.322 Weights of the observed quantities

The observed quantities in these observation equations are the differences of the astrogeodetic minus the gravimetric deflections of the vertical. The weight matrix will again be the inverse of the variance-covariance matrix  $V_{ag}$ . This  $V_{ag}$  matrix is found by summation of the variance-covariance matrices for the astrogeodetic deflections  $V_a$  and the gravimetric deflections  $V_g$ . That is,

$$V_{ag} = V_a + V_g$$

The variance-covariance matrix  $V_a$  of the astrogeodetic deflections will be formed by first forming the matrix  $V'$  which is the summation of the



variance-covariance matrices of the astronomic observations  $V_\Phi$  and that of the geodetic coordinates  $\varphi$  and  $\lambda$ ,  $V_\varphi$ , and then  $V_a$  is computed as

$$V_a = \begin{vmatrix} I & C_\varphi \end{vmatrix} V' \begin{vmatrix} I \\ C_\varphi \end{vmatrix}$$

where

$C_\varphi$  is a diagonal matrix whose diagonal element is the cosines of the latitude of the point in question

The variance-covariance matrix of the gravimetric deflections  $V_g$  will be found from the variances of the gravity anomalies propagated through the Vening-Meinesz equation.

### 3.4 Astrogeodetic Undulations

#### 3.41 Modeling the Astrogeodetic Undulations

To use the astrogeodetic undulations for the determination of the parameters of the world geodetic system, we need to find a mathematical model which will relate the undulations with as many of these parameters as possible. We start from the expression

$$N = \frac{GM}{a_n \gamma} \sum_{n=2}^{\infty} \frac{a_n}{r} \cdot \bar{C}_{n0}^* \cos m\lambda + \bar{S}_{n0} \sin m\lambda - \bar{P}_{nm}(\mu) \quad (3.4-1)$$

where [Mueller, 1964]

- N is the geop-spherop separation
- GM is the gravitational constant times the mass of the earth
- $a_e$  is the semidiameter of the reference ellipsoid
- $\bar{C}, \bar{S}$  are the fully normalized harmonic coefficients of the gravity field
- \*
- denotes differences between the actual and the reference fields
- r is the radius distance of the point in question from the center of coordinates
- $\gamma$  is the normal gravity at the point in question
- $P_{nm}(\mu)$  is the fully normalized Legendre's polynomial

The separation N is measured from a reference surface defined by the values of GM,  $a_e$ ,  $\gamma$ , and the harmonic coefficients of this reference surface. In computations the  $\bar{C}_{20}$  and  $\bar{C}_{40}$  harmonic coefficients of a level ellipsoid are usually used to define the reference surface. The difference between this reference surface, which is a special kind of Helmert's spheroid [Moritz, 1967, p. 81], and a level ellipsoid is the effect of the even-degree zonal harmonics of the level ellipsoid from degree six to infinity. This effect can be evaluated by the harmonic series of the even-degree harmonic coefficients which is a rapidly decreasing series; it decreases as powers of  $e^2$ , where  $e$  is the first eccentricity. The first term of this series is of the order of 4 cm, and the whole effect is not much more than that. Thus, we can consider our undulations as measured from a level ellipsoid without appreciable error.

The harmonic series involved in (3.4 - 1) is always convergent only for  $r > a_e$ , and it is considered generally divergent for  $r < a_e$ , or convergent down to the attracting masses for some special mass distributions as those discussed by [Moritz, 1961] and [Morrison, 1966 and 1967]. It is also known that when we substitute the mean value of gravity over the earth  $\gamma$  and put  $a_e/r = 1$ , in equation (3.4 - 1) we get Stokes' equation for the gravimetric undulations [Molodensky et al., 1962]. The relative error of Stokes' formula for

undulations is well known to be of the order of  $3 \times 10^{-3} N$ , and thus it is considered negligible [Cook, 1953]. We then put

$$\frac{GM}{a_e \gamma} = R$$

and equation (3.4 - 1) becomes

$$N = R \sum_{n=2}^{\infty} [\bar{C}_{nn}^* \cos m\lambda + \bar{S}_{nn} \sin m\lambda] \bar{P}_{nn}(\mu). \quad (3.4 - 2)$$

The undulations expressed by equation (3.4 - 2) are gravimetric undulations referred to the level ellipsoid implied by the  $\bar{C}_{20}$  used as reference but computed with a spherical approximation. Equation (3.4 - 2) must also be further simplified as we cannot carry the summation up to infinity, but only up to some finite number  $N_{max}$ . This introduces another approximation, the effect of which will be examined in a later discussion.

In order that the astrogeodetic undulation be comparable with the gravimetric one, the former must be changed from the reference ellipsoid of the particular datum to the level ellipsoid used for the gravimetric undulations. The change of the undulation  $dN$  at a point  $P$ , due to shifts of the center and to the changes of the parameters of the reference ellipsoid, is given by Rapp [1966]

$$\begin{aligned} dN = & \cos\varphi \cos\lambda dx_0 + \cos\varphi \sin\lambda dy_0 + \sin\varphi dz_0 \\ & - W da + \frac{a(1-f) \sin^2\varphi}{W} df \end{aligned} \quad (3.4 - 3)$$

where

$\varphi, \lambda$	are the geodetic coordinates of $P$
$dx_0, dy_0, dz_0$	are the total shifts of the datum center from the geocenter
$da, df$	are the differences of the semidiameter and of the flattening of the datum ellipsoid from the a priori mean earth ellipsoid
$a, f$	are the parameters of the datum ellipsoid

In order to use the same parameters as in the triangulation model, we again make the substitution (3.2 - 17b), and equation (3.4 - 3) becomes

$$\begin{aligned}
 dN = & \cos \varphi \cos \lambda \, dx_{01} + \cos \varphi \sin \lambda \, dy_{01} + \sin \varphi \, dz_{01} \\
 & + \frac{1}{W_0} [\cos \varphi \cos \lambda \cos \varphi_0 \cos \lambda_0 + \cos \varphi \sin \lambda \cos \varphi_0 \sin \lambda_0 + \\
 & + (1 - e^2) \sin \varphi \sin \varphi_0 - WW_0] da \\
 & - \frac{3\sqrt{5}a(1-f) \sin^2 \varphi_0}{2W_0^3} (\cos \varphi \cos \lambda \cos \varphi_0 \cos \lambda_0 + \cos \varphi \sin \lambda \cos \varphi_0 \sin \lambda_0) + \\
 & + \frac{3\sqrt{5}}{2} (1-f) \sin \varphi (M_0 \sin^2 \varphi_0 - 2N_0) \sin \varphi_0 + \frac{a \sin \varphi}{W} \Big] d\bar{C}_{20}
 \end{aligned}$$

or

$$\begin{aligned}
 dN = & \cos \varphi \cos \lambda \, dx_{01} + \cos \varphi \sin \lambda \, dy_{01} + \sin \varphi \, dz_{01} \quad (3.4 - 4) \\
 & + \frac{1}{W_0} [\cos \varphi \cos \varphi_0 \cos (\lambda - \lambda_0) + (1 - e^2) \sin \varphi \sin \varphi_0 - WW_0] da \\
 & - \frac{3\sqrt{5}}{2} (1-f) \left[ a \frac{\sin^2 \varphi_0}{W_0^3} \cos \varphi \cos \varphi_0 \cos (\lambda - \lambda_0) + \frac{\sin^2 \varphi}{W} \right] + \\
 & + \sin \varphi \sin \varphi_0 [M_0 \sin^2 \varphi_0 - 2N_0] d\bar{C}_{20}
 \end{aligned}$$

Another change,  $dN_R$ , due to the rotations of the datum must be added to equation (3.4 - 3). This is the change of the position of P (Fig. 3.4-1) along the normal, due to the rotations. The coordinates  $x_1$  of P after the rotations will be

$$x_1 = x + M(x - x_0) \quad (3.4 - 5)$$

where

- $x_1$  is the vector of the coordinates of the point  $P'$  which is the point P after rotations
- $x$  is the vector of the coordinates of the point P
- $x_0$  is the vector of the geodetic coordinates of the datum origin
- $M$  is the rotation matrix as in equation (3.2 - 2)

The change of the coordinates in the same system will again be

$$x_1 - x = M_1(x - x_0) = M_1 \Delta x \quad (3.4 - 6)$$

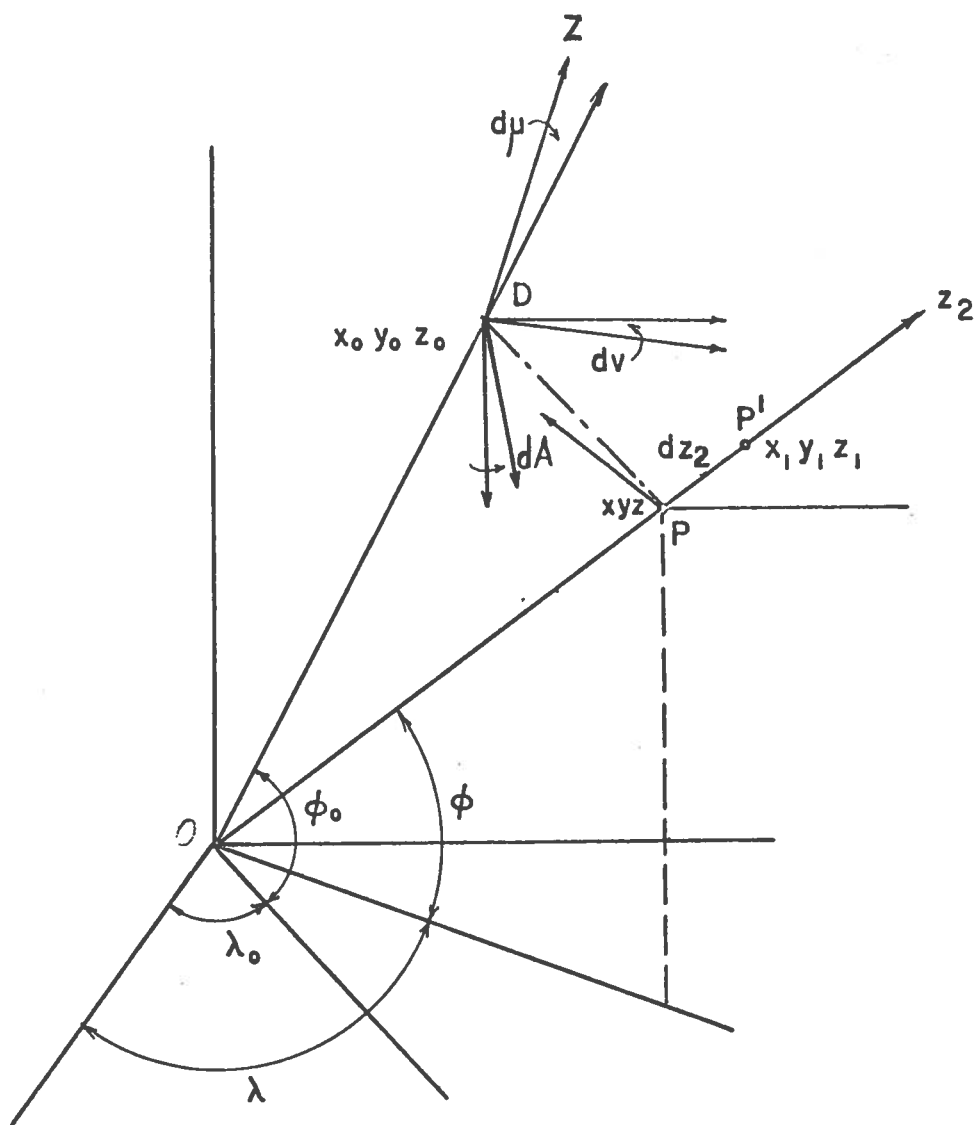


Fig. 3.4-1 Effect of rotations of a datum on astrogeodetic undulations

The change along the normal of the point P is the change of the coordinate system whose  $z_2$  axis coincides with the normal at point P. The change of the coordinates in that system will be

$$\Delta x_2 = R_2(90 - \varphi) R_3(\lambda) M_1 \Delta x \quad (3.4 - 7)$$

The change of the point P due to the rotation  $dA$ , which is the rotation around the geodetic normal at the datum origin, is, with a spherical approximation, along a line perpendicular to the axis Z and to the line DP; therefore it is a change in a direction perpendicular to the plane DOP and thus to the normal  $Pz_2$ , so it does not have any effect on the coordinate  $z_2$  of that system. By putting zero for  $dA$  in the expression (3.2 - 5) for  $M_1$  and separating the change to the coordinate  $z_2$ , we find that

$$\begin{aligned} dz_2 = dN_2 = & (\cos\varphi_0 \cos\varphi \sin\lambda \, d\nu + \cos\lambda_0 \sin\varphi \, d\mu + \sin\varphi_0 \sin\lambda_0 \sin\varphi \, d\nu) \Delta x \\ & (-\cos\varphi_0 \cos\lambda \cos\varphi \, d\nu + \sin\lambda_0 \sin\varphi \, d\mu - \sin\varphi_0 \cos\lambda_0 \sin\varphi \, d\nu) \Delta y \\ & (-\cos\lambda_0 \cos\varphi \cos\lambda \, d\mu - \sin\varphi_0 \sin\lambda_0 \cos\varphi \cos\lambda \, d\nu - \sin\lambda_0 \cos\varphi \sin\lambda \, d\mu + \\ & + \sin\varphi_0 \cos\lambda_0 \cos\varphi \sin\lambda \, d\nu) \Delta z \end{aligned} \quad (3.4 - 8)$$

Separating our parameters we have

$$\begin{aligned} dN_R = & [(\cos\lambda_0 \Delta x + \sin\lambda_0 \Delta y) \sin\varphi - \cos\varphi \cos(\lambda - \lambda_0) \Delta z] \, d\mu \\ & + [(\sin\lambda \Delta x - \cos\lambda \Delta y) \cos\varphi_0 \cos\varphi + (\sin\lambda_0 \Delta x - \cos\lambda_0 \Delta y) \sin\varphi_0 \sin\varphi \\ & + \sin\varphi_0 \cos\varphi \sin(\lambda - \lambda_0) \Delta z] \, d\nu \end{aligned} \quad (3.4 - 9)$$

Another way to derive the above equation is by projecting the change due to the rotations of the coordinates of the point P onto the geodetic normal of that point. The unit vector  $\hat{e}_n$  along the geodetic normal at point P is

$$\hat{e}_n = \begin{vmatrix} \cos\varphi \cos\lambda \\ \cos\varphi \sin\lambda \\ \sin\varphi \end{vmatrix} \quad (3.4 - 10)$$

The change of the coordinates  $dx_R$ ,  $dy_R$ ,  $dz_R$  of the point P due to the rotations are, for equation (3.2 - 7),

$$\begin{aligned}
dx_R &= (\sin \varphi_0 \Delta y - \cos \varphi_0 \sin \lambda_0 \Delta z) dA - (\cos \lambda_0 \Delta z) d\mu & (3.4-11) \\
&\quad - (\cos \varphi_0 \Delta y + \sin \varphi_0 \sin \lambda_0 \Delta z) d\nu \\
dy_R &= (-\sin \varphi_0 \Delta x + \cos \varphi_0 \cos \lambda_0 \Delta z) dA - (\sin \lambda_0 \Delta z) d\mu \\
&\quad + (\cos \varphi_0 \Delta x + \sin \varphi_0 \cos \lambda_0 \Delta z) d\nu \\
dz_R &= (\cos \varphi_0 \sin \lambda_0 \Delta x - \cos \varphi_0 \cos \lambda_0 \Delta y) dA \\
&\quad + (\cos \lambda_0 \Delta x + \sin \lambda_0 \Delta y) d\mu \\
&\quad + (\sin \varphi_0 \sin \lambda_0 \Delta x - \sin \varphi_0 \cos \lambda_0 \Delta y) d\nu
\end{aligned}$$

The change of the undulation  $dN_R$  is then

$$dN_R = \hat{e}_n \cdot \begin{vmatrix} dx_R \\ dy_R \\ dz_R \end{vmatrix} \quad (3.4-12)$$

We substitute equations (3.4-10) and (3.4-11) into equation (3.4-12) and after some rearrangement we obtain

$$\begin{aligned}
dN_R &= [(\sin \varphi \cos \varphi_0 \sin \lambda_0 - \cos \varphi \sin \lambda \sin \varphi_0) \Delta x & (3.4-13) \\
&\quad + (\cos \varphi \cos \lambda \sin \varphi_0 - \sin \varphi \cos \varphi_0 \cos \lambda_0) \Delta y \\
&\quad + (\cos \varphi \sin \lambda \cos \varphi_0 \cos \lambda_0 - \cos \varphi \cos \lambda \cos \varphi_0 \sin \lambda_0) \Delta z] dA \\
&\quad + [(\cos \lambda_0 \Delta x + \sin \lambda_0 \Delta y) \sin \varphi - \\
&\quad - \cos \varphi \cos (\lambda - \lambda_0) \Delta z] d\mu + [(\sin \lambda \Delta x - \cos \lambda \Delta y) \cos \varphi_0 \cos \varphi + \\
&\quad + (\sin \lambda_0 \Delta x - \cos \lambda_0 \Delta y) \sin \varphi_0 \sin \varphi + \\
&\quad + \sin \varphi_0 \cos \varphi \sin (\lambda - \lambda_0) \Delta z] d\nu
\end{aligned}$$

This equation reduces to equation (3.4-9) when the coefficient of  $dA$  is zero. In order that the geometric result, which indicated that the rotation  $dA$  does not have any effect on the undulation, be in accordance with the above analytic result the coefficient of  $dA$  must be identically zero. With a spherical approximation we have

$$\begin{aligned}
\Delta x &= x - x_0 = r \cos \varphi \cos \lambda - r_0 \cos \varphi_0 \cos \lambda_0 \\
\Delta y &= y - y_0 = r \cos \varphi \sin \lambda - r_0 \cos \varphi_0 \sin \lambda_0 & (3.4-14) \\
\Delta z &= z - z_0 = r \sin \varphi - r_0 \sin \varphi_0
\end{aligned}$$

where

$r, r_c$  are the radius vectors of points P and D respectively.

After the substitution of equation (3.4 - 14) into equation (3.4 - 13), the coefficient of  $dA$  vanishes. Therefore equation (3.4 - 13) reduces to equation (3.4 - 9) and both approaches give the same result.

If we had not introduced a spherical approximation in equations (3.4 - 14), equation (3.4 - 15) would have given for the coefficient of the rotation  $dA$

$$\epsilon_r = (N_0 \sin \varphi_0 - N \sin \varphi) e^2 \cos \varphi \cos \varphi_0 \sin (\lambda - \lambda_0) dA$$

This is zero for

$$\begin{aligned} \text{or} \quad \varphi &= \varphi_0 \\ \lambda &= \lambda_0 \end{aligned}$$

It is maximum, if we limit the extension of a datum within a hemisphere, when

$$\begin{aligned} \varphi &= 45^\circ \\ \varphi &= 0^\circ \\ \lambda - \lambda_0 &= 90^\circ \end{aligned}$$

For a rotation  $dA = 1''$ , this is only 11 cm, which we may neglect.

With the changes  $dN$  and  $dN_R$  of the astrogeodetic undulations computed above, we can write

$$N = N_{AG} + dN + dN_R \quad (3.4 - 16)$$

We substitute equations (3.4 - 4) and (3.4 - 9) into equation (3.4 - 16) and the mathematical model which relates the astrogeodetic undulations with the parameters of the world geodetic system:



$$\begin{aligned}
& \cos\varphi \cos\lambda \, dx_{01} + \cos\varphi \sin\lambda \, dy_{01} + \sin\varphi \, dz_{01} \\
& + [(\cos\lambda_0 \, \Delta x + \sin\lambda_0 \, \Delta y) \sin\varphi - \cos\varphi \cos(\lambda - \lambda_0) \, \Delta z] \, d\mu \\
& + [(\sin\lambda \, \Delta x - \cos\lambda \, \Delta y) \cos\varphi_0 \cos\varphi + (\sin\lambda_0 \, \Delta x - \cos\lambda_0 \, \Delta y) \sin\varphi_0 \sin\varphi \\
& \quad + \sin\varphi_0 \cos\varphi \sin(\lambda - \lambda_0) \, \Delta z] \, d\nu \\
& + \frac{1}{W_0} [\cos\varphi \cos\varphi_0 \cos(\lambda - \lambda_0) + (1 - e^2) \sin\varphi \sin\varphi_0 - WW_0] \, da \\
& - \frac{3}{2} (1 - f) \left[ a \left( \frac{\sin^2 \varphi_0}{W_0^3} \cos\varphi \cos\varphi_0 \cos(\lambda - \lambda_0) + \frac{\sin^2 \varphi}{W} \right) + \sin\varphi \sin\varphi_0 (M_0 \sin^2 \varphi_0 - 2N_0) \right] dC_{\infty} \\
& - R \sum_{n=2}^{N_{\max}} (C_{nn}^* \cos n\lambda + S_{nn} \sin n\lambda) P_{nn}(\mu) = -N_{AG} + V_{\Delta g}
\end{aligned} \tag{3.4-17}$$

In this model the observed quantities are the astrogeodetic undulations  $N_{AG}$ . The above mathematical model is already linear, thus no special linearization is necessary; however, the normal equations can take different forms depending on the way that we treat the parameters (parameters with weights or not). These cases will be treated in Chapter 4.

To decrease the effect of the neglected terms of order higher than  $N_{\max}$ , we will not use point values of the astrogeodetic undulations but mean values over some surface; for example,  $5^\circ \times 5^\circ$  blocks bounded by meridians and parallels. This way we consider some of the effect of the terms from  $N_{\max}$  to infinity; roughly, we neglect a large percentage of the terms from order  $N_{\max}$  to approximately order  $360^\circ/2S^\circ$  and a very small part of the effect of the terms of order greater than  $360^\circ/2S^\circ$ , where  $S^\circ$  denotes the dimension of  $S^\circ \times S^\circ$  blocks in which mean undulations are evaluated. The computation of these mean values is a complicated problem and depends on the adjustment method applied, the form in which the undulations are given, and the size of the block.

In most existing cases, the undulations are given for discrete nonuniformly distributed points which have not been adjusted by rigorous methods, but summary procedures or intuitive corrections have been undertaken [Ölander, 1954]. Furthermore, the undulations are very often given in the form of an undulation map which contains more information than the initial

discrete point but where the mathematical rigorousness has been further distorted.

We assume that the adjustment of the astrogeodetic undulations is carried out by rigorous adjustment procedures such as those described by [Fischer, 1966] and [Thorson, 1967]. Thus we end up with the adjusted undulations  $u$  for the intersections of meridians and parallels at  $1^\circ$  intervals, and we have also estimated the variance-covariance matrix of the adjusted undulations which actually is the variance-covariance matrix of the parameters of the adjustment. The mean undulation of a  $5^\circ \times 5^\circ$  block, for example, can be estimated by

$$\bar{N}_{Ag} = \frac{\sum_{i=1}^{36} c_i u_i}{\sum_{i=1}^{36} c_i} \quad (3.4-18)$$

where

- $u$  is the vector of the undulations contained in the block
- $c$  is the vector of 36 elements which are  $1/25$ ,  $1/50$ ,  $1/100$ , or  $0$ , depending on the position of the undulation (Fig. 3.4-2). (If an undulation refers to the intersection of a meridian and parallel which both lie in the block, the element is  $1/25$ ; if one of them is on the border, it is  $1/50$ ; and if the intersection is at a corner of the block, it is  $1/100$ . It is  $0$  when the intersection has not been determined.)

It is understood that for different sized blocks we have different  $c$  vectors. The size of the blocks where mean undulations will be computed must be examined based on the discussions of Rapp [1967b] and Pellinen [1966]. From the observation equations (3.4-10), normal equations can be formed. They will have the form

$$N_4 \begin{vmatrix} dx \\ da \\ dC \end{vmatrix} = U_4 \quad (3.4-19)$$

where the parameters have the same meaning as in (3.2-20). This set of normal equations can be solved alone, and they are actually a form of the well-known astrogeodetic determination of the size and shape of the earth.

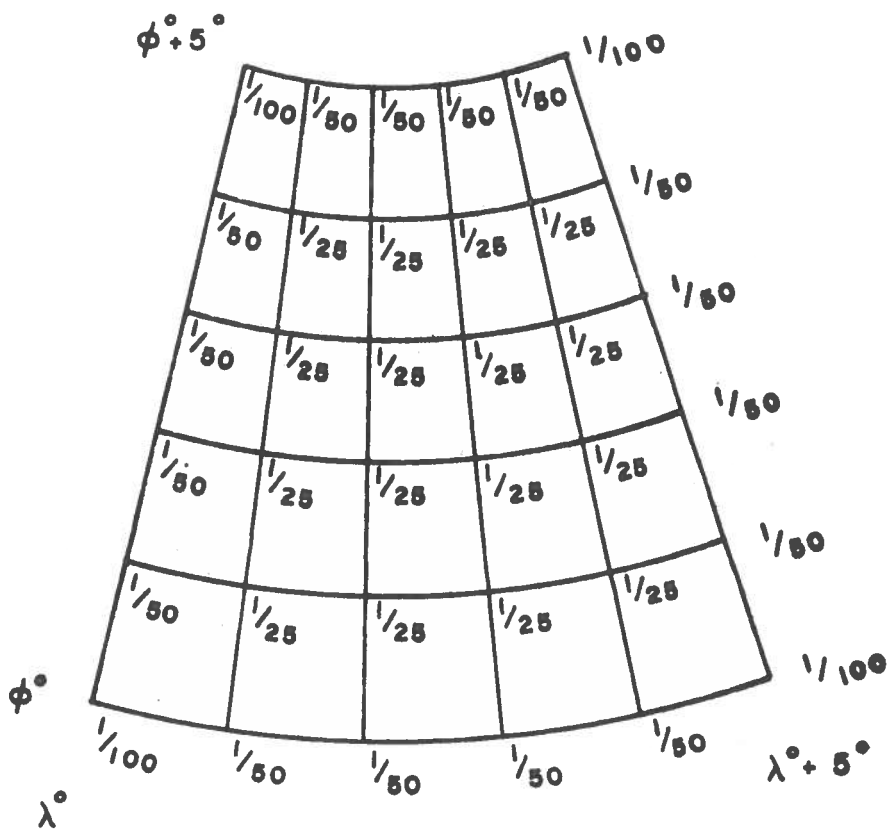


Fig. 3.4-2 Contribution of astrogeodetic undulations at 1° intervals to the mean undulation of a 5°x5° block

### 3.42 Weighting of the Astrogeodetic Undulations

The mean astrogeodetic undulations in  $5^\circ \times 5^\circ$  blocks will be computed by equation (3.4-18), that is, by a summation where we assume that the mean undulation of a  $1^\circ \times 1^\circ$  block is approximated by the undulation at the intersection of the meridian and parallel at the middle of the square. This assumption has been checked for twelve  $5^\circ \times 5^\circ$  blocks taken from the Geoid Contours in North America [Army Map Service, 1967] by comparison of the mean values computed as above with the ones computed through careful estimation of means in  $1^\circ \times 1^\circ$  blocks bordered by meridians and parallels. We tried to select blocks of the undulation map that were as representative as possible, and the result was that the two estimates differed usually by less than 0.1 m and only one difference was as large as 0.3 m. Thus from this limited sample we conclude that the representation error introduced by this assumption is negligible and the undulations at the intersections of meridians and parallels at  $1^\circ$  intervals can be treated as mean values of  $1^\circ \times 1^\circ$  blocks.

If  $\Sigma_n$  is the variance-covariance matrix of the adjusted undulations, the variance-covariance of the mean will be

$$\Sigma_N = G \Sigma_n G' \quad (3.4-20)$$

where

$G$  is a matrix whose rows are the vectors  $c / \sum_1^n c_i$  of equation (3.4-11) properly expanded with zeros to match the dimension of  $\Sigma_n$

In addition to the above, an error of omission is committed when undulations at some intersections have not been observed. To estimate the variance of this error first we computed the standard error with which a mean undulation of  $1^\circ \times 1^\circ$  block represents the mean value within a  $5^\circ \times 5^\circ$  block. Again from a limited sample we find that an undulation of an intersection represents the mean of the block with a variance of  $16 \text{ m}^2$ . As this variance is very small we make the assumption that the variance of the mean undulation of a block due to

the omission is inversely proportional to the number of the  $1^\circ \times 1^\circ$  blocks observed; however, a more reasonable assumption could be made after careful investigation of the data. Thus we add another term in equation (3.4 - 13) which will account for the error of omission. The variance of the mean can now be expressed as

$$\Sigma'_N = \Sigma_N + \left( I - \frac{\Sigma G}{25} \right) \times 16 \quad (3.4 - 14)$$

where

$\Sigma G$  is a diagonal matrix whose diagonal elements are the summations of the elements of the corresponding rows of matrix G of equation (3.4 - 13)

At the moment, as the adjustment of the astrogeodetic deflections has not been performed by rigorous methods, we can have only some estimates of the errors involved based on the discussions of [Ölander, 1954] and [Rice, 1962]. The latter, using a station interval of 22 km, gives the following equations for an estimation of the probable error arising from different sources

1. astronomic observations

a. random error  $\pm 0.011 \sqrt{k} \text{ m}$

b. systematic error  $\pm 0.024 \sqrt{k} \text{ m}$

2. error from linear interpolation  $\pm 0.014 \sqrt{k} \text{ m}$   
of deflections

3. geodetic longitude  $\pm 1.89 \times 10^{-6} k^{5/3} \text{ m}$

where

k is the distance from the origin in km

These estimates represent more or less the situation of errors in a geoidal section of the United States to which they refer, and they cannot reasonably hold for other areas. However, by changing the spacing between the stations so that it will be closer to general cases, we can have an approximate idea of the errors involved. Thus for a later application we accepted the same astronomic

errors and the same estimate for the change of the slope of the geoid. Then with a spacing for the astronomic stations of 50 km, which is more representative [Thorson, 1967], and with the standard error for the triangulation estimated by Simmons' equations as  $k^{2/3}/23.4$  [Simmons, 1951], we have the following estimate of standard errors:

from astronomic observations	$0.035 \sqrt{k}$ m
from linear interpolation of deflections	$0.075 \sqrt{k}$ m
from geodetic position	$3.88 \times 10^{-6} \times k^{5/3}$ m

where

$k$  is the distance from the datum point in km

Combining this result with that of equation (3.4 - 14), we have estimates for the standard errors of the mean undulation in a  $5^\circ \times 5^\circ$  block due to different sources:

astronomic observations	$\pm 0.035 \sqrt{k}$ m
interpolation	$\pm 0.075 \sqrt{k}$ m
geodetic position	$\pm 3.88 \times 10^{-6} k^{5/3}$
incomplete coverage	$\pm (1 - q/100)^{\frac{1}{2}} \times 4$ m

where

$q$  is the percentage of the block that is observed.

The above estimates have been used for evaluating the variances of mean undulations in a  $5^\circ \times 5^\circ$  block which will be used in a later application. The equation for estimating the variance of a mean undulation finally takes the form

$$S_N^2 = 0.00685 k + 15.00 k^{10/3} \times 10^{-12} + (1 - q/100) \times 16 \quad (3.4 - 15)$$

### 3.5 Geometric Use of Satellite Observations

#### 3.51 Geometric Models for Satellite Observations

The satellite observations used in a geometrical sense consist of simultaneous or quasi-simultaneous events of optical or electronic satellite observations. The optical observations are either corrected simultaneous topocentric right ascensions  $\alpha$  and declinations  $\delta$  obtained from properly reduced photographic plates [Veis, 1960; Mueller, 1964; Kaula, 1966; Hotter, 1967] or corrected range observations obtained from measurements with lasers [Lehr et al., 1966; Williams et al., 1965].

The electronic satellite observations, considered here in the geometrical sense, are corrected simultaneous ranges or range differences obtained from SECOR, C-Band, or Doppler observations [Mueller, 1964; Peat, 1967].

A mathematical model for each one of the above observation groups or for their combination is derived from the condition that the three vectors, formed by the geocenter and the ground station, the geocenter and the satellite position, and the ground station and the satellite position, form a close triangle (Fig. 3.5-1). This condition in vector notation is [Krakiwsky et al., 1967]

$$\mathbf{F}_{1j} = \mathbf{\bar{X}}_j - \mathbf{\bar{X}}_1 - \mathbf{\bar{X}}_{1j} = 0 \quad (3.5-1)$$

where

$$\mathbf{\bar{X}}_j = \begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix}$$

is a vector composed of the rectangular coordinates of an arbitrary satellite position;

$$\mathbf{\bar{X}}_1 = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

is a vector composed of the rectangular coordinates of an arbitrary ground station;

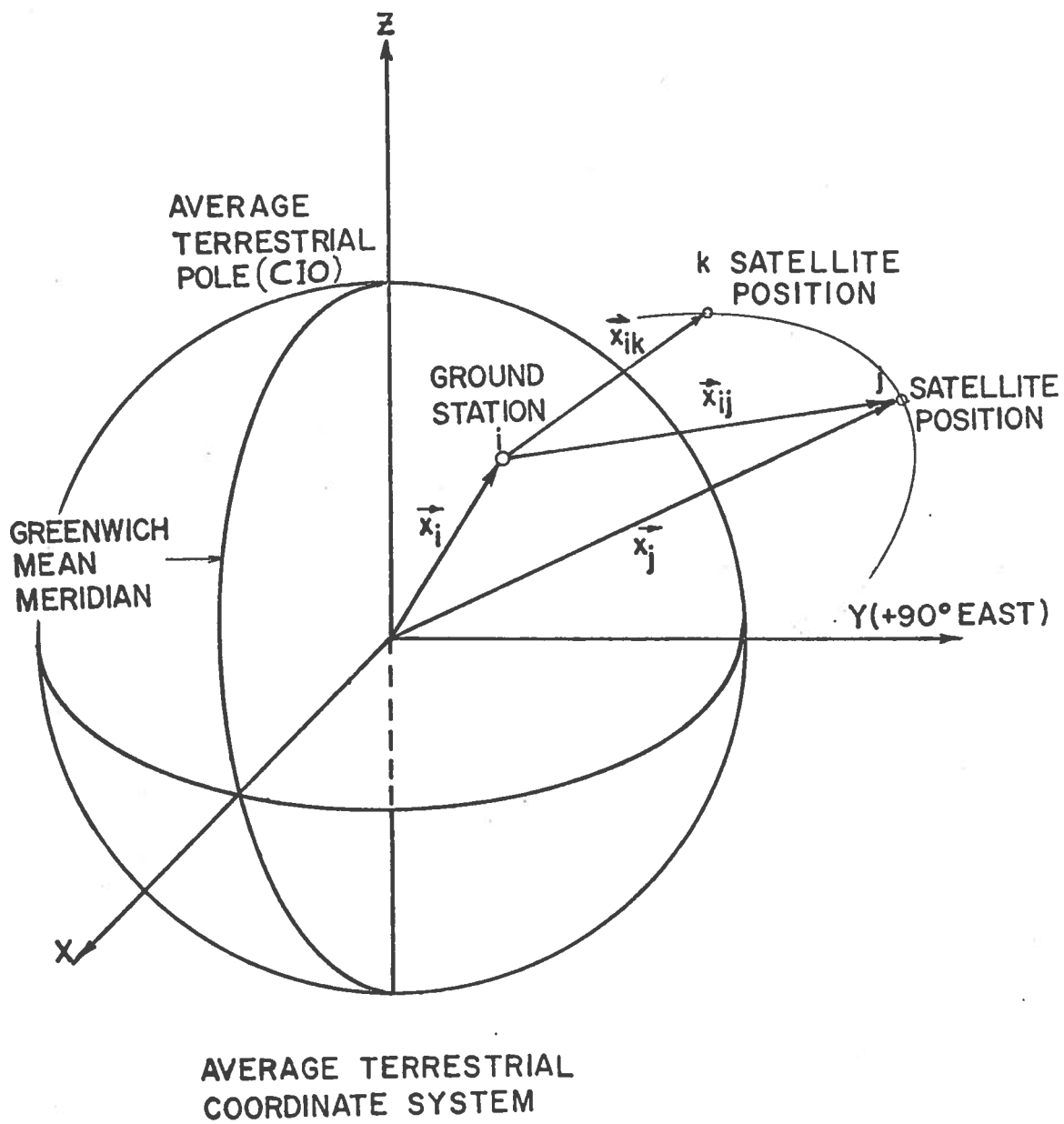


Fig. 3.5-1 Satellite position on the geocentric coordinate system



$$\vec{X}_{1j} = S \begin{vmatrix} r_{1j} & \cos \delta_{1j} & \cos \alpha_{1j} \\ r_{1j} & \cos \delta_{1j} & \sin \alpha_{1j} \\ r_{1j} & \sin \delta_{1j} & 0 \end{vmatrix}$$

where

$r_{1j}, \delta_{1j}, \alpha_{1j}$  are the topocentric range, declination, and right ascension from i to j respectively

S is a matrix which transforms the vector from the true celestial to the average terrestrial coordinate system.

The above model contains parameters and observed quantities and can be treated either after separation of the parameters from the observed quantities [Veis, 1960; Mueller, 1964] or by the general treatment of mixed models [Scheffé, 1959; Uotila, 1967b].

According to the second treatment the linearized form of the above model is

$$AX + BV + W = 0 \quad (3.5 - 2)$$

where

X are corrections to the parameters

V are corrections to the observed quantities

A, B are matrices of partial derivatives with respect to parameters and the observed quantities

For our case the matrices of the partial derivatives are

$$A = \frac{\partial F_{1j}}{\partial \vec{X}_j, \partial X_1} = \begin{vmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{vmatrix}$$

and

$$B = \frac{\partial F_{1j}}{\partial \delta, \partial \alpha, \partial r} = S R_3(-\alpha) R_2(-90^\circ + \delta) C$$

where

$$C = \begin{vmatrix} r & 0 & 0 \\ 0 & -r \cos \delta & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

Finally,

$$W = \vec{X}_j^o - \vec{X}_i^o - \vec{X}_{ij}^b$$

where

$\vec{X}_j^o, \vec{X}_i^o$  are approximate values for the satellite and station coordinates  
 $\vec{X}_{ij}^b$  is evaluated with the observed values

When only directions have been observed, the first two equations from (3.5 - 2) are used; when only ranges have been observed, only the third equation is used; and when directions and ranges have been observed simultaneously, all three equations are used.

When range differences have been observed, a mathematical model can be formulated by expressing these differences  $D_{ijk}$  with respect to the coordinates of the ground station  $\vec{X}_i$  and those of the satellite positions  $\vec{X}_j$  and  $\vec{X}_k$  (Fig. 3.5 - 1).

The model in matrix form is

$$[(\vec{X}_i - \vec{X}_j)' (\vec{X}_i - \vec{X}_j)]^{\frac{1}{2}} - [(\vec{X}_i - \vec{X}_k)' (\vec{X}_i - \vec{X}_k)]^{\frac{1}{2}} - D_{ijk} = 0 \quad (3.5 - 3)$$

This model requires corrected range differences. In practice the range differences are usually computed together with some error model parameters, for example, the correction  $dF$  to the reference frequency. These kinds of parameters can be introduced and the only change will be that in eliminating the satellite position we will eliminate the additional parameters as well so that we will finally obtain a form of normal equations containing only the station coordinate unknowns.

In the case of range differences, the pertinent matrices are

$$A = \begin{vmatrix} -\frac{X_i - X_j}{r_{ij}} & -\frac{Y_i - Y_j}{r_{ij}} & -\frac{Z_i - Z_j}{r_{ij}} & +\frac{X_i - Y_k}{r_{ik}} & +\frac{Y_i - Y_k}{r_{ik}} & +\frac{Z_i - Z_k}{r_{ik}} \\ +\frac{X_i - X_j}{r_{ij}} & -\frac{X_i - X_k}{r_{ik}} & \frac{Y_i - Y_j}{r_{ij}} & -\frac{Y_i - Y_k}{r_{ik}} & \frac{Z_i - Z_j}{r_{ij}} & -\frac{Z_i - Z_k}{r_{ik}} \end{vmatrix}$$

$$B = -I$$

and

$$W = [(\hat{X}_i^o - \hat{X}_j^o)' (\hat{X}_i^o - \hat{X}_j^o)]^{\frac{1}{2}} - [(\hat{X}_i^o - \hat{X}_k^o)' (\hat{X}_i^o - \hat{X}_k^o)]^{\frac{1}{2}} - D_{ijk}^b$$

From the above observation equations the normal equations are

$$\begin{vmatrix} 0 & 0 & A' \\ 0 & -P & B' \\ A & B & 0 \end{vmatrix} \begin{vmatrix} dX \\ V \\ k \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ W \end{vmatrix} = 0$$

After elimination of the corrections to the observations V and the correlates k, the normal equations become

$$A' (B P^{-1} B')^{-1} A dX + A' (B P^{-1} B')^{-1} W = 0$$

At this stage the parameters dX are the corrections to the ground stations and to the satellite positions. The satellite positions are not of interest to us and can be eliminated. After this elimination we obtain equations which contain as unknowns only corrections to the geocentric coordinates of the ground stations.

Finally, the normal equations from geometric treatment of satellite observations will be of the form

$$N_5 dX = U_5 \quad (3.5 - 4)$$

where

dX are again corrections to the geocentric coordinates

Even if all three kinds of observations have been performed simultaneously, no solution can be achieved. The information contained in these observations is enough to define our space network as a solid body of determined size and directions but whose absolute place in the geocentric coordinate system is not defined. Thus the normal equations are singular. To remove this singularity, it is sufficient to provide information which will fix the position of the network; this is usually done by fixing or by providing weights for the coordinates of one station. If now only directions have been observed, our space network is defined only in shape and orientation but not in size or in position. Thus, in

addition to position, three parameters defining the orientation must be provided. A possible set of orientation parameters would be three independent direction cosines.

In our solution for the world geodetic system where all kinds of geometric observations exist, the normal equations contributed by the geometric satellite observations will be singular because of lack of information on the position. This information will be provided by the dynamical satellite observations after their combination so that no additional information is necessary.

### 3.52 Weighting of Geometric Satellite Observations

The weight for the geometric satellite observations will be the inverse of their variance-covariance matrix. In all practical cases, however, correlations between different satellite observations are neglected. Sometimes the individual variance-covariances which come from the preprocessing of the observations are substituted by one variance representative of the entire observational system or at least of the group of observations from one station.

In general, the weights will emerge after careful analysis of the observations in comparison with some internal and external tests peculiar to different observational systems.

## 3.6 Dynamic Satellite Solution

### 3.61 Dynamic Use of Satellite Observations

Here the two groups of satellite observations, namely the optical and the electronic, are basically used to determine the orbits of the observed satellites and with them or from them corrections to spherical harmonic coefficients of the geopotential, to station coordinates, and to other physical constants.

The method of this determination is basically that of an iteration or perturbation approach, which uses the differences between computed and observed satellite positions to determine corrections to the dynamical and to the observational model.

Thus a general dynamical solution could contain corrections for the following parameters:

$X_p$	corrections to nominal orbital parameters
$dC$	corrections to nominal gravitational coefficients
$X_0$	corrections to nominal GM, air drag and solar radiation parameters, etc.
$X$	corrections to nominal station coordinates
$X_E$	corrections to nominal instrumental error model parameters

The processing of satellite data for dynamic solutions is so complicated that only the long experience of the people working in this field could overcome the tremendous difficulties which arise in every step. Thus it is more practical and more economical to assume that normal equations for dynamic solutions from uniform or semiuniform groups of data will be prepared by various organizations. There are many basic problems in formulating a combination of normals of satellite dynamic solutions, for example, the number of terms of the gravitational model or the method of evaluation of the partial derivatives, etc. But there are also many details which must be solved before a successful combination solution can be achieved. For example, the state of iteration at which the combination is going to take place, or the fact that the Smithsonian defines the zonal harmonics in a conventional form and the nonzonals in a fully normalized form.

Here we will try to outline only a possible framework of a combination. For that we choose three different types of dynamic solutions, namely the one of the Smithsonian Astrophysical Observatory (SAO), that of the U. S. Naval Weapons Laboratory (NWL), and that of the Jet Propulsion Laboratory (JPL); and we will examine their individual formulations and the resulting form after their combination.

### 3.62 The Smithsonian Astrophysical Observatory (SAO) Dynamic Solution

The SAO solution is actually performed in three steps:

- (1) computation of orbits,

(2) determination of zonal harmonic coefficients,  
 (3) computation of tesseral harmonic coefficients and station coordinates.  
 The first and third steps have been described by Gaposchkin [1966a and 1966b],  
 and the second is given in a series of papers by Kozai [1963, 1966, 1967].

### 3.621 Computation of orbits

The observations and the parameters are related through the simple geometry shown in Fig. 3.6-1 and expressed by the vector equation in the inertial coordinate system

$$\vec{r}(t) - \vec{R}(t) = \vec{\rho}(t) \quad (3.6-1)$$

where

- $\vec{\rho}(t)$  is the vector from the ground station to the satellite where  $\vec{\rho}'$  is its observed value
- $\vec{r}(t)$  is the vector from the geocenter to the satellite
- $\vec{R}(t)$  is the position vector of the ground station which in this coordinate system is a function of time

The vector  $\vec{r}(t)$  can be found from the theory of the satellite motion. It is a function of orbital elements  $E_1$ , spherical harmonic coefficients  $C_1$ , other perturbing forces (air drag, radiation pressure, etc.), and time:

$$\vec{r} = \vec{r}(E_1, C_1; t)$$

The vector  $\vec{R}(t)$  is related to the vector  $\vec{X}$  which refers to the average terrestrial system by the relation

$$\vec{R}(t) = R_3(-\hat{\theta}) R_2(x(t)) R_1(y(t)) \vec{X} \quad (3.6-2)$$

where

- $x(t), y(t)$  are the coordinates of the true pole, in arc, at time  $t$
- $\hat{\theta}$  is the angle between the  $X$  axis of the average terrestrial system and the direction of the mean equinox  $\Upsilon$  of 1950

Then equation (3.6-1) becomes

$$\vec{r}(E_1, C_1; t) - R_3(-\hat{\theta}) R_2(x(t)) R_1(y(t)) \vec{X} = \vec{\rho}(t) \quad (3.6-3)$$

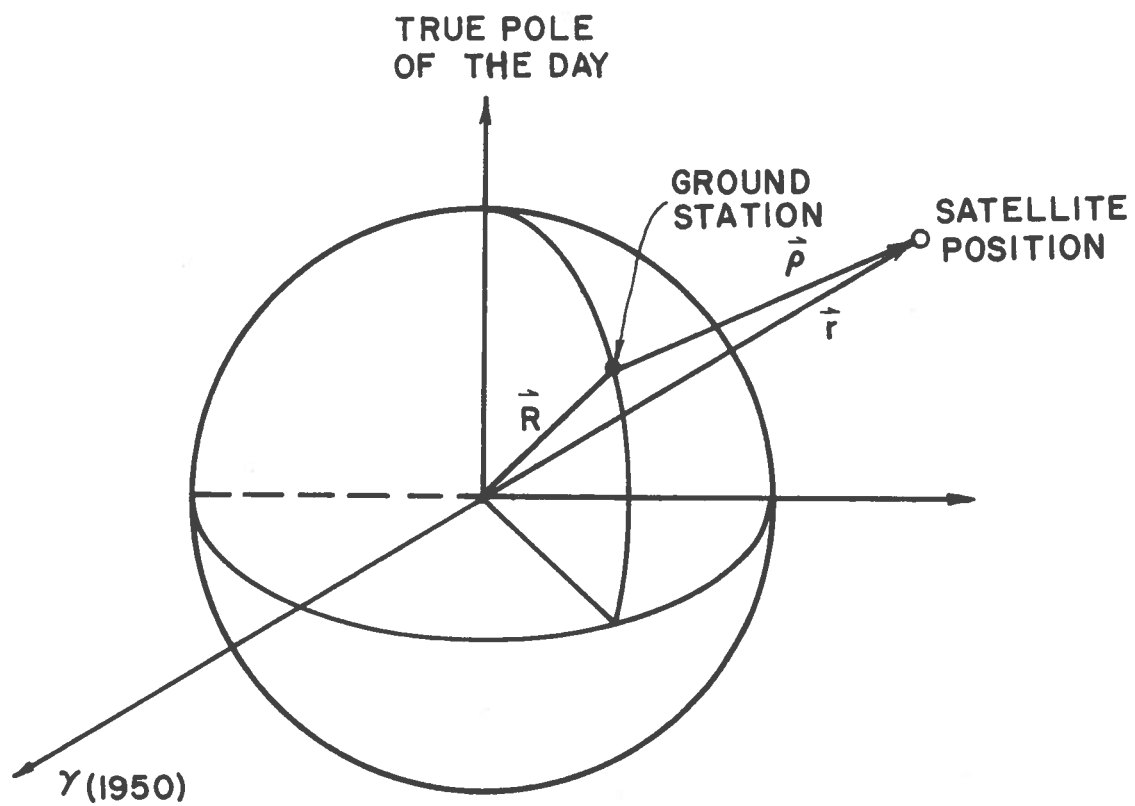


Fig. 3.6-1 Satellite position on the SAO reference system

Because the observations are not actually the components of the  $\vec{\rho}(t)$  vector, a transformation must be applied which will transform the  $\vec{\rho}(t)$  into the observed quantities, and equation (3.6 - 3) becomes

$$\vec{f} [\vec{r}(E_1, C_1; t) - R_3(-\hat{\theta}) R_2(x(t)) R_1(y(t)) \vec{X}] = \vec{f}(\vec{\rho}(t))$$

By linearizing these equations, we obtain

$$A \left[ \frac{\partial \vec{r}}{\partial E_1} \Delta E_1 + \frac{\partial \vec{r}}{\partial C_1} \Delta C_1 - R_3(-\hat{\theta}) d\vec{X} \right] = A [\rho' - \bar{r}_0(E_1^0, C_1^0; t) + R_3(-\hat{\theta}) R_2(x(t)) R_1(y(t)) X_0] \quad (3.6 - 4)$$

where

$$A = \partial \vec{f} / \partial \vec{\rho}$$

For the determination of the orbits, the gravity model as well as other force models are kept fixed; an option is left, however, for corrections to ground stations. By orbit determination we shall mean, in this case, the determination of the numerical values of constants of empirical expressions consisting of simple polynomials, trigonometric functions, and hyperbolic terms of the form

$$\begin{aligned} \mathcal{E}_1(t) &= \sum_{j=0}^n \mathcal{E}_j^1 (t - T_0)^j + \sum_{j=0}^n \delta_j^1 \sin(\alpha_j^1 + \beta_j^1 t) \\ &+ \sum_{j=0}^n {}^0H_j^1 \exp \{ {}^1H_j^1 [ {}^2H_j^1 - t ] \} \end{aligned} \quad (3.6 - 5)$$

which are time series representations of the mean orbital elements

$$\begin{aligned} \omega &= \omega(t) \\ \Omega &= \Omega(t) \\ i &= i(t) \\ e &= e(t) \\ M &= M(t) \\ \bar{n} &= dM/dt \end{aligned} \quad (3.6 - 6)$$

To these initial mean elements, long-period and secular oblateness perturbations ( $\delta \mathcal{E}^z$ ), lunar perturbations ( $\delta \mathcal{E}^L$ ), and tesseral harmonic perturbations ( $\delta \mathcal{E}^T$ )



may be computed and added if they have not been included in the definition of the mean elements.

Expressions (3.6 - 6) become

$$\begin{aligned}
 \omega &= \omega(t) + \delta\omega^Z + \delta\omega^L + \delta\omega^T \\
 \Omega &= \Omega(t) + \delta\Omega^Z + \delta\Omega^L + \delta\Omega^T \\
 i &= i(t) + \delta i^Z + \delta i^L + \delta i^T \\
 e &= e(t) + \delta e^Z + \delta e^L + \delta e^T \\
 M &= M(t) + \delta M^Z + \delta M^L + \delta M^T \\
 a &= a(n, i, e) + \delta a^T \\
 \bar{n} &= dM/dt
 \end{aligned} \tag{3.6 - 7}$$

The computations of long-period and secular oblateness perturbations based on Kozai's development is described in [Gaposchkin, 1966]. According to this description, the long-period perturbations are given as

$$\begin{aligned}
 \delta\omega^{Z1} &= P(1, 1) \cos \omega + P(1, 2) \sin 2\omega + P(1, 3) \cos 3\omega \\
 \delta\Omega^{Z1} &= P(2, 1) \cos \omega + P(2, 2) \sin 2\omega + P(2, 3) \cos 3\omega \\
 \delta i^{Z1} &= P(3, 1) \sin \omega + P(3, 2) \cos 2\omega + P(3, 3) \sin 3\omega \\
 \delta e^{Z1} &= P(4, 1) \sin \omega + P(4, 2) \cos 2\omega + P(4, 3) \sin 3\omega \\
 \delta M^{Z1} &= P(5, 1) \cos \omega + P(5, 2) \sin 2\omega + P(5, 3) \cos 3\omega
 \end{aligned} \tag{3.6 - 8}$$

where

$P(ij)$  are functions of the even zonal harmonics when  $j = 2$ , and of the odd zonal harmonics when  $j = 1, 3$

The secular motion of the perigee and of the node are both functions of even zonal harmonics. The expression for the secular motion of the argument of perigee has the form

$$\frac{1}{2\pi\bar{n}} \frac{d\omega}{dt} = \sum_{n=2}^{\infty} \frac{J_n}{p^n} f_n^{11}(i, e) + \frac{J_2}{p^4} f^{12}(i, e) + f^{13}(i, e, \bar{n}) \tag{3.6 - 9}$$

The expression for the secular motion of the longitude of the ascending node has the form

$$\frac{1}{2\pi n} \frac{d\Omega}{dt} = \sum_{n=2}^{\infty} \frac{J_n}{p^n} f_n^{2i}(i, e) + \frac{J_{22}}{p^4} f^{22}(i, e) + f^{23}(i, e, \bar{n}) \quad (3.6-10)$$

Explicit expressions are given in [Gaposchkin, 1966, pp. 131-135]. For the computation of these perturbations, coefficients from previous solutions of zonal harmonics are used. Expressions based on a development by Izsak are given for the perturbations of the orbital elements due to the moon in [Gaposchkin, 1966, section 10].

The short-period perturbations due to the tesseral harmonics are given in [Gaposchkin, 1966, section 11]. The discussion is based on Kaula's development, the harmonic coefficients are in fully normalized form, and each perturbation of each element is a trigonometric series of the form

$$\delta \mathcal{E}_{inm} = \left( \frac{\bar{C}_{nm}}{\bar{S}_{nm}} \right) \sum_p \sum_q \delta \mathcal{E}_{ipq}^* \times \text{trigonometric terms in } n, m, p, q, \omega, M, \Omega \quad (3.6-11)$$

where

$\mathcal{E}$  is a particular orbital element  
 $\delta \mathcal{E}_{ipq}^*$  are slowly changing coefficients, functions of the mean elements  $a, e, i$

The form of the coefficients  $\delta \mathcal{E}^*$  is

$$\delta \mathcal{E}_{nmpq}^* = \frac{GM \times F(i, e, a)}{[(n-2p)\dot{\omega} + (n-2p+q)\dot{\bar{n}} + m(\dot{\Omega} - \dot{\theta})]^k} \quad (3.6-12)$$

$k = 1, 2$

where

$\dot{\omega}, \dot{\Omega}, \dot{\theta}$  are secular variations

By examining the denominator in this equation we see that for some combinations of  $n, m, p, q, \dot{\omega}, \dot{\bar{n}},$  and  $\dot{\Omega}$  the denominator becomes very small and the perturbations very large. These are the so-called resonant terms. As

$\dot{\omega}$  and  $\dot{\Omega}$  are small compared to  $\dot{\theta}$ , we can see that amplification due to a small divisor occur when  $m\dot{\theta} \cong \bar{n}$ . The selection of the coefficients  $\delta\mathcal{E}^*$  which will be included in the computation is made in such a way that all effects greater than 5 m on any of the satellite orbits are included.

The total perturbation on an orbital element is given by summation over all harmonic coefficients

$$\begin{aligned} \delta\mathcal{E}_i = & \sum_{n=2}^{\infty} \sum_{m=1}^n \left\{ \bar{C}_{nm} \left[ \sum_{p=0}^n \sum_{q=-\infty}^{\infty} \delta\mathcal{E}_{nmpq}^* \begin{cases} \pm \mathcal{G} & \text{or } \pm \mathcal{C} \\ \pm \mathcal{C} & \text{or } \pm \mathcal{G} \end{cases} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \right] \right. \\ & \left. + \bar{S}_{nm} \left[ \sum_{p=0}^n \sum_{q=-\infty}^{\infty} \delta\mathcal{E}_{nmpq}^* \begin{cases} \pm \mathcal{C} & \text{or } \pm \mathcal{G} \\ \pm \mathcal{G} & \text{or } \pm \mathcal{C} \end{cases} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \right] \right\} \end{aligned} \quad (3.6-13)$$

where

$$\mathcal{C} = \mathcal{C}_{nmpq}(\omega, M, \Omega, \theta) = \cos[(n-2p)\omega + (n-2p+q)M + m(\Omega-\theta)]$$

$$\mathcal{G} = \mathcal{G}_{nmpq}(\omega, M, \Omega, \theta) = \sin[(n-2p)\omega + (n-2p+q)M + m(\Omega-\theta)]$$

with some special rule for selecting  $\mathcal{C}$ ,  $\mathcal{G}$ , and  $\pm$ .

The total perturbation in M, for example, is

$$\begin{aligned} \delta M = & \sum_{n=2}^{\infty} \sum_{m=1}^n \left\{ \bar{C}_{nm} \left[ \sum_{p=0}^n \sum_{q=-\infty}^{\infty} \delta M_{nmpq}^* \begin{pmatrix} n-m \text{ even} & n-m \text{ odd} \\ \mathcal{G} & -\mathcal{C} \end{pmatrix} \right] \right. \\ & \left. + \bar{S}_{nm} \left[ \sum_{p=0}^n \sum_{q=-\infty}^{\infty} \delta M_{nmpq}^* \begin{pmatrix} n-m \text{ even} & n-m \text{ odd} \\ -\mathcal{C} & -\mathcal{G} \end{pmatrix} \right] \right\} \end{aligned}$$

After the computation of the tesseral perturbations, the second harmonic oblateness short-period perturbations are added in the form

$$\begin{aligned} \Omega &= \Omega + \delta\Omega^s \\ \sin i &= \sin i + \delta i^s \cos i \\ \cos i &= \cos i - \delta^s \sin i \\ \sin \ell &= \sin \ell + \delta \ell^s \cos \ell \\ \cos \ell &= \cos \ell - \delta \ell^s \sin \ell \\ r &= a(1 - e \cos E) + \delta r^s \end{aligned} \quad (3.6-14)$$

where

E is the eccentric anomaly  
 $\ell = \omega + f$   
 f is the true anomaly

Full expressions for all the  $\delta \mathcal{E}^s$  second harmonic oblateness short-period perturbations are given in [Gaposchkin, 1966, p. 146].

The observations are also reduced to the adopted coordinate system and corrected for light travel time, diurnal aberration, and parallactic refraction.

With the corrected orbital elements, the satellite vector is computed from

$$\vec{r}(t) = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} r (\cos \ell \cos \Omega - \sin \ell \cos i \sin \Omega) \\ r (\cos \ell \sin \Omega + \sin \ell \cos i \cos \Omega) \\ r \sin \ell \sin i \end{bmatrix} \quad (3.6-15)$$

Our initial model (3.6-4), because we keep the gravity model fixed, becomes

$$A \left[ \frac{\partial \vec{r}}{\partial \omega} \Delta \omega + \frac{\partial \vec{r}}{\partial \Omega} \Delta \Omega + \frac{\partial \vec{r}}{\partial i} \Delta i + \frac{\partial \vec{r}}{\partial e} \Delta e + \frac{\partial \vec{r}}{\partial M} \Delta M + \frac{\partial \vec{r}}{\partial a} \left( \frac{2a}{3\pi} \right) \Delta \left( \frac{dM}{dt} \right) - R_3(-\hat{\theta}) dX \right] = A [\rho' - \rho_0] \quad (3.6-16)$$

where

$\rho_0$  is the vector from the station to satellite computed with the help of equation (3.6-15)

The matrix A which transforms our model from the adopted orbital system to the observations system  $(\alpha, \delta, \rho)$  is in general given by

$$A = \frac{1}{\rho} \begin{bmatrix} -\cos \alpha \sin \delta & -\sin \alpha \sin \delta & \cos \delta \\ -\sin \alpha & \cos \alpha & 0 \\ \rho_x/\rho & \rho_y/\rho & \rho_z/\rho \end{bmatrix} \quad (3.6-17)$$

where

$\rho_x, \rho_y, \rho_z$  are the three components of the vector  $\vec{\rho}(t)$

The partial derivatives needed in (3.6-16) are computed from equation (3.6-15) as follows:

$$\frac{\partial \vec{r}}{\partial \omega} = \frac{\partial \vec{r}}{\partial f} = \begin{vmatrix} \frac{\partial r_x}{\partial \omega} \\ \frac{\partial r_y}{\partial \omega} \\ \frac{\partial r_z}{\partial \omega} \end{vmatrix} = \begin{vmatrix} r(-\sin \ell \cos \Omega - \cos \ell \cos i \sin \Omega) \\ r(-\sin \ell \sin \Omega + \cos \ell \cos i \cos \Omega) \\ r \cos \ell \sin i \end{vmatrix} \quad (3.6-18)$$

$$\frac{\partial \vec{r}}{\partial \Omega} = \begin{vmatrix} \frac{\partial r_x}{\partial \Omega} \\ \frac{\partial r_y}{\partial \Omega} \\ \frac{\partial r_z}{\partial \Omega} \end{vmatrix} = \begin{vmatrix} -r_y \\ r_x \\ 0 \end{vmatrix}$$

$$\frac{\partial \vec{r}}{\partial i} = \begin{vmatrix} \frac{\partial r_x}{\partial i} \\ \frac{\partial r_y}{\partial i} \\ \frac{\partial r_z}{\partial i} \end{vmatrix} = \begin{vmatrix} r_z \sin \Omega \\ -r_z \cos \Omega \\ r \cos i \sin \ell \end{vmatrix}$$

$$\frac{\partial \vec{r}}{\partial e} = \begin{vmatrix} \frac{\partial r_x}{\partial e} \\ \frac{\partial r_y}{\partial e} \\ \frac{\partial r_z}{\partial e} \end{vmatrix} = \hat{r} \left[ \frac{a e \sin^2 E}{1 - e \cos E} - a \cos E \right] + \frac{\partial \vec{r}}{\partial f} \left[ \frac{\sin f}{1 - e \cos E} + \frac{\sin f}{1 - e^2} \right]$$

$$\frac{\partial \vec{r}}{\partial M} = \begin{vmatrix} \frac{\partial r_x}{\partial M} \\ \frac{\partial r_y}{\partial M} \\ \frac{\partial r_z}{\partial M} \end{vmatrix} = \hat{r} \frac{2\pi a e \sin E}{1 - e \cos E} + \frac{\partial \vec{r}}{\partial v} \frac{2\pi \sin f}{\sin E(1 - e \cos E)}$$

As we are not interested in corrections to the orbital elements, but in corrections to the coefficients of the empirical polynomial of equations (3.6-5), we substitute the corrections of the orbital elements with their differentials from equations (3.6-5). Thus for  $\omega$  we have

$$\begin{aligned} \omega = & \omega_0 + \omega_1 t + \omega_2 t^2 + \dots + S_1^\omega \sin[\alpha_1^\omega + \beta_1^\omega t] + S_2^\omega \sin[\alpha_2^\omega + \beta_2^\omega t] + \dots \\ & + {}^0H_1^\omega \exp \{ {}^1H_1^\omega t_n [ {}^2H_1^\omega - T ] \} + {}^0H_2^\omega \exp \{ {}^1H_2^\omega t_n [ {}^2H_2^\omega - T ] \} \end{aligned} \quad (3.6-19)$$

where

$t$  is the time from the epoch of the orbit

$T$  is the time in Modified Julian Days

and the differential

$$\begin{aligned} \Delta\omega = & \Delta\omega_0 + \Delta\omega_1 t + \dots + \Delta S_1^\omega \sin[\alpha_1^\omega + \beta_1^\omega t] + S_1^\omega \cos[\alpha_1^\omega + \beta_1^\omega t] \Delta\alpha_1^\omega \\ & + S_1^\omega \cos[\alpha_1^\omega + \beta_1^\omega t] \Delta\beta_1^\omega t + \dots + \Delta {}^0H_1^\omega \exp \{ \dots \} \dots \end{aligned} \quad (3.6-20)$$

We substitute these expressions into (3.6-16) and obtain the final form of our observation equation

$$\begin{aligned} A \left[ \frac{\partial \vec{r}}{\partial \omega} \sum_k f_k \Delta\omega_k + \frac{\partial \vec{r}}{\partial \Omega} \sum_k f_k \Delta\Omega_k + \frac{\partial \vec{r}}{\partial i} \sum_k f_k \Delta i_k + \frac{\partial \vec{r}}{\partial e} \sum_k f_k \Delta e_k \right. \\ \left. + \sum_k \left( \frac{\partial \vec{r}}{\partial M} f_k - \frac{2}{3} \frac{\vec{r}}{n} f_k^* \right) \Delta M_k - R_3(-\theta) \Delta X \right] = A(\rho' - \rho_0) = \begin{vmatrix} d\delta \\ \cos \delta d\alpha \\ d\rho/\rho \end{vmatrix} \end{aligned} \quad (3.6-21)$$

where

$f_k$  is the partial derivative with respect to the  $k$ th coefficient of the empirical function (3.6-19)

For the determination of orbits the observations are weighted inversely to their standard errors.

Observation equations (3.6-21) are formed for every satellite observation. From them normal equations are formed and solved with the usual methods of least squares. The process is iterated by recomputing the elements that are affected by changes in the orbit and the observations are

tested with a rejection criterion of three times the standard error of the previous iteration. The number of parameters is arbitrary, and its selection is based on the satellites and on the quality of the gravity model used.

A refinement of the computation of orbits, which also helped in the determination of tesseral harmonics, was the so-called method of rotated residuals, that is, the expression of the same observation equations in a coordinate system defined by the orbital plane and its normal. In this coordinate system, motion and residuals along the orbit  $u$  and perpendicular to the orbit  $w$  are well separated and more clearly related to the timing errors or to the uncertainties of the tesseral harmonic coefficients. For the SAO equations, the matrix  $B$  which transforms the first two equations of (3.6 - 21) into the above orbital coordinate system is

$$B = \begin{vmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{vmatrix} = \begin{vmatrix} B_1 \\ B_2 \end{vmatrix} \quad (3.6 - 22)$$

where

$$\cos \varphi = \hat{e}_w \cdot \hat{e}_c$$

$$\sin \varphi = -\hat{e}_w \cdot \hat{e}_b$$

with

$$\hat{e}_w = \frac{\hat{e}_A \times \frac{\dot{\vec{r}}}{\dot{n}}}{\left| \hat{e}_A \times \frac{\dot{\vec{r}}}{\dot{n}} \right|}$$

$$\hat{e}_u = \hat{e}_w \times \hat{e}_A$$

$$\frac{\dot{\vec{r}}}{\dot{n}} = \frac{a}{\sqrt{1-e^2}} \begin{vmatrix} -(\sin u + e \sin \omega) \\ \cos u + e \cos \omega \\ 0 \end{vmatrix}$$

$$u = \ell = \omega + f$$

$$\hat{e}_A, \hat{e}_B, \hat{e}_C$$

are unit vectors in the direction of the observed satellite position, in the plane of the equator  $\pi/2$  from  $\hat{e}_A$ , and normal to  $\hat{e}_A, \hat{e}_B$ .

Izsak developed the transformation of the two equations in vector notation for the optical observations of SAO, that is, for observations of right-ascension and declination; these equations are given as follows in [Gaposchkin, 1966] where the corrections to the stations have not been considered.

$$\begin{aligned} \frac{1}{\sigma_j} du = & \frac{1}{\sigma_j} \hat{e}_u \cdot \left\{ (\hat{e}_n \times \vec{r}) \sum_k f_k \Delta \omega_k + (\hat{e}_z \times \vec{r}) \sum_k f_k \Delta \Omega_k \right. \\ & + r \sin \ell \hat{e}_n \sum_k f_k \Delta i_k + \left[ (\hat{e}_n \times \vec{r}) \frac{a}{r} \frac{\sin E}{\sqrt{1-e^2}} - \vec{a} \right] \sum_k f_k \Delta e_k \\ & \left. + \sum_k \left( 2\pi \frac{\vec{r}}{n} f_k - \frac{2}{3} \frac{\vec{r}}{n} f_k^* \right) \Delta M_k \right\} \end{aligned} \quad (3.6-23)$$

$$\begin{aligned} \frac{1}{\sigma_j} dw = & \frac{1}{\sigma_j} \hat{e}_w \cdot \left\{ (\hat{e}_n \times \vec{r}) \sum_k f_k \Delta \omega_k + (\hat{e}_z \times \vec{r}) \sum_k f_k \Delta \Omega_k \right. \\ & + r \sin \ell \hat{e}_n \sum_k f_k \Delta i_k + \left[ (\hat{e}_n \times \vec{r}) \frac{a}{r} \frac{\sin E}{\sqrt{1-e^2}} - \vec{a} \right] \sum_k f_k \Delta e_k \\ & \left. - \frac{2}{3} \frac{\vec{r}}{n} \sum_k f_k^* \Delta M_k \right\} \end{aligned}$$

where

$\hat{e}_n$  is unit vector normal to orbital plane

$\hat{e}_z$  is unit vector in the direction of the sidereal  $z$  axis

$$\begin{vmatrix} du \\ dw \end{vmatrix} = B \begin{vmatrix} d\delta \\ \cos \delta d\alpha \end{vmatrix} = \begin{vmatrix} \cos \varphi d\delta - \sin \varphi \cos \delta d\alpha \\ \sin \varphi d\delta + \cos \varphi \cos \delta d\alpha \end{vmatrix}$$

In weighting equations (3.6-23), for the first iteration we use the same weighting, that is, the weights are inverses of the standard errors. But in the second and last iteration, in order to account for timing errors, equations in  $du$  are multiplied by the factor  $\mu$ :

$$\mu = \sqrt{\frac{\sum dw^2}{\sum du^2}}$$



### 3.622 The determination of zonal harmonic coefficients

The determination of zonal harmonic coefficients is based on the computed orbits. Specifically, for the determination of the even zonal harmonic coefficients the secular motions for the longitude of the ascending node and the argument of perigee are used, with equations (3.6 - 10) and (3.6 - 9) serving as observation equations.

The observation equations are formulated for conventional zonal harmonics, but the change to fully normalized coefficients is only a matter of proper scaling of the terms of the equations. Luni-solar and air drag perturbations must be taken into account when using these equations.

The observation equations are already linear with respect to the unknowns; thus a solution could be performed for correction to some nominal values or for the values themselves of the parameters. However, for better numerical results, and in view of simultaneous solutions with the tesseral harmonics, a solution for corrections to a set of nominal values is preferable. The observations are weighted reciprocally proportional to their variances and solved with the usual least squares process.

Kozai, in a recent solution, introduced another weighting approach to account for some of the neglected higher-order terms on the estimates and on the estimated variances [Kozai, 1967]. There he made the assumption that the values of  $J_{22}$  to  $J_{27}$  are  $0.5 \times 10^{-7}$ , estimated the error committed by their omission, and weighted the equations reciprocally proportional to the new variances.

The determination of the odd zonal harmonic coefficients is based on the long-period perturbations. The first four of equations (3.6 - 8) are used as observation equations.

The above discussion concerning the even harmonics also applies to the determination of the odd zonal harmonic coefficients.

### 3.623 Determination of the tesseral harmonics and station coordinates

The determination of the tesseral harmonics and station coordinates is also based on the computed satellite orbits. To formulate the mathematical model for the least squares determination of tesseral harmonics, we start from equation (3.6-16). This time we are not interested in correcting the orbital elements but in correcting the tesseral harmonic coefficients and the station coordinates; thus we make the following substitution:

$$\Delta \mathcal{E}_1 = \sum_{n=2}^{\infty} \sum_{m=0}^n \frac{\partial \mathcal{E}_1}{\partial \bar{C}_{nm}} \Delta \bar{C}_{nm} \quad (3.6-24)$$

The partial derivatives in this equation can be evaluated from equation (3.6-13) with which we have established the effect of the tesseral harmonic on the orbital elements. For example,

$$\frac{\partial \mathcal{M}}{\partial \bar{C}_{nm}} = \frac{\partial \delta \mathcal{M}}{\partial \bar{C}_{nm}} = \sum_{p=0}^n \sum_{q=-\infty}^{\infty} \delta M_{nmpq}^* \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \quad (3.6-25)$$

Equation (3.6-16) becomes

$$\begin{aligned} A \left[ \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial \bar{r}}{\partial \omega} \cdot \frac{\partial \omega}{\partial \bar{C}_{nm}} + \frac{\partial \bar{r}}{\partial \Omega} \cdot \frac{\partial \Omega}{\partial \bar{C}_{nm}} + \frac{\partial \bar{r}}{\partial I} \cdot \frac{\partial I}{\partial \bar{C}_{nm}} + \frac{\partial \bar{r}}{\partial e} \cdot \frac{\partial e}{\partial \bar{C}_{nm}} + \frac{\partial \bar{r}}{\partial M} \cdot \frac{\partial \mathcal{M}}{\partial \bar{C}_{nm}} \right. \right. \\ \left. \left. + \frac{\partial \bar{r}}{\partial a} \cdot \frac{\partial a}{\partial \bar{C}_{nm}} \right) d\bar{C}_{nm} + \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial \bar{r}}{\partial \omega} \cdot \frac{\partial \omega}{\partial \bar{S}_{nm}} + \frac{\partial \bar{r}}{\partial \Omega} \cdot \frac{\partial \Omega}{\partial \bar{S}_{nm}} \right. \right. \\ \left. \left. + \frac{\partial \bar{r}}{\partial e} \cdot \frac{\partial e}{\partial \bar{S}_{nm}} + \frac{\partial \bar{r}}{\partial M} \cdot \frac{\partial \mathcal{M}}{\partial \bar{S}_{nm}} + \frac{\partial \bar{r}}{\partial a} \cdot \frac{\partial a}{\partial \bar{S}_{nm}} \right) d\bar{S}_{nm} - R_3(-\hat{\theta}) dX \right] \\ = A [\rho' - \rho_0] \end{aligned} \quad (3.6-26)$$

Since the orbit computations have been performed in a coordinate system defined by the orbital plane and its normal and many quantities in that system are readily available, the solution for tesseral harmonics is also performed in that coordinate system. To transform equation (3.6-26) into that coordinate system, we have to multiply its first two rows by the matrix B, equation (3.6-22). According to Izsak's derivation and his vector notation, the transformed

equation (3.6 - 26) can be written

$$du = \hat{e}_u \cdot \left\{ (\hat{e}_u \times \vec{r}) \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial \omega}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \frac{\partial \omega}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + (\hat{e}_z \times \vec{r}) \right. \quad (3.6 - 27)$$

$$\left. \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial \Omega}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \frac{\partial \Omega}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + r \sin \ell \hat{e}_u \sum_{n=2}^{\infty} \sum_{m=2}^n \left( \frac{\partial i}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \right. \right.$$

$$\left. \frac{\partial i}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + \left[ (\hat{e}_u \times \vec{r}) \frac{a}{r} \frac{\sin E}{\sqrt{1-e^2}} - \bar{a} \right] \sum_{n=2}^{\infty} \sum_{m=2}^n \left( \frac{\partial e}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \right.$$

$$\left. \frac{\partial e}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + \left( 2\pi \frac{\dot{\vec{r}}}{\bar{n}} - \frac{2}{3} \frac{\vec{r}}{\bar{n}} \right) \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial M}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \frac{\partial M}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) \}$$

$$- B_1 \bar{A} R_3 (-\hat{\theta}) dX$$

$$dw = \hat{e}_w \cdot \left\{ (\hat{e}_u \times \vec{r}) \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial \omega}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \frac{\partial \omega}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + (\hat{e}_z \times \vec{r}) \right.$$

$$\left. \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial \Omega}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \frac{\partial \Omega}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + r \sin \ell \hat{e}_u \sum_{n=2}^{\infty} \sum_{m=0}^n \left( \frac{\partial I}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \right. \right.$$

$$\left. \frac{\partial I}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + \left[ (\hat{e}_u \times \vec{r}) \frac{a}{r} \frac{\sin E}{\sqrt{1-e^2}} - \bar{a} \right] \sum_{n=2}^{\infty} \sum_{m=2}^n \left( \frac{\partial e}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \right.$$

$$\left. \frac{\partial e}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) + \left( 2\pi \frac{\dot{\vec{r}}}{\bar{n}} - \frac{2}{3} \frac{\vec{r}}{\bar{n}} \right) \sum_{n=2}^{\infty} \sum_{m=2}^n \left( \frac{\partial M}{\partial \bar{C}_{nm}} d\bar{C}_{nm} + \frac{\partial M}{\partial \bar{S}_{nm}} d\bar{S}_{nm} \right) \}$$

$$- B_2 \bar{A} R_3 (-\hat{\theta}) dX$$

where

$\bar{A}$  is the first two rows of the matrix A in equation (3.6 - 17)

From these observation equations normal equations are formed and solved using standard least squares techniques. The quantity that is minimized is the summation of the squares of the differences between the observed quantities  $dU$ ,  $dW$  and their computed values  $du$ ,  $dw$

$$\sum_i [(dU_i - du_i)^2 + (dW_i - dw_i)^2] = \min$$

At the starting point there are no approximate values for the corrections of the coefficients so that  $du$  and  $dw$  are zero. The initial along-track and across-track residuals  $dU$  and  $dW$  are provided by the differential orbit improvement program (DOI 3), together with the instantaneous orbital elements. The length of an orbital arc represented by a single empirical polynomial varies between one and four weeks, and the number of satellite observations in that interval varies from 60 to 2500. The program can simultaneously handle up to 250 orbital arcs. The tesseral harmonics which will be included in the solution depend on the distribution of the observations, the number of satellites and their orbits, and the number of observing stations, since the total number of unknowns is limited to 100.

The RMS's of the original and improved residuals are computed and their ratio, called the respective improvement factor of the solution, is computed by the program. We must remember that the SAO observations, being only direction observations, cannot provide a solution alone. A scale must be provided which has been introduced by defining the GM value, which is used to determine the semimajor axes of the orbits according to Kepler's law [Veis, 1967]. By combining simultaneous laser observations or ground traverses, scale can be introduced and a correction for a nominal value of GM can be obtained by the SAO solution.

### 3.63 Naval Weapons Laboratory (NWL) Dynamic Solution

The NWL satellite observations are mainly Doppler observations. Frequency observations are made every four seconds during each satellite pass. The computation process of the NWL system is described in detail in two NWL reports [Naval Weapons Laboratory, 1967 and 1968] on which the following discussion is mainly based. The geodetic solution is again based on orbit computations which in this case is done by numerical integration. The existing NWL program has the ability to use many kinds of observation data, but we will limit ourselves to discussion of Doppler observations, since Doppler is the principle type of geodetic satellite observation processed by

the NWL. The raw observations first undergo a filtering stage for detection of erroneous data and deletion of superfluous data [Gross, 1968].

The received frequencies are related to the emitted ones by the equation

$$f = f_s [1 - (\dot{\rho}/c)] + \delta f_r \quad (3.6 - 28)$$

where

- $f$  is the received frequency
- $f_s$  is the emitted frequency
- $\dot{\rho}$  is the rate of change of distance between satellite and observation station
- $c$  is the velocity of light
- $\delta f_r$  is the ionospheric refraction correction

The NWL uses the mean equinox and the mean earth's spin axis of a date to establish an inertial coordinate system. In this system we have again equation (3.6 - 1)

$$\vec{\rho}(t) = \vec{r}(t) - \vec{R}(t) \quad (3.6 - 29)$$

where

- $\vec{r}(t)$  is the geocentric satellite position vector
- $\vec{\rho}(t)$  is the topocentric position vector of the satellite
- $\vec{R}(t)$  is the geocentric station vector

Then  $\dot{\rho}_1$ , which is the rate of change of the slant range  $|\vec{\rho}_1|$ , is

$$\dot{\rho}_1 = \hat{\rho} \cdot (\dot{\vec{r}} - \dot{\vec{R}})$$

where

- $\hat{\rho} = \frac{\vec{r} - \vec{R}}{|\vec{r} - \vec{R}|}$  is the unit vector in the topocentric direction of the satellite
- $\dot{\vec{r}}$  is the satellite velocity in the inertial system
- $\dot{\vec{R}}$  is the station velocity vector in the inertial system which can be written

$$\dot{\vec{R}} = (NP)^{-1} \vec{\omega} \times \vec{R}$$

where

P is the general precession

N is the nutation matrix [Mueller, 1969]

$$\vec{\omega} = \begin{vmatrix} 0 \\ 0 \\ \omega \end{vmatrix} \quad \text{with } \omega \text{ the rotation rate of the earth}$$

Thus equation (3.6 - 28) becomes

$$f = f_s \left[ 1 - \hat{\rho} \cdot \frac{(\dot{\vec{r}} - \dot{\vec{R}})}{c} \right] + \delta f_r \quad (3.6 - 30)$$

If we linearize this equation we have

$$\frac{\partial f}{\partial f_s} \delta f_s + \frac{\partial f}{\partial r} \delta r + \frac{\partial f}{\partial p} \delta p + \frac{\partial f}{\partial X} \delta X = f - f_c \quad (3.6 - 31)$$

where

$\delta f_s$  are corrections to the emitted frequencies

$\delta r$  are corrections to satellite positions and satellite velocities

$\delta p$  are corrections to (a) gravity coefficients, (b) atmospheric drag effects, (c) solar radiation effects

$\delta X$  are corrections to ground station positions

$f_c$  are computed values of  $f$

As we can see from equation (3.6 - 30), to evaluate equation (3.6 - 31) we first need

$\vec{r} = (r_x, r_y, r_z)$  the satellite position

$\dot{\vec{r}} = (\dot{r}_x, \dot{r}_y, \dot{r}_z)$  the satellite velocity

These are provided by the equations of the satellite motion. The motion of the satellite in an inertial coordinate system is represented by the vector differential equation

$$\ddot{\vec{r}} = \vec{F} \quad (3.6 - 32)$$

According to the force model used by the NWL, equation (3.6 - 32) becomes

$$\ddot{\vec{r}} = \vec{G}_E + \vec{G}_M + \vec{G}_S + \vec{D}_G + \vec{R}_p + \vec{T}_S + \vec{T}_M \quad (3.6 - 33)$$

where

- $\vec{G}_E$  is the acceleration due to the earth's gravitational field
- $\vec{G}_M$  is the acceleration due to the moon's gravitational field
- $\vec{G}_S$  is the acceleration due to the sun's gravitational field
- $\vec{D}_G$  is the acceleration due to atmospheric drag
- $\vec{R}_p$  is the acceleration due to solar radiation pressure
- $\vec{T}_S$  is the acceleration due to the solar tidal bulge
- $\vec{T}_M$  is the acceleration due to the moon tidal bulge

The particular models for the above accelerations are described in detail in [Naval Weapons Laboratory, 1968, Chapter 5]. Within the computer program "ASTRO," developed at NWL through many years, equation (3.6 - 33) is numerically integrated to give values of  $\vec{r}$  and  $\dot{\vec{r}}$ , first at equal time intervals and then at the observation times. To perform the integration, Cowell's formula and a prediction-corrector technique is used—of 12th order for positions and 4th order for velocities with a 45 sec interval. Then Lagrangian interpolation is used to get values of  $\vec{r}$  and  $\dot{\vec{r}}$  at the observation times.

We can also write

$$\frac{\partial f}{\partial p} = \sum_{j=x, y, z} \left[ \frac{\partial f}{\partial r_j} \cdot \frac{\partial r_j}{\partial p} + \frac{\partial f}{\partial \dot{r}_j} \cdot \frac{\partial \dot{r}_j}{\partial p} \right] \quad (3.6 - 34)$$

After equation (3.6 - 34), equation (3.6 - 31) becomes

$$\frac{\partial f}{\partial f_s} \delta f_s + \frac{\partial f}{\partial r} \delta r + \sum_{j=x, y, z} \left( \frac{\partial f}{\partial r_j} \cdot \frac{\partial r_j}{\partial p} + \frac{\partial f}{\partial \dot{r}_j} \cdot \frac{\partial \dot{r}_j}{\partial p} \right) \delta p + \frac{\partial f}{\partial X} \delta X = f - f_c \quad (3.6 - 35)$$

The terms  $\partial f / \partial f_s$ ,  $\partial f / \partial r_j$ , and  $\partial f / \partial \dot{r}_j$  are readily available from equation (3.6 - 30) where the ionospheric refraction correction is considered constant

$$\frac{\partial f}{\partial \vec{r}} = -\frac{f_s}{c} \left\{ \frac{1}{|\vec{\rho}|} [(\vec{r} - \vec{R}) - (\hat{\rho} \cdot (\vec{r} - \vec{R})) \hat{\rho}] \right\} \quad (3.6 - 36)$$

$$\frac{\partial f}{\partial \vec{r}} = -\frac{f}{c} \hat{\rho}$$

From there they can be numerically evaluated with the values of  $\vec{r}$  and  $\dot{\vec{r}}$  at the observation times.

To evaluate the remaining partial derivatives we put

$$\xi_1 = \frac{\partial \vec{r}}{\partial p_1}$$

After differentiation we get

$$\ddot{\xi}_1 = \sum_{j=x,y,z} \left[ \frac{\partial \vec{r}}{\partial r_j} \xi_{1j} + \frac{\partial \vec{r}}{\partial r_j} \dot{\xi}_{1j} \right] + \frac{\partial \vec{r}}{\partial p_1} \quad (3.6 - 37)$$

The partial derivatives  $\partial \vec{r} / \partial p_1$  can easily be evaluated by the adopted force model when the force model contains them explicitly, and they are otherwise zero.

The partial derivatives  $\partial \vec{F} / \partial \vec{r}$  and  $\partial \vec{F} / \partial \dot{\vec{r}}$  are obtained by differentiation of the force model with respect to positions and velocities after the model has been expressed as a function of them. These partials are derived in [Naval Weapons Laboratory, 1968, Chapter 6]. The coefficients of equation (3.6 - 37) are then known so that it can be integrated. The components of this equation are also found by numerical integration at equal time intervals and interpolation for the observation times.

The last group of partial derivatives are those with respect to the ground stations. These can be found from Equation (3.6 - 30) after expression of the station vector in the inertial coordinate system  $\vec{R}$  as a function of the position vector in earth-fixed coordinates  $\vec{X}$ . For this purpose we take

$$\vec{R} = R_3 (-\theta) \vec{X} \quad (3.6 - 38)$$

and the partials are

$$\frac{\partial f}{\partial \vec{X}} = \frac{f_s}{c} \left( \left[ \hat{\rho} \cdot \frac{\vec{r} - \vec{R}}{|\vec{\rho}|} \right] \hat{\rho} - \frac{\vec{r} - \vec{R}}{|\vec{\rho}|} - \hat{\rho} \times (NP)^{-1} \vec{\omega} \right)' R_3 (-\theta) \quad (3.6 - 39)$$



Thus all the terms of the observation equations have been evaluated. The observations are weighted reciprocally proportional to their variances obtained during the filtering stage. From the observation equations normal equations are formed, the parameters  $\delta f_s$ ,  $\delta r$ , and the parameters of the drag and radiation pressure force models are eliminated, and the normal equations are stored.

Many groups of such normals summed together are solved to provide the solution for geodetic parameters. The "ASTRO" computer program can simultaneously provide corrections for a complete set of harmonic coefficients of degree and order 21, plus two parameters for each satellite to account for resonance effects, and approximately 100 stations.

Again, Doppler data alone cannot provide a solution and a singularity occurs when all station longitudes and the initial right ascensions of each orbit are considered unknowns. The longitude of one station is held fixed in all Doppler solutions; a correction to the nominal value of GM is provided by the Doppler solution [Anderle et al., 1967].

### 3.64 Jet Propulsion Laboratory (JPL) Solution

The orbital solution of JPL has many similarities to the solution of the NWL, but it also has some special characteristics peculiar to the type of satellite being observed and thus to the type of information being collected.

The JPL is basically observing unmanned spacecraft, usually called probes, travelling approximately 10,000 mi from the earth to interplanetary distances and the edge of the solar system [Jet Propulsion Laboratory, 1968]. The main source of data at JPL is the two-way Doppler, observed by 85- or 210-ft antennas, from the so-called Deep Space Network.

The orbit determination function is basic to the JPL solution. The equations of motion are integrated in terms of the inertial rectangular coordinates, as is the case in the NWL solution. However, the force model is somewhat different [Warner et al., 1964]. The equation of motion is written as

$$\vec{r} = \vec{G}_c + \vec{G}_b + \vec{G}_{cb} + \vec{G}_{obl} + \vec{P}_{pf} + \vec{R}_p \quad (3.6 - 40)$$

where

- $\vec{G}_c$  is the acceleration of the probe due to a central body
- $\vec{G}_b$  is the acceleration of the probe due to n bodies
- $\vec{G}_{cb}$  is the acceleration of the central body due to the n bodies
- $\vec{G}_{obl}$  is the acceleration of the probe due to the oblateness of the earth and/or moon
- $\vec{P}_{pf}$  is the acceleration of the probe during the powered portion of its flight
- $\vec{R}_p$  is the acceleration of the probe due to the pressure of solar radiation

Detailed equations for the above force model are given in [Warner, 1964]. We only remark that, unlike the other cases, the gravitational potential of the earth is represented by the first three zonal harmonics only, as the higher-order terms have a negligible effect on the probes at great distances from the earth. However, the attraction and even the oblateness of other bodies become important to such flights. In the program ODP (Orbit Determination Program) the position and velocity are obtained by a stepwise Cowell numerical integration at intervals depending on the distance of the probe; Everett interpolation is used to obtain values at the observation or any other times. The program can handle 13 types of observations, although two-way Doppler is the main type processed at the JPL.

The linearized mathematical model is

$$A_1 \delta q_1 + B_1 \delta x = \delta f_1 \quad (3.6 - 41)$$

where

- $\delta q_1$  are corrections to parameters unique to the ith mission, i. e., the initial position and velocity vectors of the probe
- $\delta x$  are corrections to parameters common to all missions, i. e., station positions, earth mass
- $\delta f_1$  is Doppler data from the ith probe

$$A_1 = \frac{\partial f_1}{\partial q_1}$$

$$B_1 = \frac{\partial f_1}{\partial x}$$

The parameters common to all missions and which have geodetic significance are

GM	earth gravitational constant
$a_e$	earth radius that scales the lunar ephemeris
$\gamma_B$	solar pressure constant
$\bar{C}_{20}, \bar{C}_{30}, \bar{C}_{40}$	the first three zonal harmonics
$R_1, B_1, L_1$ or $x_1, x_2, x_3$	station coordinates shown in Fig. 3.6-2

Equations for the above partial derivatives are given in [Warner et al., 1964].

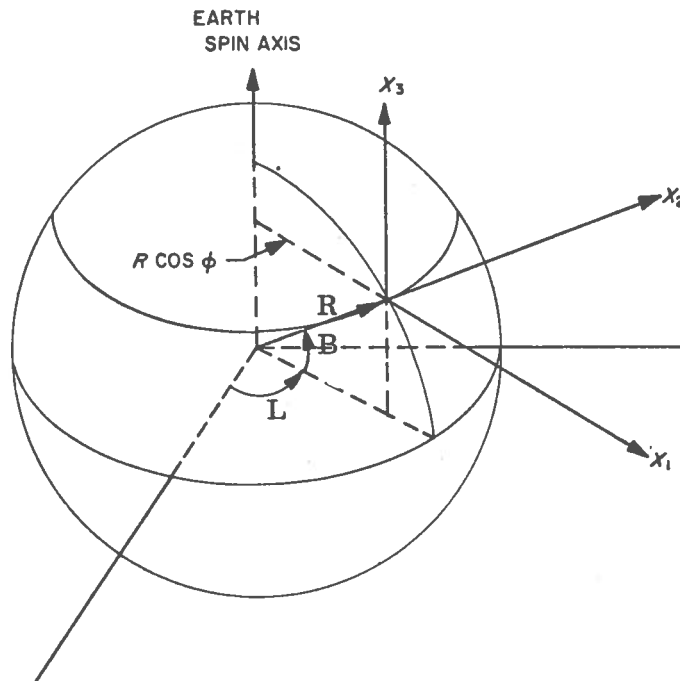


Fig. 3.6-2 R, B, L and  $x_1, x_2, x_3$  coordinate systems

Weights  $P_1$  are assigned to the data, based on the preprocessing phase, and to the parameters which will be estimated, based on the estimated reliability of their a priori value.

From the observation equations, normal equations are formed:

$$\begin{pmatrix} A_1^T P_1 A_1 & A_1^T P_1 B_1 \\ B_1^T P_1 A_1 & B_1^T P_1 B_1 \end{pmatrix} \begin{pmatrix} \delta q_1 \\ \delta x_1 \end{pmatrix} = \begin{pmatrix} A_1^T P_1 \delta f_1 \\ B_1^T P_1 \delta f_1 \end{pmatrix} \quad (3.6 - 42)$$

We call the solution from the  $i$ th mission

$$\hat{q}_1 = q_1 + \delta q_1$$

$$\hat{x}_1 = x_1 + \delta x_1$$

We form the difference of the  $i$ th solution  $\hat{x}_1$  and a standard value  $x_0$

$$\delta \hat{x}_1 = \hat{x}_1 - x_0$$

From each experiment we also form the matrix

$$M_1 = B_1^T P_1 B_1 - (A_1^T P_1 B_1)^T (A_1^T P_1 A_1)^{-1} A_1^T P_1 B_1 \quad (3.6 - 43)$$

Then the least squares estimate  $x^*$  of the common parameters over all missions and thus over all groups of observations  $f_1, f_2, \dots, f_n$  is given by

$$x^* = x_0 + \delta x^*$$

where

$$\delta x^* = \left( \sum_{i=1}^n M_i \right)^{-1} \sum_{i=1}^n M_i \delta \hat{x}_i \quad (3.6 - 44)$$

The characteristics of the JPL solution for the parameters important for the world geodetic system are:

- (a) The JPL has provided the best solution for the gravitational constant GM.
- (b) The recovery of the zonal harmonics is not of geodetic significance because of the great distance of the probes.
- (c) The coordinates of the stations are not all determined with the same accuracy; the estimates of longitudes are stronger than those

of latitudes. More specifically, it has been demonstrated that the estimation of  $x_1$  is better than  $x_2$ , and there is no information at all affecting  $x_3$  [Vegos et al., 1968].

### 3.65 Combination of Satellite Dynamics

From what we have said so far, an outline of a possible solution of the problem of combination of satellite dynamics could be as follows. Considering the number of ground stations, the number of satellites, and the total number of observations, together with the final required accuracies, the degree of a basic gravity model could be decided [Guier et al., 1965]. At present, however, the high-degree harmonic coefficients are in general better determined by terrestrial methods than by satellites. Therefore, the degree of this basic gravity model will be decided by the accuracies of the terrestrial solutions. This question will be examined again in Chapter 4. In addition to this basic model, allowance must be provided for additional terms for possible resonance effects. Beyond this, force models for the gravitational potential of the moon and the sun, atmospheric drag, solar radiation pressure, and luni-solar tidal effects must be agreed upon and used by all participating agencies.

Then agencies like SAO and NWL, using the same values of the various constants, will prepare normal equations for corrections to ground stations, to harmonic coefficients, and to the constant GM. The NWL normal equations are already in this system; the SAO normals will be formed from the summation of the normals of the tesseral harmonic and zonal harmonic solutions where the coefficient  $\bar{C}_{20}$  will also be carried as an unknown.

Systems like JPL that have only a few stations, and in a different coordinate system, a few significant harmonics, and a few very significant constants can be combined in two ways. When the corrections to JPL ground station coordinates are expressed as  $x_1$ ,  $x_2$ ,  $x_3$  and when polar motion has been included, their relations with the coordinates of the world geodetic system are simple, linear, and separate for each coordinate. We can then substitute and change the JPL system to make it consistent with the definitions

of the parameters of the other two groups. For that we must also eliminate the parameters used by JPL which are not of geodetic interest. When the corrections to the coordinates of JPL ground stations are in the form of geocentric latitude, longitude, and radius, we can use the solution vector and its covariance matrix to form a set of observation equations and then normal equations. A mathematical model on which we can base this solution is the following:

$$\begin{aligned}
 F_1 &= \frac{Z}{(X^2 + Y^2 + Z^2)^{\frac{1}{2}}} - \sin B = 0 \\
 F_2 &= \frac{Y}{X} - \tan L = 0 \\
 F_3 &= (X^2 + Y^2 + Z^2)^{\frac{1}{2}} - R = 0 \\
 &\quad GM_S - GM_J = 0 \\
 F_4 &\quad \bar{C}_{20S} - \bar{C}_{20J} = 0 \\
 &\quad \bar{C}_{30S} - \bar{C}_{30J} = 0 \\
 &\quad \bar{C}_{40S} - \bar{C}_{40J} = 0
 \end{aligned} \tag{3.6 - 45}$$

where the subscript S denotes a value of a parameter from the satellite solution and the subscript J denotes the value from a JPL solution. Equations similar to the last can be formed for every parameter for which we want to force the value from the JPL solution to be equal to the value from the satellite solution. This model will give observation equations of the mixed type

$$AX + BV + W = 0$$

The partial derivatives with respect to the parameters are

$$\begin{aligned}
 \frac{\partial F_1}{\partial X} &= - \frac{X}{R^3} \\
 \frac{\partial F_1}{\partial Y} &= - \frac{Y}{R^3} \\
 \frac{\partial F_1}{\partial Z} &= \frac{R^2 - Z}{R^3}
 \end{aligned} \tag{3.6 - 46}$$

$$\frac{\partial F_2}{\partial X} = - \frac{Y}{X^2}$$

$$\frac{\partial F_2}{\partial Y} = \frac{1}{X}$$

$$\frac{\partial F_2}{\partial Z} = 0$$

$$\frac{\partial F_3}{\partial X} = \frac{X}{R}$$

$$\frac{\partial F_3}{\partial Y} = \frac{Y}{R}$$

$$\frac{\partial F_3}{\partial Z} = \frac{Z}{R}$$

Derivatives with respect to the observed quantities are

$$\frac{\partial F_1}{\partial B} = - \cos B \quad (3.6 - 47)$$

$$\frac{\partial F_2}{\partial L} = - \frac{1}{\cos^2 L}$$

$$\frac{\partial F_3}{\partial R} = - 1$$

The partial derivatives of the equations of group  $F_4$  with respect to the parameters are zero or 1 and those with respect to the observed quantities are zero or -1.

We form normal equations and then eliminate the correlates. The normal equations obtained can now be added to the normal equations of the other satellite solutions.

Again we will have the problem of relative weighting which must be solved according to the discussion in Section 3.8.

The normal equations will be of the form

$$N_s \begin{vmatrix} dX \\ dC \\ da \end{vmatrix} = U_s \quad (3.6 - 48)$$

That is, they will contain corrections for the geocentric coordinates, the spherical harmonics, and the elements of the general terrestrial ellipsoid. It will not be necessary to fix any direction or scale because the direction will be provided by the optical observations and the scale by the strong determination of GM.

### 3.7 Some Additional Constraints

We are seeking a solution which will provide the level ellipsoid to be used as a reference surface, as well as the geocentric coordinates of some physical points and the harmonic coefficients of the gravity field of the earth. This reference ellipsoid will have its center at the center of mass of the earth and will have the same angular velocity  $\omega$ , the same difference of moments of inertia  $C - \bar{A}$ , the same mass  $M$ , and the same potential  $W_0$  as the geoid; this means that it will be the mean earth ellipsoid.

To assure the above, some constraints between the unknown parameters must be applied. The coincidence of the center of the ellipsoid with the center of mass of the earth has been achieved by forcing to zero all the first-degree and second-degree, first-order harmonic coefficients of the geopotential. The rotation velocity  $\omega$  is the one defined in [Explanatory Supplement, 1961, p. 76] which is well beyond the noise of the observations; thus it can be considered constant and without errors [Rapp, 1967c]. For the remaining parameters the following three equations will be used to constrain the values of the unknowns. From [Moritz, 1967, p. 111]:

$$df = \frac{3}{2} dJ_2 = -\frac{3}{2} dC_{20} = -\frac{3\sqrt{5}}{2} d\bar{C}_{20} \quad (3.7-1)$$

$$d\gamma_e = \Delta g_0 + \frac{1}{3} \gamma_e df = \Delta g_0 - \frac{\sqrt{5}}{2} \gamma_e d\bar{C}_{20} \quad (3.7-2)$$

From [Rapp, 1967c]:

$$GM = \gamma_e a^2 (1-f) + \omega^2 a^3 (1-f) \left( 1 + \frac{e' q_0'}{6q_0} \right) \quad (3.7-3)$$



where

$$\begin{aligned} e' & \text{ is the second eccentricity of the ellipsoid} \\ q_0 & = \frac{1}{2} [\alpha_0 - 3 \cot \alpha_0 (1 - \cot \alpha_0)] \\ q'_0 & = - \frac{3(1 - \alpha_0 \cot \alpha_0)}{\sin^2 \alpha_0} - 1 \\ \alpha_0 & = \tan^{-1} e' \end{aligned}$$

The first two equations have been used to eliminate two unknowns,  $df$  and  $dy_e$ , from previous models. Only the third equation will actually be used as an additional constraint in the adjustment process. After linearization, equation (3.7 - 3) takes the form

$$\begin{aligned} a^2 (1 - f) dy_e + \left[ 2a\gamma_e (1 - f) + 3a^2 \omega^2 (1 - f) \left( 1 + \frac{e' q'_0}{6q_0} \right) \right] da \\ - \left[ \gamma_e a^2 + \omega^2 a^3 \left( 1 + \frac{e' q'_0}{6q_0} \right) \right] df + \omega^2 a^3 (1 - f) \frac{q'_0}{6q_0} de' - d(GM) + W = 0 \end{aligned} \quad (3.7 - 4)$$

From the relation

$$e'^2 = \frac{e^2}{1 - e^2}$$

we have

$$de' = \frac{de^2}{2e'(1 - e^2)^2} \quad (3.7 - 5)$$

and from the relation

$$e^2 = 2f - f^2$$

we have already derived that

$$de^2 = - 3\sqrt{5} (1 - f) d\bar{C}_{\infty} \quad (3.7 - 6)$$

Substituting (3.7 - 2), (3.7 - 5), and (3.7 - 6) into (3.7 - 4), we have

$$\begin{aligned} a^2 (1 - f) \Delta g_0 + a(1 - f) \left[ 2\gamma_e + 3a\omega^2 \left( 1 + \frac{e' q'_0}{6q_0} \right) \right] da \\ + \frac{3}{2} \sqrt{5} \left[ \frac{2+f}{3} \gamma_e a^2 + \omega^2 a^3 \left( 1 + \frac{e' q'_0}{6q_0} - \frac{q'_0}{6q_0 e' (1 - e^2)} \right) \right] d\bar{C}_{\infty} - d(GM) + W = 0 \end{aligned} \quad (3.7 - 7)$$

where

$W$  is the misclosure of equation (3.7 - 3) estimated with the approximate values of the parameters  $GM, \omega, a, \gamma_0, f, e'$  of the approximate level ellipsoid

The simplest way to introduce this constraint into our system is to consider it as an observation equation and give proper weight to the misclosure  $W$ . We can also treat it as a constraint following the method described by [Uotila, 1967b]. In this case we write equation (3.7 - 7) in matrix form:

$$CX + W = 0$$

and the normal equations from previous models

$$NX = U$$

These can be combined in the system

$$\begin{bmatrix} N & C' \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ -k \end{bmatrix} = \begin{bmatrix} U \\ -W \end{bmatrix}$$

The solution of this system is usually given as a correction  $\delta X$  to the solution  $X^*$  obtained by the adjustment of the previous models. The correction is given by

$$\delta X = N^{-1}C'[CN^{-1}C']^{-1}[-W - CX^*] \quad (3.7 - 8)$$

If we use this constraint as an observation equation and we form normal equations, they will have the form

$$N_7 \begin{bmatrix} \Delta g_0 \\ da \\ dC \end{bmatrix} = U_7 \quad (3.7 - 9)$$

That is, they will contain unknowns related to the parameters of the terrestrial ellipsoid, to the spherical harmonic, and to  $\Delta g_0$ .

### 3.8 Combination of Normal Equations

We summarize in Table 3.8 - 1 the groups of normal equations arising from all groups of observations and the kind of parameters that each group contains. We separated the parameters into four groups, and we indicate

Table 3.8 - 1

Type of Observation and Parameters Involved

Type of Observation	Geocentric Coordinates	Harmonic Coefficients	General Terrestrial Ellipsoid	Datum Shifts and Rotations
Triangulation	dX	dC	da	dx
{ Undulations		dC	da	dx
{ Deflections		dC	da	dx
Geometric Satellites	dX			
Satellite Dynamics	dX	dC	da	
Gravity Anomalies		dC	da	
Additional Conditions		dC	da	

with dX when the group contains geocentric coordinates as unknowns, with dC when the group contains harmonic coefficients, with da when it contains elements of the mean earth ellipsoid, and with dx when it contains the shifts of the origins, the rotations of the geodetic systems, and the differences in scale. The parameter  $\Delta g_0$  is not considered here because it has been eliminated by that stage. The simultaneous solutions of all these groups of normal equations will provide the four groups of parameters.

We suppose that in the formation of these normals, the same a priori variance of unit weight was used. Also, if correct weights have been assigned to the observations, the solution of the sum of all reduced normal equations is the same as the solution of all observations together [Bjerhammar, 1967]. If the separate solutions of the reduced normals indicate that incorrect weights

have been assigned so that the existing variances of unit weight are not homogeneous, a scaling of the normals can be applied [Köhnlein, 1965; Krakiwsky et al., 1967].

First the validity of the individual solutions must be tested; if the a priori variance is consistent with the assigned variance-covariance matrix and the model is free of model errors, the quantity  $nS_0^2/\sigma_0^2$  should be distributed as a chi-square distribution with  $n$  degrees of freedom,  $\chi^2(n)$  where

- $n$  are the degrees of freedom of the quadratic form of the residuals
- $\sigma_0^2$  is the a priori variance of unit weight
- $S_0^2$  is the same quantity estimated by the adjustment in question

We can then establish an interval  $(a, b)$  such that

$$\Pr \left\{ a < \frac{nS_0^2}{\sigma_0^2} < b \right\} = 0.95$$

We accept the hypothesis that  $S_0^2$  is an unbiased estimate of  $\sigma_0^2$  if  $nS_0^2/\sigma_0^2$  is between these limits and reject this hypothesis otherwise. This confidence interval for  $S_0^2$  is

$$\frac{a\sigma_0^2}{n} < S_0^2 < \frac{b\sigma_0^2}{n} \quad (3.8-1)$$

Because the number of degrees of freedom with which we are usually working is very large, we may use the fact that for large degrees of freedom the variable

$$\sqrt{2\chi^2} - \sqrt{2n-1}$$

is approximately distributed as the standard normal distribution  $N(0,1)$  [Selby, 1968]. Thus  $\chi_a^2$  may be computed by

$$\chi_a^2 = \frac{1}{2} [x_a + \sqrt{2n-1}]^2$$

where

$x_a$  is a point of the cumulative normal distribution  $N(0,1)$  with probability  $1 - a/2$

For different degrees of freedom and  $\sigma_c^2 = 1$ , the confidence intervals are shown below.

<u>n</u>	<u>a/n</u>	<u>b/n</u>
30	0.560	1.567
100	0.738	1.291
500	0.879	1.127
1000	0.914	1.089
2000	0.939	1.063
10000	0.972	1.028

If, from a group of normals, the estimated variance of unit weight is accepted as an unbiased estimate of the a priori variance of unit weight, these normals are added to the other unchanged; if the test indicated that the estimated variance of unit weight differs significantly from the a priori one, this group of normals is multiplied by a factor  $P_1$ , given as

$$P_1 = \frac{1}{S_1^2}$$

and then the normals are added to the other groups.

After the solution of the combined system, the homogeneity of the variances of unit weight can be tested by applying the same test on the new estimate of the variance of unit weight.

The above discussion holds as long as only random errors are involved in these individual adjustments and the normals are free of systematic errors. We must realize, however, that the systematic errors in our models are sometimes significant and that these errors are not usually decreased by increasing the number of observations. The estimated variances of the parameters are then too small and the combination results in an overweighting

of the most numerous relative to the least numerous [Kaula, 1966b]. Then, instead of the above method, a more reasonable approach would be to determine more realistic variances of the estimated parameters. From these we could form a fictitious variance of unit weight so that the weight coefficient matrix would create these variances [Rapp, 1967]. Another approach followed by some investigators is the trial of many combinations of weights basing the decision for the selection on the behavior and properties of the different solutions. A detailed error analysis and a careful evaluation of the systematic errors involved helps to make more proper decisions at that stage.

#### 4. ERROR ESTIMATES FOR THE PRESENT DETERMINATION OF THE COMPONENTS OF THE WORLD GEODETIC SYSTEM

Today there are no general solutions in the sense of Chapter 3; there are, however, many separate solutions and interesting combinations in groups of two where we can see the uncertainties of the present determinations and make some estimates of the uncertainties that would result from a combined solution.

The uncertainties presented here were obtained by procedures which approximate the actual combinations. In these procedures only the main effects, so to speak, were considered in a combination, while secondary effects were ignored. For example, in combining terrestrial and satellite results for the determination of spherical harmonic coefficients, we have neglected the effect of the improvement of the station coordinates on the harmonic coefficients resulting from the same combination.

Although we do not believe that this approximate procedure constitutes, by any means, a method for the determination of variances of the components of the world geodetic system, we were forced to use them by lack of real data.

##### 4.1 Accuracy of the Present Determination of Geocentric Coordinates and Shifts, Rotations, and Scale of Datums

The first component of the world geodetic system consists of the geocentric coordinates of a set of physical points and their variances. The best way of determining these geocentric coordinates is through satellite observations; consequently the existing satellite solutions can be utilized as

the basic step in the determination of the accuracy of that set. We consider the SAO and the NWL solutions which have provided geocentric coordinates for approximately 60 stations shown in Fig. 4.1-1.

The SAO station coordinates are part of a set of geodetic parameters called a 1966 SAO Standard Earth [Lundquist et al., 1966]. They are based on optical satellite observations, and they have been determined together with the tesseral harmonics, essentially up to the 8th degree and order, with 45 additional selected coefficients from degree 9 to 15. For the computation of perturbations, zonal harmonics up to 14th degree were used. The SAO coordinates refer to a geocentric coordinate system with its  $z$  axis through the Conventional International Origin and its  $X$  axis implicitly determined by the defined longitude of the U. S. Naval Observatory ( $77^{\circ}03'55''94$ ) to which tabulations of Universal Time correspond [Lundquist et al., 1966]. The coordinates have an estimated standard deviation approaching 10 m [Lundquist, 1966, p. 6].

The NWL stations are part of the NWL-8 geodetic parameters. They are based on Doppler satellite observations and have been determined simultaneously with all gravity coefficients up to the 12th degree and order together with zonal harmonics up to the 19th degree. Gravity coefficients for six degrees for each of three orders have also been selected for each satellite to account for the dominant resonance effects. The NWL station coordinates also refer to a geocentric system with its  $Z$  axis through the Conventional International Origin. The  $X$  axis, however, is defined by the longitude of the one station which is kept fixed in the NWL solution. The coordinates have an estimated standard deviation of 8 m [Anderle et al., 1967].

There is also some information for the geocentric coordinates contained in the geodetic coordinates of the same points established by the triangulation, which information can be used through equation (3.2-18). In a rigorous combination of satellite and terrestrial data for station coordinates, we could form normal equations of type (3.2-20) using as a model equations (3.2-18) where the observed quantities are the geodetic coordinates. Then we could



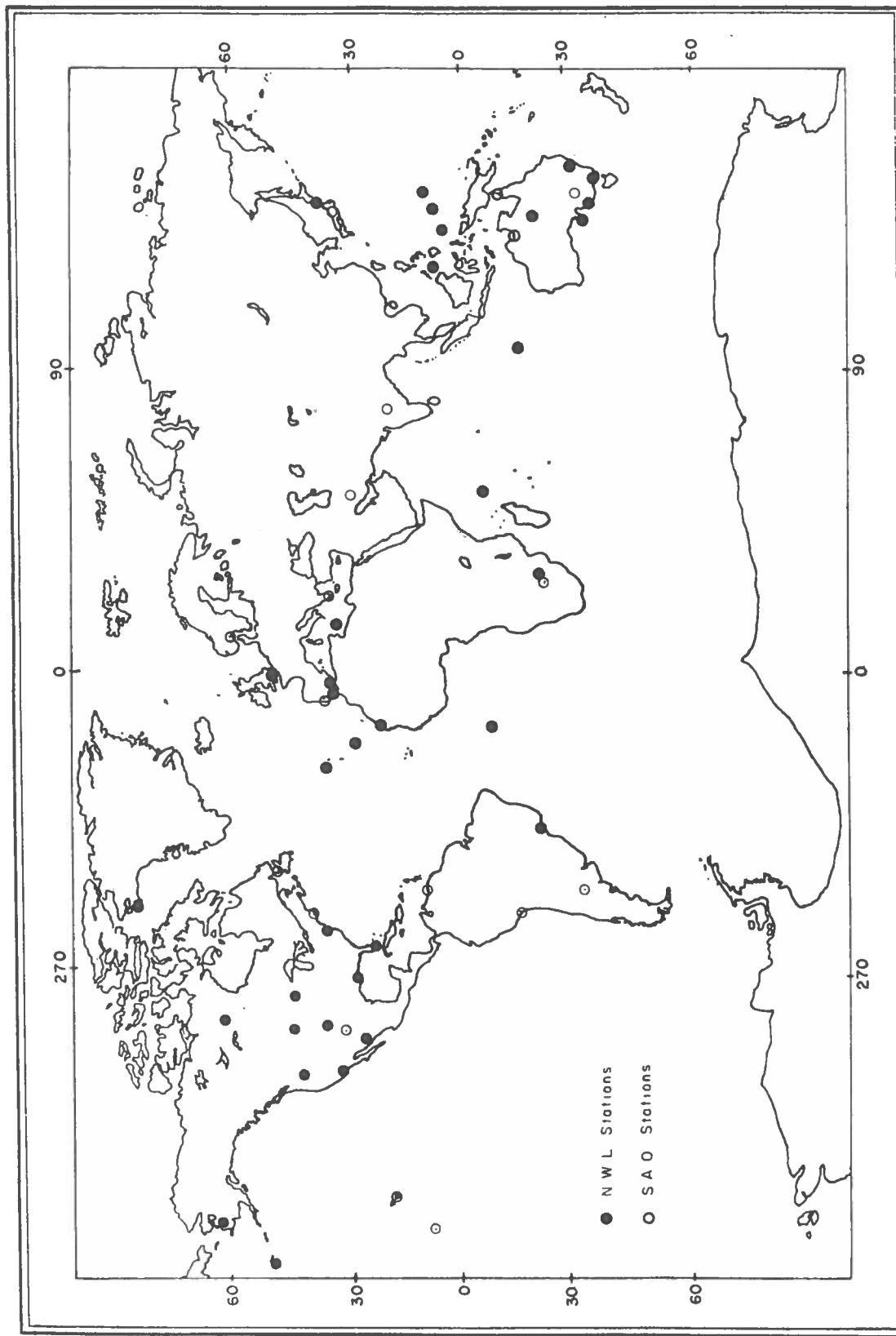


Fig. 4.1-1 SAO and NWL observing stations

add these normals to the normals of the other groups of observations of the world geodetic system. The solution of the new system would provide improved geocentric coordinates and would recover the values of the datum transformation parameters as well as other parameters. Because we do not have the data to perform such a rigorous combination, we will examine what information may be obtained from a combination of coordinates obtained from satellite observations and coordinates obtained from terrestrial triangulation.

Using the geocentric coordinates provided by satellite solutions and the geodetic coordinates provided by the triangulation, an adjustment can be made which will furnish improved geocentric coordinates, and it will recover the values of the datum transformation parameters.

To examine this determination we consider 15 stations, the coordinates of which have been determined from satellite observations and triangulation. Of the 15 stations shown in Fig. 4.1-2, four are SAO stations and eleven are NWL stations.

The geocentric coordinates of the NWL stations are published only as  $\phi$ ,  $\lambda$ ,  $h$  coordinates, where  $h$  is the geometric height above the ellipsoid [Anderle et al., 1967, table 10]. These coordinates refer to a geocentric ellipsoid of the Mercury datum [NASA, 1968]. These coordinates are given in Table 4.1 - 1. From them geocentric Cartesian coordinates have been computed.

The geocentric coordinates of the SAO stations are given in [Lundquist et al., 1966]. Cartesian geocentric coordinates for all stations with their variances are given in Table 4.1 - 2. Covariances are not available and have not been considered.

The geodetic coordinates  $\phi, \lambda, h$  of the same stations on the North American Datum are taken from [NASA, 1968] with the exception of station No. 2200 which has been taken from [Anderle et al., 1967, table 12] and its astrogeodetic undulation from the Geoid Charts of North and Central America 1967 published by the Army Map Service. These coordinates are given in

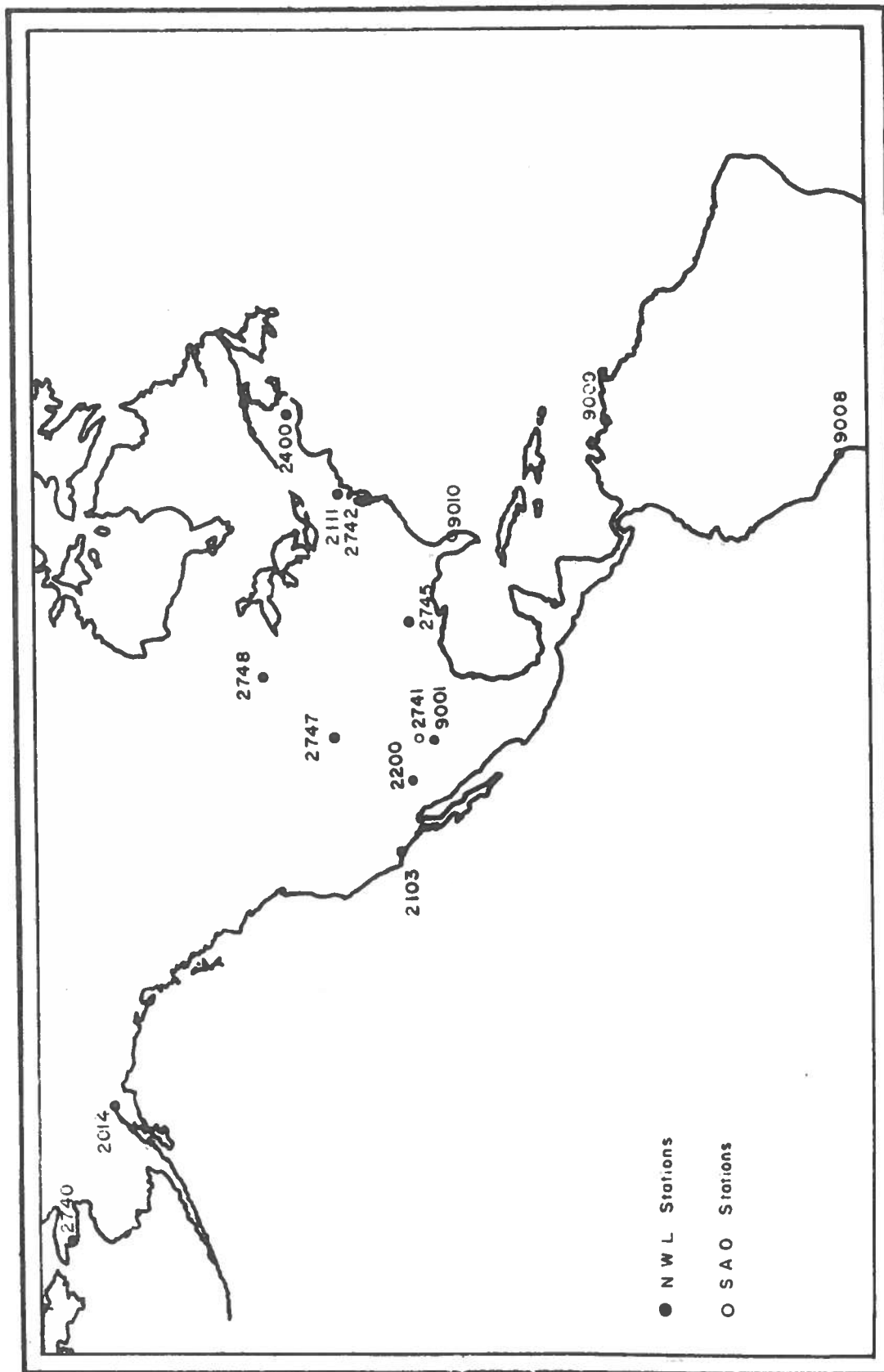


Fig. 4.1-2 SAO and NWL stations on North American datum

Table 4.1 - 1  
Coordinates of NWL Stations Referred to an Ellipsoid  
of  $a = 6\,378\,166\text{ m}$ ,  $1/f = 298.3$

$\phi$	$\lambda$	Name	h
61°16'59.9"	210°10'27.78"	ANCHOR	44. m
32164409	253144471	LACRES	1145.
39094829	283061080	APLMND	80.
34064095	240530814	POIMUC	-71.
44242122	292010985	WINHMA	-35.
64294747	194360097	NUMEAL	-11.
32252467	253264863	NEWMEX	1603.
39013993	283102703	BELTSV	-16.
33253191	269050874	STNVIL	-16.
41080010	255075351	WARAFB	1838.
47563863	262370727	GRANDF	217.

Table 4.1 - 2

## Geocentric Coordinates of the Points Used

$X_m$	$Y_m$	$Z_m$	Name	Variances $m^2$	Longi- tude	Lati- tude	Code No.	
-1535759.	-5166995.	3401041.	IORGAN100.	100. 100.	32.7423253.	44.8	9001	
1942769.	-5804078.	-1796961.	IGARRA100.	100. 100.	-16.466288.	506	9007	
2251820.	-5816915.	1327164.	ICURAC100.	100. 100.	12.089291.	162	9009	
976282.	-5601389.	2980242.	IJUPIRI100.	100. 100.	27.020279.	887	9010	
-2656166.	3-1544347.9	5570647.9	ANCHOR	64. 64.	64.	61.283210.	174	2014
-1556230.	8-5169426.2	3387250.0	LACRES	64. 64.	64.	32.279253.	246	2103
1122623.	6-4823044.8	4006471.3	APLMND	64. 64.	64.	39.163283.	103	2111
-2572078.	2-4618379.2	3556661.1	POIMUC	64. 64.	64.	34.111240.	886	2200
1711147.	3-4231112.3	4440422.5	WINHMA	64. 64.	64.	44.406292.	019	2400
-2664310.	4-694144.4	5733783.4	NOMEAL	64. 64.	64.	64.497194.	600	2740
-1535751.	2-5167000.0	3401044.8	NEWMEX	64. 64.	64.	32.424253.	447	2741
1130761.	7-4830821.3	3994722.6	BELTSV	64. 64.	64.	39.028283.	174	2742
-85022.	7-5327955.6	3493453.7	STNVIL	64. 64.	64.	33.426269.	086	2745
-1234832.	3-4651139.7	4174820.7	WARAFB	64. 64.	64.	41.133255.	132	2747
-549925.	5-4245029.5	4712898.2	GRANDF	64. 64.	64.	47.944262.	619	2748

Table 4.1 - 3. The accuracy of the triangulation can be estimated by a combination of Simmons' equation (3.2 - 27), which gives the standard error on the horizontal plane, and equation (3.4 - 15), which gives the standard error of the astrogeodetic undulations, and thus accounts for the vertical displacement.

Table 4.1 - 3  
Geodetic Coordinates on the North American Datum

Code No.	$\phi$	$\lambda$	Undula- tion m	Orthometric Height H m
9001	32252456	253265117	-0.3	ORGAN 16513
9007	-16280509	288302284	165.	IQUIRA 24519
9009	12052161	291094255	50.	ICURAC 7.4
9010	27011288	279531301	11.4	IJUPTR 15.1
2014	61170198	210103746	-6.2	ANCHOR 68.
2103	32164375	253144825	-1.8	LACRES 1203.
2111	39094783	283061107	1.2	APLMND 145.
2200	34064082	240531253	-30.	POIMUC 3.
2400	44242091	292010917	8.	WINHMA 21.
2740	64295029	194361234	-21.9	NOMEAL 14.
2741	32252440	253265202	-1.3	NEWMEX 1655.
2742	39013946	283102725	1.2	BELTSV 50.
2745	33253157	269051070	4.9	STNVIL 44.
2747	41080008	255075721	8.3	WARAFB 1882.
2748	47563860	262370991	3.	GRANDE 277.

The errors of orthometric heights have been neglected because they are considered very small in comparison with the standard errors of the astrogeodetic undulations.

The North American Datum has been computed according to the development method; therefore the coordinates given cannot be used in equations (3.2 - 19) which have been derived with the assumption that the triangulation has been computed according to the projective method. To change the coordinates of a triangulation from coordinates computed with the development method to approximate coordinates computed in the projective method, the Molodensky correction must be applied [Bomford, 1962, p. 135]. The Molodensky correction is computed by integration along the triangulation chain and thus requires knowledge of both the triangulation chain and the astrogeodetic undulations. This information is not available to us. For most of the NWL stations a correction has been computed to approximate projective station coordinates based on the assumption of a linear change in geoid height along a great circle from datum origin to the station [Anderle et al., 1967, table 21]. We tried to verify these corrections, but as details on the equations and the data used were not given we were unable to do so. We decided therefore to use the geodetic coordinates as they are. Thus the transformation parameters determined this way are those connecting the realization of the present North American datum, as it is defined by the coordinates of the triangulation stations and their variances and the geocentric coordinate system, defined by the geocentric coordinates and associated variances of the same stations computed by satellite observations. Using the coordinates from Table 4.1 - 3, Cartesian coordinates have been computed by equation (3.2 - 1). They are given in Table 4.1 - 4. The variance of the horizontal position has been computed by Simmons' equation for each point. This variance was assumed to be the same in all azimuths. The components in the meridian and the prime vertical planes are given in the first two columns under the heading "variances." The third column contains the variance in the vertical direction computed by equation (3.2 - 15).

Covariances in this horizon system have not been considered. Using Tables 4.1 - 2 and 4.1 - 4, two types of adjustments have been performed. We have used equation (3.2 - 6) as a mathematical model to determine two sets of transformation parameters. The first transforms the North American coordinate system to the SAO system and the second transforms the North American system to the NWL coordinate system.

Table 4.1 - 4  
Geodetic Cartesian Coordinates of the Points Used

Code No.	x m	y m	z m	Name	Variances m <sup>2</sup>
9001	-1232722.9	-2167147.9	3400868.4	IURGAN	18. 18. 7.
9007	1942779.4	-5804219.6	-1797093.2	IQUIRA	235. 235. 136.
9009	2251841.7	-5817089.7	1326996.9	ICURAC	124. 124. 47.
9010	976311.9	-5601550.6	2880064.1	IJUPTR	52. 52. 17.
2014	-2656168.4	-1544504.2	5570468.2	ANCHOR	125. 125. 47.
2103	-1556191.8	-5169591.7	3387072.4	LACRES	19. 19. 7.
2111	1122667.7	-4823204.9	4006284.1	APLMND	43. 43. 14.
2200	-2572045.3	-4618551.7	3556477.6	POIMUC	42. 42. 14.
2400	1711195.3	-4231271.2	4440234.2	WINHMA	66. 66. 22.
2740	-2664795.9	-694297.5	5733602.3	NOMFAL	158. 158. 68.
2741	-1535706.0	-2167128.9	3400865.7	NEWMEX	18. 18. 7.
2742	1130804.9	-4830982.3	3994535.9	BELISV	43. 43. 14.
2745	-84974.6	-2328110.7	3493278.0	SINVIL	17. 17. 7.
2747	-1234787.0	-4651305.3	4174632.9	WARAFB	8. 8. 4.
2748	-549891.5	-4245194.0	4712715.2	GRANDE	18. 18. 7.



By using equation (3.2 - 6), we have determined total shifts, because, as has already been explained, when only coordinates in the two systems are available, the two parts of the total shift cannot be separated and corrections for the semidiameter and the flattening cannot both be recovered. In determining the transformation parameters, the geocentric and the geodetic coordinates have both been considered as observed quantities with variances implied by the variances of Tables 4.1 - 2 and 4.1 - 4 respectively. The covariances generated when the computed variances in the horizon system are transformed to variances in rectangular coordinates have been carried through all the adjustments.

We initially determined the transformation parameters using the four SAO stations. The results of these determinations are shown in Table 4.1 - 5. First, an adjustment was performed using only the three total shifts, the results of which are shown in the first part of Table 4.1 - 5. Second, we made the same adjustment using all the seven transformation parameters. This time we computed the angular transformation parameters according to the definitions of Veis, Molodensky and Bursa. By comparing equations (3.2 - 5) with (3.2 - 2b) or (3.2 - 2c), we have the relations between Veis' rotations and Molodensky's and Bursa's. These are

$$\begin{aligned} da_3 &= \sin\varphi_0 dA - \cos\varphi_0 d\nu \\ da_2 &= \cos\varphi_0 \lambda_0 dA + \cos\lambda_0 d\mu + \sin\varphi_0 \sin\lambda_0 d\nu \\ da_1 &= \cos\varphi_0 \cos\lambda_0 dA - \sin\lambda_0 d\mu + \sin\varphi_0 \cos\lambda_0 d\nu \end{aligned} \quad (4.1 - 1)$$

We were then able to transform Veis' parameters to Molodensky's and show that Veis' rotations and Molodensky's or Bursa's are equivalent.

In Table 4.1 - 5a, we give the correlation matrices of the parameters of these adjustments. First, the correlation matrix of the three shifts is given. Then three correlation matrices corresponding to the transformation parameters of Veis, Molodensky and Bursa are given. We also tested the significance of several parameters by applying an F test. The F test [Scheffé, 1959] uses

Table 4.1-5

North American Datum Shifts, and Shifts, Rotations  
and Scale Corrections from SAO Stations

Solution with Three Parameters			
$dx_0 = -27.0 \text{ m} \pm 6.9$			
$dy_0 = 157.7 \text{ m} \pm 6.3$			
$dz_0 = 167.2 \text{ m} \pm 6.9$			
$\sigma_0 = 1. \text{ m}$		$V'PV = 9.9$	
$\hat{\sigma}_0 = 1.05 \text{ m}$		$d.o.f. = 9$	
Solution with Seven Parameters			
Parameter	Veis	Molodensky	Bursa
$dx_0$	$-37.9 \text{ m} \pm 7.2$	$-37.9 \text{ m} \pm 7.2$	$-75.6 \text{ m} \pm 24.2$
$dy_0$	$164.1 \pm 7.2$	$164.1 \pm 7.2$	$173.0 \pm 16.2$
$dz_0$	$174.8 \pm 7.1$	$174.8 \pm 7.1$	$120.8 \pm 20.6$
$dA, da_3$	$-1''.17 \pm 0.6$	$-1''.70 \pm 0.8$	$-1''.70 \pm 0.8$
$d\mu, da_2$	$1.29 \pm 0.8$	$-0.08 \pm 0.5$	$-0.08 \pm 0.5$
$d\nu, da_1$	$1.24 \pm 0.7$	$1.29 \pm 0.8$	$1.29 \pm 0.8$
$\epsilon$	$5.73 \times 10^{-6} \pm 2.1$	$5.73 \times 10^{-6} \pm 2.1$	$5.73 \times 10^{-6} \pm 2.1$
$\sigma_0 = 1$		$V'PV = 3.0$	
$\hat{\sigma}_0 = 0.77$		$d.o.f. = 5$	
		$V'PV = 7.3$	
		$d.o.f. = 6$	
3 shifts + 3 rotations			
From equation (4.1 - 1)			
$da_3 = -1''.70$		$da_2 = -0''.07$	$da_1 = 1''.29$

Table 4.1-5a  
Correlation Matrices of Datum Transformation Parameters  
from SAO Stations

Solution with Three Parameters

1.000	0.047	-0.002
	1.000	0.051
		1.000

Solution with Veis' Seven Parameters

1.000	-0.009	0.021	0.492	-0.046	-0.023	-0.347
	1.000	0.022	0.010	-0.489	-0.057	0.189
		1.000	0.309	-0.191	-0.025	0.575
			1.000	0.545	-0.555	0.013
				1.000	-0.656	-0.049
					1.000	-0.005
						1.000

Solution with Molodensky's Seven Parameters

1.000	-0.009	0.021	0.218	-0.526	-0.095	-0.347
	1.000	0.022	0.042	-0.075	0.495	0.189
		1.000	0.130	-0.300	-0.223	0.575
			1.000	0.181	-0.689	0.003
				1.000	-0.226	-0.006
					1.000	-0.051
						1.000

Solution with Bursa's Seven Parameters

1.000	0.426	0.654	0.952	-0.363	-0.684	-0.038
	1.000	0.396	0.460	0.134	-0.593	0.773
		1.000	0.657	0.176	-0.943	-0.175
			1.000	0.181	-0.689	0.003
				1.000	-0.226	-0.006
					1.000	-0.051
						1.000

the statistic defined by

$$F = \frac{n - u}{q} \frac{V'PV_{\omega} - V'PV_{\Omega}}{V'PV_{\Omega}} \quad (4.1 - 2)$$

where

$n$  is the number of observations

$u$  is the number of parameters in the unrestricted model

$q$  is the number of parameters to be tested

$V'PV_{\Omega}$  is the quadratic form of the residuals in the unrestricted model

$V'PV_{\omega}$  is the quadratic form of the residuals under the hypothesis to be tested

When the hypothesis to be tested is that a set of  $q$  parameters are equal to zero, the statistic  $F$  is distributed as  $F_{q, n-u}$ . The test consists of selecting a significance level  $\alpha$  and rejecting the hypothesis if and only if  $F > F_{\alpha; q, n-u}$ . Based on Table 4.1 - 5 and the table of values of  $F$  for  $\alpha = 0.05$  given in [Scheffé, 1959, pp. 426-427], we computed the following tests.

For the adjustment with three rotations and the scale

$$F = \frac{5}{4} \frac{9.0 - 3.0}{3.0} = 2.88 \quad F_{0.05; 4, 5} = 5.19$$

For the adjustment with three rotations

$$F = \frac{6}{3} \frac{7.3 - 3.0}{3.0} = 2.86 \quad F_{0.05; 3, 6} = 4.76$$

For the scale only

$$F = \frac{5}{1} \frac{9.3 - 7.3}{7.3} = 1.78 \quad F_{0.05; 1, 5} = 6.61$$

From the above we conclude that when datum parameters were determined from SAO stations, the scale correction and the rotations were not significant. The insignificance of the rotations and of the scale probably means that the data is not able to detect these very small corrections. The above tests are the same for all three groups of parameters as the residuals and their quadratic forms are identical for all three.

We also determined the same parameters using the NWL stations. The results are given in Tables 4.1 - 6 and 4.1 - 6a. Now using Table 4.1 - 6, we made the same tests as above with the following results.

For the three rotations and the scale

$$F = \frac{26}{4} \frac{15.1 - 7.9}{7.9} = 5.92 \quad F_{0.05; 4, 28} = 2.74$$

For the three rotations

$$F = \frac{27}{3} \frac{9.7 - 7.9}{7.9} = 2.1 \quad F_{0.05; 3, 27} = 2.96$$

For the scale

$$F = \frac{26}{1} \frac{15.1 - 9.7}{9.7} = 14.5 \quad F_{0.05; 1, 28} = 4.3$$

From the above test we conclude that when the datum transformation parameters are determined from the NWL stations, the rotations are not significant but the scale correction is.

By observing Tables 4.1 - 5, 4.1 - 5a, 4.1 - 6, 4.1 - 6a, we also notice that Bursa's parameters provide a weaker determination of shifts and generally larger correlations. Thus Molodensky's parameters are preferable when rotations around the axis of the geodetic system are desired.

Another group of adjustments was performed using equation (3.2 - 6). This time only the geodetic coordinates of Table 4.1 - 4 were considered to be

Table 4.1 - 6

North American Datum Shifts, and Shifts, Rotations,  
and Scale Correction from NWL Stations

Solution with Three Parameters			
$dx_0 = -40.2 \text{ m} \pm 2.1$			
$dy_0 = 161.8 \text{ m} \pm 2.0$			
$dz_0 = 182.1 \text{ m} \pm 2.0$			
$\sigma_0 = 1. \text{ m}$		$V'PV = 15.1$	
$\hat{\sigma}_0 = 0.71 \text{ m}$		$d.o.f. = 30$	

Solution with Seven Parameters			
Parameter	Veis	Molodensky	Bursa
$dx_0$	$-42.2 \text{ m} \pm 1.7$	$-42.2 \text{ m} \pm 1.7$	$-23.9 \text{ m} \pm 6.5$
$dy_0$	$162.8 \pm 1.6$	$162.8 \pm 1.6$	$152.0 \text{ m} \pm 6.0$
$dz_0$	$182.2 \pm 1.7$	$182.2 \pm 1.7$	$194.7 \text{ m} \pm 6.8$
$dA, da_3$	$0.''45 \pm 0.2$	$0.''78 \pm 0.2$	$0.''78 \pm 0.2$
$d\mu, da_2$	$-0.15 \pm 0.2$	$0.08 \pm 0.2$	$0.08 \pm 0.2$
$d\nu, da_1$	$-0.64 \pm 0.2$	$-0.14 \pm 0.2$	$-0.14 \pm 0.2$
$\epsilon$	$-2.21 \times 10^{-6} \pm 0.9$	$-2.21 \times 10^{-6} \pm 0.9$	$-2.21 \times 10^{-6} \pm 0.9$
$\sigma_0 = 1.$		$V'PV = 7.9$	
$\hat{\sigma}_0 = 0.55$		$d.o.f. = 26$	
		$V'PV = 9.7$	3 shifts + 3 rotations
		$d.o.f. = 27$	

From equation (4.1 - 1)

$da_3 = 0.''78$	$da_2 = 0.''08$	$da_1 = -0.''14$
-----------------	-----------------	------------------

Table 4.1-6a  
Correlation Matrices of Datum Transformation Parameters  
from NWL Stations

Solution with Three Parameters

1.000	-0.006	0.021
	1.000	0.094
		1.000

Solution with Veis' Seven Parameters

1.000	-0.016	0.017	-0.120	0.013	0.212	0.005
	1.000	0.086	-0.006	-0.046	-0.044	-0.182
		1.000	0.022	0.360	0.071	-0.053
			1.000	0.131	0.160	0.005
				1.000	0.295	0.091
					1.000	-0.014
						1.000

Solution with Molodensky's Seven Parameters

1.000	-0.016	0.017	-0.263	0.048	0.007	0.005
	1.000	0.086	0.035	0.037	0.043	-0.182
		1.000	-0.048	-0.114	0.363	-0.053
			1.000	0.141	-0.161	0.016
				1.000	-0.301	-0.010
					1.000	0.094
						1.000

Solution with Bursa's Seven Parameters

1.000	0.233	0.253	0.703	0.750	-0.299	0.103
	1.000	0.231	0.034	0.220	-0.700	0.586
		1.000	0.128	0.340	-0.812	-0.598
			1.000	0.141	-0.161	0.016
				1.000	-0.301	-0.010
					1.000	0.094
						1.000

observed quantities. To the previous observation equations we added the ones formed from the following simple model:

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} L_x \\ L_y \\ L_z \end{vmatrix}$$

where

$L_x, L_y, L_z$  are adjusted values of  $X, Y, Z$

The linearization of this gives the following observation equations

$$\begin{vmatrix} dX \\ dY \\ dZ \end{vmatrix} = \begin{vmatrix} L_x^b \\ L_y^b \\ L_z^b \end{vmatrix} - \begin{vmatrix} X_o \\ Y_o \\ Z_o \end{vmatrix} + \begin{vmatrix} V_x \\ V_y \\ V_z \end{vmatrix}$$

In these equations the observed quantities were the coordinates provided by satellite solutions with their variances. Covariances were not available and were neglected. The unknowns for this adjustment were the datum transformation parameters and the geocentric coordinates. It is understood and has been verified by the solutions that the datum transformation parameters are identical with the ones given by the previous adjustment. The corrections to geocentric coordinates given by this adjustment represent the effect of the terrestrial triangulation on the geocentric coordinates determined from satellites. We performed two separate determinations; one using only the SAO stations and one using only the NWL stations. The adjusted geocentric coordinates and their standard errors are given in Table 4.1 - 7. The coordinates of the SAO stations have been determined by an adjustment with the SAO stations only. Thus they refer to the SAO geocentric coordinate system. The coordinates of the NWL stations have been determined by an adjustment using only the NWL stations and therefore they refer to the NWL geocentric coordinate system.

In the adjustments performed we noticed that the variances of unit weight were in all cases within the tolerance limits of the a priori variance.



Table 4.1 - 7  
Adjusted Geocentric Coordinates

Coordinate:		X	$\hat{\sigma}$	Y	$\hat{\sigma}$	Z	$\hat{\sigma}$
SAO Stations	IORGAN	-1,535 765 m	$\pm 6.8$ m	-5,166 996 m	$\pm 7.5$ m	3,401 042 m	$\pm 6.8$ m
	IQUIPA	1,942 767	$\pm 7.2$	-5,804 076	$\pm 7.6$	-1,796 957	$\pm 7.2$
	ICURAC	2,251 824	$\pm 6.4$	-5,816 921	$\pm 6.5$	1,327 163	$\pm 6.5$
	IJUPITER	976 287	$\pm 5.8$	-5,601 384	$\pm 6.2$	2,880 238	$\pm 5.7$
NWL Stations	ANCHOR	-2,656 189	$\pm 3.7$	-1, 544 346	$\pm 3.8$	5,570 649	$\pm 3.6$
	LACRES	-1,556 235	$\pm 2.6$	-5, 169 425	$\pm 2.4$	3,387 254	$\pm 2.7$
	APLMND	1,122 623	$\pm 3.1$	-4, 823 047	$\pm 2.8$	4,006 469	$\pm 3.0$
	POIMUC	-2,572 081	$\pm 3.0$	-4, 618 382	$\pm 3.0$	3,556 661	$\pm 3.1$
	WINHMA	1,711 149	$\pm 3.4$	-4, 231 116	$\pm 3.3$	4,440 420	$\pm 3.4$
	NOMEAL	-2,664 813	$\pm 3.9$	- 694 143	$\pm 3.9$	5,733 784	$\pm 3.8$
	NEWMEX	-1,535 748	$\pm 2.6$	-5,166 993	$\pm 2.4$	3,401 048	$\pm 2.6$
	BELTSV	1,130 760	$\pm 3.1$	-4,830 825	$\pm 2.8$	3,994 721	$\pm 3.0$
	STNVIL	- 85 020	$\pm 2.6$	-5,327 975	$\pm 2.3$	3,493 460	$\pm 2.5$
	WARAFB	-1,234 828	$\pm 2.1$	-4,651 141	$\pm 2.0$	4,174 815	$\pm 2.1$
	GRANDF	- 549 931	$\pm 2.5$	-4,245 033	$\pm 2.3$	4,712 897	$\pm 2.2$

$$\text{SAO } \hat{\sigma}_0 = 0.77 \text{ m}$$

$$\text{NWL } \hat{\sigma}_0 = 0.55 \text{ m}$$

This implies that we reject the hypothesis that the variance of unit weight is significantly different from unit. From Table 4.1 - 7 it is evident that the existing ground connections supported by the astrogeodetic undulations improve the present solutions for station coordinates, and it is worthwhile to include them in the solution.

The above adjustment approximates a regular combination of satellite and survey observations; this approximation suffers from the neglect of the

covariances between the satellite coordinates and between the survey coordinates. The omission of satellite covariances does not appear to be very serious as the correlations among the components of one station are very small and those among the coordinates of different stations are entirely negligible [Gaposchkin, 1966a]. The correlations among triangulation stations and astrogeodetic undulations have been omitted because of a lack of information on that subject, and nothing can be said until the studies now under way are concluded [Pope, 1969]. Intuitively, however, the correlation between geodetic coordinates of such widely separated stations must be small, and thus the fact that they have been excluded will not invalidate the above results.

#### 4.2 Accuracy of Present Determination of Spherical Harmonic Coefficients of the Potential

The next component of the world geodetic system to be examined is the set of spherical harmonic coefficients of the geopotential. As we have seen, there are three main sources of information for the harmonics, namely, satellites, terrestrial gravity, and astrogeodetic undulations; and we will examine the accuracy obtained from each source as well as from their combination.

##### 4.21 Determination of Harmonic Coefficients from Satellites

First we examine the determination of spherical harmonic coefficients from satellites. Since the zonals and tesserals are determined separately, we will first examine the determination of the zonal harmonics.

The zonal harmonics, according to Section 3.6, are determined from the secular motions of the node and perigee and from the long periodic perturbations. Every such determination is always a determination of a limited number of coefficients, and it is effected by the neglected higher-order terms to such an extent that the estimation by least squares is questionable. A further problem is that only a limited number of satellite orbits on which to base the analysis are available. Thus the analyst must make a choice [King-Hele, 1962]. He

may either compute the greatest possible number of coefficients and thus account for the effects of as many of them as possible, or he may allow some redundancy by estimating a smaller set of coefficients. In the former case the accuracy of the estimates is decreased because the adjustment does not contain any redundancy, while in the latter case the effect of neglected higher-order terms may degrade the solution. King-Hele [1965] solves for a moderate number of coefficients by least squares or by minimizing the maximum residual (mini-max), and he increases the estimates of the uncertainties to account for the effect of the neglected higher-order terms; Kozai [1967] solves for a large number of unknowns retaining at the same time the least squares process and the uncertainty estimates which come from it; and Smith [1965] solves for as many of the coefficients as there are equations and obtains accuracy estimates by propagating the observational errors into the solution vector. There is also a difference in the method of assigning weights among the above investigators. King-Hele and Smith use a kind of scaling of the observation equations by making the coefficient of  $J_{20}$  by 1.5 or 1, respectively, and use the resultant observation equations with equal weights. Kozai does not perform any scaling and assigns weights according to the observational variances increased by the squares of the effects of the coefficients of order 22 to 27 on the assumption that their values are  $0.5 \times 10^{-7}$ . In addition, Kozai, in all his determinations of the even zonal harmonics uses the secular motions both of the node and the perigee [Kozai, 1967], though King-Hele in [King-Hele et al., 1963 and 1965] and Smith [1965] use only the motion of the node. Furthermore, it has been argued that data from the perigee motions are not compatible with that from the nodal motion and must not be used as they are much more sensitive to the neglected higher-order harmonics [Cook, 1965].

The three more recent sets of even zonal harmonics with their uncertainties are given in Table 4.2 - 1. In the last two columns the estimated accuracies of the coefficients are given for conventional and fully normalized

Table 4.2-1

Even Zonal Harmonics and Their Uncertainties  
in Units of  $10^{-6}$

	King-Hele 1965			Smith 1965	Kozai 1967	$\sigma_J$	$\sigma_c$
$J_2$	1082.64 $\pm 0.02$	1082.68* $\pm 0.10$ **		1082.64 $\pm 0.08$	1082.639 $\pm 0.007$	$\pm 0.06$	$\pm 0.027$
$J_4$	-1.52 $\pm 0.03$	-1.61 $\pm 0.10$		-1.70 $\pm 0.25$	-1.608 $\pm 0.021$	$\pm 0.09$	$\pm 0.030$
$J_6$	0.57 $\pm 0.07$	0.71 $\pm 0.10$		0.73 $\pm 0.40$	0.542 $\pm 0.041$	$\pm 0.13$	$\pm 0.036$
$J_8$	0.44 $\pm 0.11$	0.13 $\pm 0.20$		-0.46 $\pm 0.42$	-0.128 $\pm 0.064$	$\pm 0.23$	$\pm 0.056$
$J_{10}$		0.09 $\pm 0.20$		-0.17 $\pm 0.29$	-0.338 $\pm 0.054$	$\pm 0.21$	$\pm 0.046$
$J_{12}$		-0.31 $\pm 0.20$		-0.22 $\pm 0.10$	0.053 $\pm 0.097$	$\pm 0.30$	$\pm 0.060$
$J_{14}$				0.19 $\pm 0.11$	-0.174 $\pm 0.099$	$\pm 0.25$	$\pm 0.046$
$J_{16}$					0.449 $\pm 0.102$		
$J_{18}$					-0.324 $\pm 0.074$		
$J_{20}$					0.334 $\pm 0.069$		

\* based on mini-max

\*\* accuracy estimates

coefficients. We also give the correlation matrices associated with these solutions in Table 4.2 - 2 to illustrate how large the correlations are in these solutions.

The above discussion shows that there is not full agreement about the method of computation, the values of the parameters, or their uncertainties. King-Hele has compared three groups of zonal harmonic coefficients by comparing the secular motions of the longitude of the ascending node which each set would produce in satellite orbits of specified semi-major axis and various inclinations [King-Hele et al., 1966]. One set of coefficients was given by King-Hele [et al., 1965], another by Kozai [1964], and the last one by Smith [1965]. He found that all three sets represent equally well the gravitational field for inclinations greater than  $26^\circ$ , and thus all sets must be of approximately the same accuracy. In Table 4.2 - 1 are given accuracy estimates for a set of King-Hele's solutions. Another accuracy estimate is given by Rapp [1967]. Basing his analysis on some precision estimates published by Köhnlein [1966a] and a comparison with terrestrial gravity anomalies, he concluded that the standard errors of the SAO 1966 coefficients should be multiplied by a factor of four to give standard deviations.

If we may apply this rule to the standard errors given by Kozai and listed in Table 4.2 - 1, we obtain values similar to the ones given by King-Hele. We therefore decided to adopt as standard deviations the mean values of King-Hele's values and the one found with Rapp's estimates.

The situation is a little better for the odd zonal harmonics, for which the three more recent solutions of King-Hele [et al., 1967], King-Hele [et al., 1968], and Kozai [1967] are listed in Table 4.2 - 3. From this table it appears that Kozai's standard errors are more reasonable, and we list them in the last two columns of Table 4.2 - 3 for conventional and fully normalized harmonics.

Satellite determinations for the nonzonal harmonics are that of SAO, of degree and order 8 with 45 additional coefficients of degree 9 through 15,

Table 4.2-2  
Correlation Matrices

From King-Hele's Solutions									
J <sub>2</sub>	J <sub>4</sub>	J <sub>6</sub>	J <sub>8</sub>	J <sub>2</sub>	J <sub>4</sub>	J <sub>6</sub>	J <sub>8</sub>	J <sub>10</sub>	J <sub>12</sub>
1.00	-0.14	0.14	0.27	1.000	0.977	0.977	0.979	0.986	0.968
	1.00	0.45	0.61		1.000	0.988	0.993	0.994	0.982
		1.00	0.50			1.000	0.988	0.998	0.970
			1.00				1.000	0.989	0.981
								1.000	0.978
									1.000

From Smith's Solution						
J <sub>2</sub>	J <sub>4</sub>	J <sub>6</sub>	J <sub>8</sub>	J <sub>10</sub>	J <sub>12</sub>	J <sub>14</sub>
	1.000	0.990	0.989	0.993	0.989	0.784
		1.000	0.989	0.993	0.991	0.788
			1.000	0.993	0.988	0.783
				1.000	0.997	0.804
					1.000	0.373
						1.000

From Kozai's Solution									
J <sub>2</sub>	J <sub>4</sub>	J <sub>6</sub>	J <sub>8</sub>	J <sub>10</sub>	J <sub>12</sub>	J <sub>14</sub>	J <sub>16</sub>	J <sub>18</sub>	J <sub>20</sub>
1.000	-0.968	0.976	-0.950	0.913	-0.823	0.715	-0.537	0.386	-0.152
	1.000	-0.986	0.977	-0.944	0.853	-0.765	0.568	-0.453	0.170
		1.000	-0.988	0.966	-0.889	0.802	-0.628	0.501	-0.215
			1.000	-0.987	0.932	-0.865	0.707	-0.594	0.303
				1.000	-0.968	0.923	-0.787	0.693	-0.392
					1.000	-0.971	0.905	-0.805	0.577
						1.000	-0.932	0.896	-0.614
							1.000	-0.912	0.797
								1.000	-0.716
									1.000

Table 4.2-3  
Odd Zonal Harmonics and Their Standard Errors  
in Units of  $10^{-6}$

	King-Hele 1967		King-Hele 1968	Kozai 1967	$\sigma_J$	$\sigma_{\bar{e}}$
$J_3$	-2.53 $\pm$ 0.02	-2.50 $\pm$ 0.01	-2.54 $\pm$ 0.01	-2.57 $\pm$ 0.007	$\pm$ 0.01	$\pm$ 0.004
$J_5$	-0.22 $\pm$ 0.04	-0.26 $\pm$ 0.01	-0.21 $\pm$ 0.01	-0.17 $\pm$ 0.012	$\pm$ 0.01	$\pm$ 0.004
$J_7$	-0.41 $\pm$ 0.06	-0.40 $\pm$ 0.02	-0.40 $\pm$ 0.01	-0.42 $\pm$ 0.023	$\pm$ 0.02	$\pm$ 0.006
$J_9$	0.09 $\pm$ 0.06	0.00 $\pm$ 0.06	0.00	-0.02 $\pm$ 0.033	$\pm$ 0.03	$\pm$ 0.008
$J_{11}$	-0.14 $\pm$ 0.05	-0.27 $\pm$ 0.06	0.00	-0.18 $\pm$ 0.045	$\pm$ 0.05	$\pm$ 0.009
$J_{13}$	0.29 $\pm$ 0.06	0.36 $\pm$ 0.08	0.00	-0.15 $\pm$ 0.055	$\pm$ 0.06	$\pm$ 0.011
$J_{15}$	-0.40 $\pm$ 0.06	-0.65 $\pm$ 0.10	-0.20 $\pm$ 0.03	-0.07 $\pm$ 0.063	$\pm$ 0.06	$\pm$ 0.011
$J_{17}$		0.30 $\pm$ 0.08	0.00	-0.05 $\pm$ 0.057	$\pm$ 0.06	$\pm$ 0.010
$J_{19}$		0.00 $\pm$ 0.11	0.00	-0.08 $\pm$ 0.051	$\pm$ 0.05	$\pm$ 0.008
$J_{21}$		0.58 $\pm$ 0.11	0.26 $\pm$ 0.05			
$J_{23} - J_{25}$			0.00			
$J_{27}$			-0.15 $\pm$ 0.10			
$J_{29}$			0.00			
$J_{31}$			0.00			

and the NWL general solution for zonals and nonzonals of degree and order 12 with additional resonant parameters. The standard errors from the SAO least squares solution for tesseral coefficients are given in [Gaposchkin, 1966b, pp. 200-201, 238]. These standard errors are a measure of the internal consistency, but they are too small to be accepted as accuracy figures. There are then only Rapp's estimates which can be considered to be reliable accuracy figures. Although the relation between the variances provided by the least squares process and the true variances of the solution is more complicated than a scale factor, it is reasonable to assume that a set of scaled variances can provide a better estimate than the one given by the least squares process.

We decided then to use Rapp's accuracy estimates [1967, pp. 37-38] for the tesseral harmonics to estimate mean standard deviations of a harmonic coefficient in a given degree. The SAO coefficients with Rapp's estimates up to  $14 \times 14$  are given in Table 4.2 - 4.



Table 4.2 - 4

## SAO Coefficients and Rapp's Accuracy Estimates

N	M	C		S	
		INPUT COEFFICIENT	STANDARD ERROR	INPUT COEFFICIENT	STANDARD ERROR
2	0	-484.1735	0.012		
2	2	2.379	0.056	-1.351	0.056
3	0	0.9623	0.032		
3	1	1.936	0.052	0.266	0.056
3	2	0.734	0.052	-0.538	0.048
3	3	0.561	0.072	1.620	0.076
4	0	0.5497	0.020		
4	1	-0.572	0.032	-0.469	0.028
4	2	0.330	0.048	0.661	0.048
4	3	0.851	0.032	-0.190	0.028
4	4	-0.053	0.138	0.230	0.120
5	0	0.0633	0.030		
5	1	-0.079	0.044	-0.103	0.044
5	2	0.631	0.048	-0.232	0.044
5	3	-0.520	0.056	0.007	0.056
5	4	-0.265	0.060	0.064	0.056
5	5	0.156	0.088	-0.592	0.088
6	0	-0.1792	0.033		
6	1	-0.047	0.028	-0.027	0.024
6	2	0.069	0.048	-0.366	0.048
6	3	-0.054	0.052	0.031	0.044
6	4	-0.044	0.080	-0.518	0.088
6	5	-0.313	0.040	-0.458	0.036
6	6	-0.040	0.096	-0.155	0.096
7	0	0.0860	0.040		
7	1	0.197	0.084	0.156	0.092
7	2	0.364	0.048	0.163	0.044
7	3	0.250	0.064	0.018	0.064
7	4	-0.152	0.064	-0.102	0.060
7	5	0.076	0.084	0.054	0.084
7	6	-0.209	0.076	0.063	0.068
7	7	0.055	0.088	0.096	0.084
8	0	0.0655	0.048		
8	1	-0.075	0.052	0.065	0.048
8	2	0.026	0.088	0.039	0.084
8	3	-0.037	0.060	0.004	0.056
8	4	-0.212	0.076	-0.012	0.084
8	5	-0.053	0.072	0.118	0.068
8	6	-0.017	0.100	0.318	0.096
8	7	-0.0087	0.052	0.031	0.060
8	8	-0.248	0.080	0.102	0.076
9	0	0.0122	0.055		
9	1	0.117	0.092	0.012	0.088
9	2	-0.0040	0.048	0.035	0.048
9	3	0.0	0.095	0.0	0.095
9	4	0.0	0.095	0.0	0.095
9	5	0.0	0.095	0.0	0.095
9	6	0.0	0.095	0.0	0.095
9	7	0.0	0.095	0.0	0.095
9	8	0.0	0.095	0.0	0.095
9	9	0.0	0.095	0.0	0.095

Table 4.2 - 4 (continued)

10	0	0.0118	0.044		
10	1	0.105	0.056	-0.126	0.052
10	2	-0.105	0.088	-0.042	0.088
10	3	-0.065	0.096	0.030	0.088
10	4	-0.074	0.072	-0.111	0.076
10	5	0.0	0.101	0.0	0.101
10	6	0.0	0.101	0.0	0.101
10	7	0.0	0.101	0.0	0.101
10	8	0.0	0.101	0.0	0.101
10	9	0.0	0.101	0.0	0.101
10	10	0.0	0.101	0.0	0.101
11	0	-0.0630	0.028		
11	1	-0.053	0.080	0.015	0.080
11	2	0.0	0.067	0.0	0.067
11	3	0.0	0.067	0.0	0.067
11	4	0.0	0.067	0.0	0.067
11	5	0.0	0.067	0.0	0.067
11	6	0.0	0.067	0.0	0.067
11	7	0.0	0.067	0.0	0.067
11	8	0.0	0.067	0.0	0.067
11	9	0.0	0.067	0.0	0.067
11	10	0.0	0.067	0.0	0.067
11	11	0.0	0.067	0.0	0.067
12	0	0.0714	0.036		
12	1	-0.163	0.080	-0.071	0.072
12	2	-0.103	0.104	-0.005	0.108
12	3	0.0	0.065	0.0	0.065
12	4	0.0	0.065	0.0	0.065
12	5	0.0	0.065	0.0	0.065
12	6	0.0	0.065	0.0	0.065
12	7	0.0	0.065	0.0	0.065
12	8	0.0	0.065	0.0	0.065
12	9	0.0	0.065	0.0	0.065
12	10	0.0	0.065	0.0	0.065
12	11	0.0	0.065	0.0	0.065
12	12	-0.031	0.065	0.000	0.065
13	0	0.0219	0.064		
13	1	0.0	0.041	0.0	0.041
13	2	0.0	0.041	0.0	0.041
13	3	0.0	0.041	0.0	0.041
13	4	0.0	0.041	0.0	0.041
13	5	0.0	0.041	0.0	0.041
13	6	0.0	0.041	0.0	0.041
13	7	0.0	0.041	0.0	0.041
13	8	0.0	0.041	0.0	0.041
13	9	0.0	0.041	0.0	0.041
13	10	0.0	0.041	0.0	0.041
13	11	0.0	0.041	0.0	0.041
13	12	-0.059	0.041	0.050	0.041
13	13	-0.059	0.041	0.077	0.041

Table 4.2 - 4 (continued)

14	0	-0.0332	-0.048		
14	1	-0.015	0.060	0.005	0.060
14	2	0.0	0.034	0.0	0.034
14	3	0.0	0.034	0.0	0.034
14	4	0.0	0.034	0.0	0.034
14	5	0.0	0.034	0.0	0.034
14	6	0.0	0.034	0.0	0.034
14	7	0.0	0.034	0.0	0.034
14	8	0.0	0.034	0.0	0.034
14	9	0.0	0.034	0.0	0.034
14	10	0.0	0.034	0.0	0.034
14	11	0.0002	0.034	-0.000	0.034
14	12	0.094	0.034	-0.028	0.034
14	13	0.0	0.034	0.0	0.034
14	14	-0.014	0.034	-0.003	0.034

The computed mean standard deviations for tesseral harmonics only and for all coefficients in a degree, according to Rapp's standard deviations, are given in Table 4.2 - 5.

Table 4.2 - 5

Mean Accuracy of Harmonic Coefficients  
(in units of  $10^{-9}$ )

Degree	Tesseral Only	All Coefficients
2	56	41
3	59	55
4	56	52
5	58	56
6	57	55
7	72	70
8	72	71
9	89	88
10	91	89
11	68	66
12	69	71
13	41	42
14	36	36

The recent NWL set of spherical harmonic coefficients and their variances are not available to us. The only information about its accuracy has been obtained from Rapp [1967b]. Rapp performed a series of tests concerning the accuracy of different sets of harmonic coefficients, but the results concerning the NWL set were inconclusive. There were instances where the NWL solution exhibited a better behavior than the SAO set of coefficients and vice versa. We then decided to consider both sets as being of approximately the same accuracy. Next we tried to estimate the accuracy of the harmonics from the combined SAO and NWL data.

In Fig. 4.2-1 we plotted the mean standard deviations from all coefficients of a degree for the satellite coefficients that were used by Rapp in his combination solutions. We also plotted the mean standard errors of his terrestrial and of his combination solutions, both of which are given in [Rapp, 1967]. This figure also shows the mean standard errors of harmonic coefficients according to the equation

$$\sigma = \frac{0.34 \times 10^{-6}}{n-1} \quad (4.2-1)$$

which is a model given by Rapp [1968a] for the standard errors of harmonic coefficients obtained from terrestrial determinations when model anomalies are used.

We made the assumption that the standard errors of all coefficients within a degree are the same and equal to the mean standard error of that degree. We also made the assumption that the coefficients are independent. As the combination solution is actually the multivariate mean of the combined quantities, the standard error of each degree of the combination solution coefficients can be computed from the equation for the variance of a weighted mean

$$\sigma_{\text{comb}}^2 = \frac{1}{\frac{1}{\sigma_s^2} + \frac{1}{\sigma_t^2}} \quad (4.2-2)$$

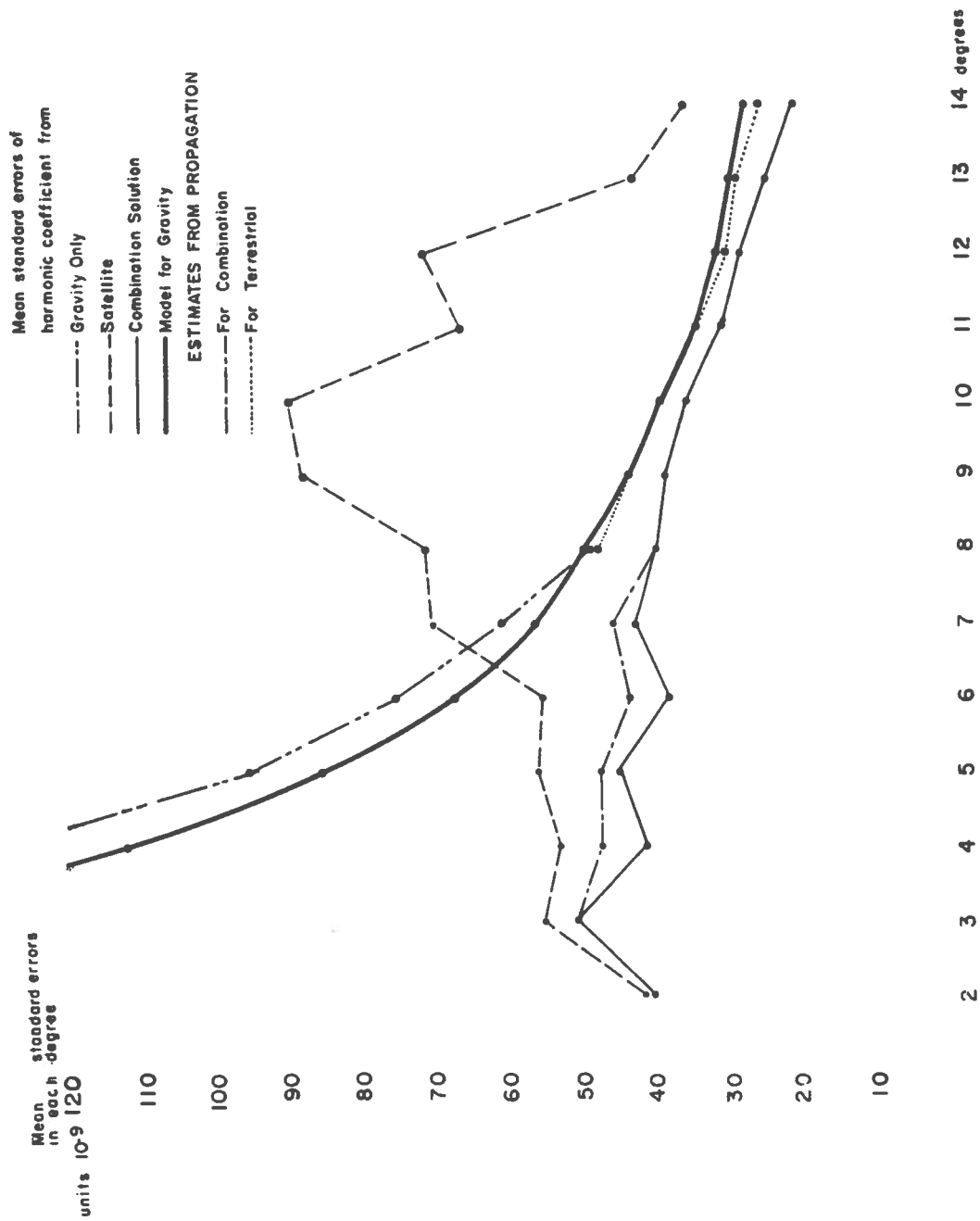


Fig. 4.2-1 Mean standard errors of harmonic coefficients obtained by rigorous procedure and by quadratic propagation

where

$\sigma_{comb}$  is the standard error of the combination solution coefficients  
 $\sigma_s, \sigma_t$  are again the satellite and the terrestrial solution coefficient standard errors

From the terrestrial and satellite standard errors we computed the mean standard errors in each degree of the combination solution up to degree 8 by the above equation. These results are plotted in Fig. 4.2-1. We noted that the mean standard errors found by error propagation are very close to the actual mean standard errors found from the combination solution. From degree 9 to 14 we made the same assumptions but we tried to find the mean standard error of the coefficients of a terrestrial solution which, when combined with the mean standard error of the satellite coefficients, will yield the mean standard errors of the combination solution. We again plotted the results and found that the mean standard errors produced by error propagation are very close to the standard errors given by Rapp's model. We conclude then that within the limitations of Rapp's combination solution the uncertainties of a combination solution can very well be approximated by a simple error propagation of the starting standard errors using equation (4.2 - 2).

Similarly, a very rough estimate of the accuracy of the spherical harmonic coefficients of the combination of SAO and NWL data will be found by a simple error propagation. If in this case we consider the accuracies of the two determinations to be the same, the resulting accuracy will be obtained by dividing the accuracy of one satellite determination by  $\sqrt{2}$ . The results are averaged with the accuracies of zonal harmonics to give the mean accuracy of a combined satellite determination in each degree.

The standard errors of the zonal harmonic determination of NWL were omitted because they are not available. Even if they had been available, it is believed that their contribution would be very small, since the zonal harmonics are already well determined.

In Fig. 4.2-2 we plotted the mean standard deviations for the zonals and tesserals of the SAO satellite determination and the results of the combination of the SAO and NWL data.

#### 4.22 Determination of Harmonic Coefficients from Terrestrial Gravity

##### Anomalies

Another very valuable source for the determination of the gravity field is the terrestrial gravity observations. The Department of Geodetic Science at The Ohio State University has been engaged in the collection, evaluation, and reduction of gravity observations on a world-wide scale [Uotila et al., 1966a]. The distribution of the surface gravity observations existing in 1964 has been published in [Uotila, 1966]. From the observations that have been collected at The Ohio State University, Uotila computed mean gravity anomalies for  $5^\circ \times 5^\circ$  blocks [Uotila, 1962]. Rapp used 805 mean anomalies from Uotila's computations and computed mean anomalies for another 621  $5^\circ \times 5^\circ$  blocks [Rapp, 1967]. These 1426  $5^\circ \times 5^\circ$  blocks constitute the observed or surveyed part of the earth's surface and the remaining 1166 constitute the unobserved or the unsurveyed part. For all the computed mean anomalies, Rapp estimated standard errors using statistical methods developed in [Kaula, 1959; Heiskanen and Moritz, 1967]. The distribution of the observed  $5^\circ \times 5^\circ$  blocks, together with a code for their standard errors according to Rapp [1968e], are shown in Fig. 4.2-3.

From these mean anomalies, Rapp [1969], applying the approach of a weighted least squares adjustment, computed a set of harmonic coefficients of 8th degree and order and their standard errors. These coefficients, given in Table 4.2 - 6, show a much better agreement with the satellite solutions than older terrestrial solutions do. To further refine his terrestrial solution, Rapp [1968c] used model terrestrial anomalies for which he estimated a standard error of 20 mgal; with these anomalies he filled the remaining 1166  $5^\circ \times 5^\circ$  blocks. This new set of data can also be represented by Fig. 4.2-3, if we

Fig.4.2-2 Mean standard deviations of spherical harmonics from satellite solutions

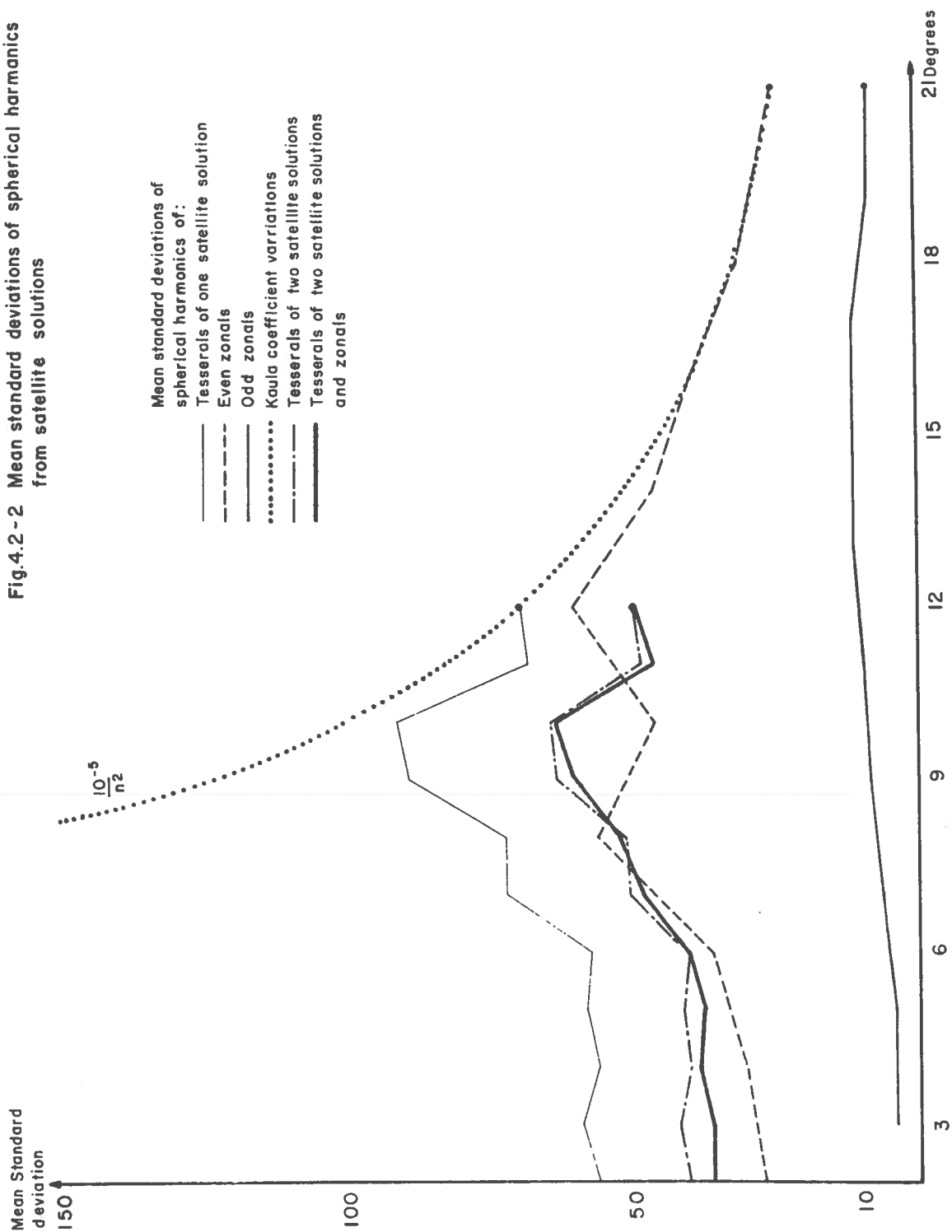






Fig. 4.2-3 Distribution and quality of terrestrial gravity data

Table 4.2 - 6

Rapp's Harmonic Coefficients from Terrestrial Gravity Data

Coefficient Degree and Order	Observed Anomalies Only				Observed + Model Anomalies			
	$\bar{C}$	$\sigma$	$\bar{S}$	$\sigma$	$\bar{C}$	$\sigma$	$\bar{S}$	$\sigma$
20	-484.21 ± .83				-485.36 ± .36			
22	4.98 ± .96		-0.43 ± .74		2.19 ± .33		-1.22 ± .32	
30	-0.30 ± .43				0.25 ± .17			
31	0.85 ± .43		-1.32 ± .45		0.79 ± .18		0.06 ± .17	
32	-0.59 ± .63		-1.08 ± .52		0.59 ± .17		-0.46 ± .17	
33	0.81 ± .37		-0.33 ± .51		1.00 ± .16		0.96 ± .17	
40	0.64 ± .25				0.68 ± .11			
41	0.01 ± .36		1.33 ± .36		-0.18 ± .12		-0.05 ± .12	
42	0.40 ± .41		1.03 ± .40		0.20 ± .12		0.38 ± .12	
43	1.10 ± .28		0.52 ± .41		1.02 ± .11		-0.21 ± .11	
44	-0.72 ± .31		0.93 ± .28		0.18 ± .11		0.31 ± .11	
50	0.44 ± .20				-0.20 ± .09			
51	-0.34 ± .24		-0.57 ± .23		-0.32 ± .09		-0.02 ± .09	
52	0.51 ± .28		-0.77 ± .33		0.02 ± .09		-0.03 ± .09	
53	0.15 ± .21		0.02 ± .31		0.00 ± .08		0.07 ± .09	
54	0.67 ± .25		-0.45 ± .23		0.23 ± .08		0.08 ± .08	
55	0.86 ± .20		-0.08 ± .21		0.13 ± .08		-0.44 ± .08	
60	-0.41 ± .18				-0.01 ± .07			
61	-0.44 ± .16		-0.13 ± .15		-0.29 ± .07		0.06 ± .07	
62	-0.49 ± .18		0.30 ± .24		-0.15 ± .07		-0.09 ± .07	
63	-0.50 ± .21		-0.01 ± .26		0.02 ± .07		0.17 ± .07	
64	-0.29 ± .17		-0.17 ± .17		-0.13 ± .07		-0.30 ± .06	
65	-0.78 ± .18		-0.84 ± .17		-0.21 ± .06		-0.58 ± .06	
66	0.08 ± .14		-0.22 ± .15		0.01 ± .06		-0.24 ± .06	
70	0.05 ± .12				0.05 ± .06			
71	0.27 ± .13		0.66 ± .13		0.14 ± .06		0.24 ± .05	
72	0.26 ± .13		0.13 ± .14		0.13 ± .06		0.14 ± .06	
73	0.71 ± .16		0.19 ± .19		0.29 ± .06		0.01 ± .06	
74	-0.39 ± .13		0.02 ± .13		-0.28 ± .06		-0.09 ± .06	
75	0.32 ± .13		0.16 ± .11		0.06 ± .05		0.15 ± .05	
76	-0.17 ± .12		0.53 ± .12		-0.14 ± .05		0.13 ± .05	
77	-0.18 ± .11		0.21 ± .09		0.03 ± .05		0.04 ± .05	

Table 4.2 - 6 (continued)

80	0.22 ± .09		0.09 ± .05	
81	-0.01 ± .10	-0.19 ± .09	-0.05 ± .05	0.03 ± .04
82	0.02 ± .08	-0.09 ± .09	0.04 ± .05	0.05 ± .05
83	-0.21 ± .10	-0.06 ± .11	-0.03 ± .05	0.06 ± .06
84	0.03 ± .09	-0.07 ± .09	-0.01 ± .04	0.05 ± .04
85	-0.12 ± .08	0.18 ± .08	-0.11 ± .04	0.17 ± .04
86	0.02 ± .07	0.13 ± .08	-0.00 ± .04	0.19 ± .04
87	0.13 ± .08	-0.15 ± .07	0.01 ± .04	-0.04 ± .04
88	0.14 ± .07	0.04 ± .07	0.00 ± .04	0.01 ± .04

assume that the blank blocks have a standard error of 20 mgal. From this data Rapp computed a second set of 8th degree and order harmonic coefficients and their standard errors, and it is this set of data that he used to combine terrestrial and satellite results for harmonic coefficients. The second set of harmonics and their standard errors are given in Table 4.2 - 6. We could test the consistency of the SAO satellite solution (Table 4.2 - 4), Rapp's terrestrial solution with model anomalies (Table 4.2 - 6), and their standard errors by forming a multivariate confidence region [Hamilton, 1964, pp. 139-142]. This would require the full variance-covariance matrices of these solutions. As full variance-covariance matrices are not available, we test each pair of parameters by examining the estimated marginal variances. The difference of any pair of coefficients of the same degree and order would have a distribution which could be approximate by the  $N(0, \sigma_T^2 + \sigma_S^2)$  distribution, where  $\sigma_T$  and  $\sigma_S$  are the standard errors of the harmonic coefficients from the terrestrial and satellite solutions. This is the case because  $\sigma_T$  and  $\sigma_S$  have been determined with such a large number of degrees of freedom that it will not make any numerical difference if we use a normal distribution instead of student's t distribution. With a significance level of 95%, the interval  $(\hat{x}_1 - \hat{x}_2 - 1.96 \sqrt{\sigma_T^2 + \sigma_S^2}, \hat{x}_1 - \hat{x}_2 + 1.96 \sqrt{\sigma_T^2 + \sigma_S^2})$ , where  $\hat{x}_1$ ,  $\hat{x}_2$  are satellite and terrestrial estimates of a harmonic, must include zero, otherwise the

hypothesis of the equality of the two means is rejected. This test can be written as  $|\hat{x}_1 - \hat{x}_2| < 1.96 \sqrt{\sigma_1^2 + \sigma_2^2}$ .

In Table 4.2 - 7 we give the absolute values of the differences between Rapp's terrestrial solution in which model anomalies have been included and the SAO satellite solution, together with the quantities  $1.96 \sqrt{\sigma_1^2 + \sigma_2^2}$ . Only the terms where the hypothesis of the equality of means is rejected by the test are listed. Out of the 75 coefficients, 22 are formally rejected. The percentage is too high, and it suggests, under the limitation of the above assumptions, two hypotheses: (a) The satellite uncertainties should be larger, (b) The terrestrial uncertainties should be larger. We want to change the standard errors of a group by a scale factor to bring them into agreement.

We have accepted the hypothesis that the least squares estimates of the variances for the SAO satellite solutions were only precision figures, and we have modified them to obtain accuracy figures. It is also reasonable to assume that the uncertainty estimates of the terrestrial solution, estimated after such a long and approximate process, could also be too small and must be scaled. We first see that among the rejected differences some are for zonal harmonics of both even and odd degrees, like the  $\bar{C}_{20}$ , which are determined with high accuracy from satellites. It would require a factor of 40 if we were to justify the difference between the  $\bar{C}_{20}$ 's by scaling the satellite standard errors. In general, because the standard errors of the satellite coefficients are very small it would require unreasonably large scale factors to justify the existing differences by scaling the satellite standard errors. Nor would the simultaneous scaling of the standard errors of the satellite and terrestrial coefficients help very much, especially in the case of the low-order harmonics. We conclude then that the variances of the harmonic coefficients from the terrestrial solution must be increased to give more reliable accuracy estimates. We noticed that if we double the given standard errors of the coefficients of the terrestrial solution then only four differences are rejected which gives a more reasonable percentage of rejected differences for

Table 4.2 - 7

Differences Between SAO Satellite and Rapp's Terrestrial Coefficients  
and Their Confidence Levels

Degree and Order	$\bar{C}$ Difference	$1.96 \sqrt{\sigma_T^2 + \sigma_S^2}$		Degree and Order	$\bar{S}$ Difference	$1.96 \sqrt{\sigma_T^2 + \sigma_S^2}$	
20	1.19	0.71	(1.70)				
30	0.71	0.34	(2.08)				
31	1.15	0.37	(3.20)				
33	0.44	0.34	(1.30)	33	0.66	0.36	(1.89)
41	0.39	0.24	(1.60)	41	0.42	0.24	(2.08)
				42	0.28	0.25	(1.10)
50	0.26	0.21	(1.41)				
51	0.24	0.20	(1.24)				
52	0.61	0.20	(3.30)	52	0.202	0.197	(1.10)
53	0.52	0.19	(3.20)				
54	0.50	0.20	(3.03)				
60	0.17	0.15	(1.10)				
61	0.24	0.17	(1.67)				
62	0.22	0.17	(1.41)	62	0.28	0.17	(1.88)
				64	0.22	0.21	(1.10)
72	0.23	0.15	(1.49)				
84	0.20	0.17	(1.62)				
88	0.25	0.18	(2.40)				

the 75 coefficients. The factors by which the given standard errors of the terrestrial solution should be multiplied so that the differences would not have been rejected are given in parentheses in Table 4.2 - 7.

Again, although we believe that the relation between the standard errors of the harmonic coefficients of this terrestrial solution and their standard deviations is more complicated than a simple scale factor, still a scaling of these standard errors will provide better accuracy figures. We then decided to multiply the standard errors by 2 to get more realistic accuracy figures.

By multiplying the standard errors by 2, equation (4.2 - 3) becomes

$$\sigma_{(c,s)} = \frac{\pm 0.68 \times 10^{-6}}{n-1} \quad (4.2-3)$$

The above result raises the question of the cause of this low accuracy. If we attributed all this to the assigned weights we will have to double the estimated standard errors for the observed and the model anomalies, and this is difficult to justify. In order to investigate this question a little further we used the adjusted gravity anomaly field given in [Rapp, 1968, pp. 26-27]. Based on this and the variances of the anomalies given in Fig. 4.2-3, we generated observed and model mean gravity anomalies having the adopted anomalies as their mean values and the variances of the observed or the model anomalies as their variances. For this purpose we used an IBM 7094 computer subroutine which generated normally distributed random numbers  $n(0,1)$ , with mean zero and variance 1.

We know that  $\frac{x - \mu}{\sigma}$  is distributed as  $n(0,1)$ . Thus we found any generated mean anomaly

$$x = n(0,1) \times \sigma + \mu$$

where

$\sigma$  is the standard error of the mean anomalies  
 $\mu$  are the adjusted anomalies

Thus we generated three sets of anomalies which had the adjusted anomalies as their means and the standard errors of the mean anomalies given by Rapp as their standard errors.

Next we determined, by least squares, harmonic coefficients of degree and order 8 from the adjusted anomaly field with a uniform standard error of 10 mgal. This set of coefficients is given in Table 4.2-8. We also determined harmonic coefficients from the three generated anomaly fields, using as standard errors those of Rapp. These sets of coefficients are given in Tables 4.2-9, 4.2-10, and 4.2-11.

Table 4.2 - 8

8 x 8 Harmonic Coefficients Determined from Rapp's Adjusted Anomalies

				C				S						
N	M	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	N	M	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	N	M	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE
2	0	0.	1.8299	0.2463	2	0	0.			2	2	0.		
2	2	-484.2029	-0.1111	-484.3140	2	2	0.	2.3650	0.2506	2	2	0.	-1.3457	-1.3457
3	0	0.	0.9816	0.1258	3	0	0.	0.9816	0.1258	3	1	0.	0.2328	0.2328
3	1	0.	1.8142	0.1258	3	1	0.	1.8142	0.1256	3	2	0.	-0.5433	-0.5433
3	2	0.	0.7221	0.1256	3	2	0.	0.7221	0.1256	3	3	0.	1.5226	1.5226
3	3	0.	0.6330	0.0838	3	3	0.	0.6330	0.0838	3	3	0.		
4	0	0.7832	-0.2785	0.5047	4	0	0.	-0.2785	0.0838	4	0	0.		
4	1	0.	-0.5150	-0.5150	4	1	0.	-0.5150	0.0840	4	1	0.	-0.4377	-0.4377
4	2	0.	0.2859	0.2859	4	2	0.	0.2859	0.0838	4	2	0.	0.5782	0.5782
4	3	0.	0.8750	0.8750	4	3	0.	0.8750	0.0838	4	3	0.	-0.1958	-0.1958
4	4	0.	0.0938	0.0938	4	4	0.	0.0938	0.0838	4	4	0.	0.2744	0.2744
5	0	0.	0.0645	0.0645	5	0	0.	0.0645	0.0628	5	0	0.		
5	1	0.	-0.1142	-0.1142	5	1	0.	-0.1142	0.0631	5	1	0.	-0.0720	-0.0720
5	2	0.	0.5360	0.5360	5	2	0.	0.5360	0.0629	5	2	0.	-0.2063	-0.2063
5	3	0.	-0.3590	-0.3590	5	3	0.	-0.3590	0.0628	5	3	0.	0.0245	0.0245
5	4	0.	-0.0487	-0.0487	5	4	0.	-0.0487	0.0628	5	4	0.	0.0750	0.0750
5	5	0.	0.0861	0.0861	5	5	0.	0.0861	0.0628	5	5	0.	-0.5689	-0.5689
6	0	0.	-0.1464	-0.1464	6	0	0.	-0.1464	0.0502	6	0	0.		
6	1	0.	-0.0365	-0.0365	6	1	0.	-0.0365	0.0505	6	1	0.	-0.0120	-0.0120
6	2	0.	0.0134	0.0134	6	2	0.	0.0134	0.0503	6	2	0.	-0.2917	-0.2917
6	3	0.	-0.0497	-0.0497	6	3	0.	-0.0497	0.0503	6	3	0.	0.0639	0.0639
6	4	0.	-0.0257	-0.0257	6	4	0.	-0.0257	0.0503	6	4	0.	-0.4133	-0.4133
6	5	0.	-0.2907	-0.2907	6	5	0.	-0.2907	0.0503	6	5	0.	-0.4509	-0.4509
6	6	0.	-0.0088	-0.0088	6	6	0.	-0.0088	0.0503	6	5	0.	-0.1843	-0.1843
7	0	0.	0.0705	0.0705	7	0	0.	0.0705	0.0418	6	6	0.		
7	1	0.	0.1050	0.1050	7	1	0.	0.1050	0.0421	7	1	0.	0.0999	0.0999
7	2	0.	0.3223	0.3223	7	2	0.	0.3223	0.0420	7	2	0.	0.1459	0.1459
7	3	0.	0.1743	0.1743	7	3	0.	0.1743	0.0419	7	3	0.	0.0042	0.0042
7	4	0.	-0.1922	-0.1922	7	4	0.	-0.1922	0.0419	7	4	0.	-0.0891	-0.0891
7	5	0.	0.0702	0.0702	7	5	0.	0.0702	0.0419	7	5	0.	0.0351	0.0351
7	6	0.	-0.1654	-0.1654	7	6	0.	-0.1654	0.0419	7	6	0.	0.0916	0.0916
7	7	0.	0.0678	0.0678	7	7	0.	0.0678	0.0419	7	7	0.	-0.0299	-0.0299
8	0	0.	0.0475	0.0475	8	0	0.	0.0475	0.0358	7	7	0.		
8	1	0.	-0.0243	-0.0243	8	1	0.	-0.0243	0.0360	8	1	0.	0.0299	0.0299
8	2	0.	0.0239	0.0239	8	2	0.	0.0239	0.0360	8	2	0.	0.0891	0.0891
8	3	0.	0.0019	0.0019	8	3	0.	0.0019	0.0359	8	3	0.	0.0439	0.0439
8	4	0.	-0.0937	-0.0937	8	4	0.	-0.0937	0.0359	8	4	0.	0.0167	0.0167
8	5	0.	-0.0629	-0.0629	8	5	0.	-0.0629	0.0359	8	4	0.	0.0824	0.0824
8	6	0.	-0.1071	-0.1071	8	5	0.	-0.0629	0.0359	8	5	0.	0.2868	0.2868
8	7	0.	0.0254	0.0254	8	6	0.	-0.1071	0.0359	8	6	0.	0.0302	0.0302
8	8	0.	-0.1044	-0.1044	8	7	0.	0.0254	0.0359	8	7	0.	-0.0169	-0.0169
9		0.			9		0.		0.0359	8	8	0.		

Table 4.2 - 9

8 x 8 Harmonic Coefficients Determined from Generated Anomaly Field No. 1

S

C

N	M	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	N	M
2	C	-484.2029	1.4945	0.3334	0.4	-1.8473	0.3323	2	2
2	2	0.	-0.4653	0.3739	0.				
2	2	0.	3.0335	0.3433	0.				
3	C	0.	0.7227	0.1794	0.				
3	1	0.	1.5095	0.1851	0.	-0.3892	0.1831	3	1
3	2	0.	0.8840	0.1768	0.	-0.9078	0.1763	3	2
3	3	0.	0.7559	0.1659	0.	1.8595	0.1732	3	3
4	C	0.7832	-0.3229	0.4602	0.				
4	1	0.	-0.1631	0.1262	0.	-0.1476	0.1213	4	1
4	2	0.	0.0810	0.1217	0.	0.7203	0.1247	4	2
4	3	0.	0.9945	0.1121	0.	-0.2807	0.1167	4	3
4	4	0.	0.0900	0.1135	0.	0.4180	0.1111	4	4
5	C	0.	0.0468	0.0905	0.				
5	1	0.	-0.2760	0.0925	0.	-0.0393	0.0907	5	1
5	2	0.	0.2555	0.0935	0.	-0.2315	0.0943	5	2
5	3	0.	-0.2270	0.0868	0.	0.2485	0.0912	5	3
5	4	0.	-0.0221	0.0854	0.	-0.1003	0.0829	5	4
5	5	0.	0.2393	0.0837	0.	-0.6799	0.0844	5	5
6	C	0.	-0.2335	0.0694	0.				
6	1	0.	-0.2423	0.0746	0.	0.0408	0.0705	6	1
6	2	0.	-0.2189	0.0722	0.	-0.3076	0.0735	6	2
6	3	0.	0.0885	0.0716	0.	0.1283	0.0752	6	3
6	4	0.	-0.1866	0.0693	0.	-0.3770	0.0672	6	4
6	5	0.	-0.3030	0.0645	0.	-0.6118	0.0667	6	5
6	6	0.	-0.0987	0.0676	0.	-0.2478	0.0659	6	6
7	C	0.	0.0607	0.0579	0.				
7	1	0.	0.1610	0.0608	0.	0.2751	0.0563	7	1
7	2	0.	0.2374	0.0589	0.	0.1839	0.0601	7	2
7	3	0.	0.1921	0.0585	0.	0.1127	0.0606	7	3
7	4	0.	-0.2571	0.0587	0.	-0.0839	0.0585	7	4
7	5	0.	0.1819	0.0537	0.	0.0555	0.0553	7	5
7	6	0.	-0.1187	0.0536	0.	0.1239	0.0539	7	6
7	7	0.	0.1033	0.0544	0.	0.1551	0.0547	7	7
8	C	0.	0.1497	0.0489	0.				
8	1	0.	-0.0472	0.0492	0.	-0.0285	0.0468	8	1
8	2	0.	-0.0787	0.0484	0.	0.0668	0.0498	8	2
8	3	0.	-0.0450	0.0474	0.	-0.0107	0.0489	8	3
8	4	0.	-0.1165	0.0466	0.	-0.1424	0.0470	8	4
8	5	0.	-0.0979	0.0450	0.	0.1992	0.0455	8	5
8	6	0.	0.0626	0.0421	0.	0.2499	0.0430	8	6
8	7	0.	-0.1253	0.0437	0.	-0.0554	0.0443	8	7
8	8	0.	0.0161	0.0449	0.	-0.0225	0.0449	8	8



Table 4.2 - 10

8 x 8 Harmonic Coefficients Determined from Generated Anomaly Field No. 2

S

C

N	M	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	N	M
2	2	C. -484.2029	1.4407	1.4407	0.	-C. 9871	-0.9871	2	2
2	2	C. -484.2029	-0.7262	-484.9291	0.			2	2
3	2	C. -484.2029	1.9685	1.9685	0.			3	1
3	2	C. -484.2029	1.2389	1.2389	0.	-0.0455	-0.0455	3	1
3	2	C. -484.2029	1.6340	1.6340	0.	-C. 6833	-C. 6833	3	2
3	2	C. -484.2029	C. 9059	0.9059	0.	1.4888	1.4888	3	3
3	2	C. -484.2029	0.6539	0.6539	0.			3	3
3	2	C. -484.2029	-0.1741	0.6091	0.			3	3
4	1	C. -484.2029	-0.5664	-0.5664	0.	-0.2456	-0.2456	4	1
4	1	C. -484.2029	0.0675	0.0675	0.	0.6303	0.6303	4	1
4	1	C. -484.2029	0.8523	0.8523	0.	-0.1864	-0.1864	4	2
4	1	C. -484.2029	0.1292	0.1292	0.	0.1167	0.1167	4	3
4	1	C. -484.2029	-0.0581	-0.0581	0.	0.1111	0.1111	4	4
4	1	C. -484.2029	-0.2037	-0.2037	0.			4	4
5	1	C. -484.2029	0.3688	0.3688	0.	-C. 0578	-C. 0578	5	1
5	1	C. -484.2029	-0.2638	-0.2638	0.	-C. 0968	-C. 0968	5	2
5	1	C. -484.2029	-0.0249	-0.0249	0.	0.1700	0.1700	5	3
5	1	C. -484.2029	0.1621	0.1621	0.	0.0678	0.0678	5	4
5	1	C. -484.2029	-0.1336	-0.1336	0.	-C. 6792	-C. 6792	5	5
5	1	C. -484.2029	-0.3279	-0.3279	0.			5	5
6	1	C. -484.2029	-0.2115	-0.2115	0.	0.2242	0.2242	6	1
6	1	C. -484.2029	0.0277	0.0277	0.	-C. 2587	-C. 2587	6	2
6	1	C. -484.2029	-0.0861	-0.0861	0.	C. 2001	0.2001	6	3
6	1	C. -484.2029	-0.3051	-0.3051	0.	-0.4820	-0.4820	6	4
6	1	C. -484.2029	-0.0227	-0.0227	0.	-0.6374	-0.6374	6	5
6	1	C. -484.2029	0.0952	0.0952	0.	-C. 2500	-C. 2500	6	6
7	1	C. -484.2029	0.1771	0.1771	0.			7	1
7	1	C. -484.2029	0.1856	0.1856	0.	C. 2285	0.2285	7	1
7	1	C. -484.2029	0.3236	0.3236	0.	0.2160	0.2160	7	2
7	1	C. -484.2029	-0.3719	-0.3719	0.	C. 1248	0.1248	7	3
7	1	C. -484.2029	0.1546	0.1546	0.	C. 0022	0.0022	7	4
7	1	C. -484.2029	-0.1003	-0.1003	0.	C. 1471	0.1471	7	5
7	1	C. -484.2029	-0.0470	-0.0470	0.	C. 1682	0.1682	7	6
7	1	C. -484.2029	0.1370	0.1370	0.	-C. 0276	-C. 0276	7	7
8	1	C. -484.2029	0.0310	0.0310	0.			8	1
8	1	C. -484.2029	-0.0050	-0.0050	0.	-0.0879	-0.0879	8	1
8	1	C. -484.2029	-0.1825	-0.1825	0.	0.0718	0.0718	8	2
8	1	C. -484.2029	-0.0691	-0.0691	0.	-C. 0639	-C. 0639	8	3
8	1	C. -484.2029	-0.0964	-0.0964	0.	-0.1243	-0.1243	8	4
8	1	C. -484.2029	0.0450	0.0450	0.	0.2303	0.2303	8	5
8	1	C. -484.2029	0.0199	0.0199	0.	C. 2573	0.2573	8	6
8	1	C. -484.2029	0.0096	0.0096	0.	-0.0108	-0.0108	8	7
8	1	C. -484.2029	-0.0081	-0.0081	0.	0.0424	0.0424	8	8
8	1	C. -484.2029	0.0437	0.0437	0.			8	8
8	1	C. -484.2029	0.0449	0.0449	0.			8	8

Table 4.2 - 11

8 x 8 Harmonic Coefficients Determined from Generated Anomaly Field No. 3

S

C

N	M	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	N	M
2	0	0.	1.8557	1.8557	0.3334				
2	2	0.	-0.1965	-484.3958	0.3739				
3	0	C.	3.1099	3.1099	0.3433				
3	2	C.	C.7202	C.7202	0.1794				
3	1	0.	1.5400	1.5400	0.1851				
3	3	0.	C.8853	0.8853	0.1768				
3	3	0.	0.5325	0.5325	0.1659				
4	0	C.7832	-0.2934	C.4858	0.1181				
4	1	C.	-0.6118	-0.6118	0.1262				
4	2	C.	C.2160	0.2160	0.1217				
4	3	C.	1.1480	1.1480	0.1121				
4	4	0.	0.2906	C.2906	0.1135				
5	0	C.	-C.0328	-0.0328	0.0505				
5	1	C.	-0.2121	-0.2121	0.0925				
5	2	C.	C.3885	0.3885	0.0935				
5	3	C.	-C.1538	-0.1538	0.0964				
5	4	C.	0.0051	0.0091	0.0854				
5	5	C.	C.1657	0.1657	0.0837				
6	0	C.	-0.2675	-0.2679	0.0694				
6	1	C.	-0.1703	-0.1703	0.0746				
6	2	C.	-0.2455	-C.2456	0.0722				
6	3	C.	-C.0566	-0.0566	0.0716				
6	4	C.	-C.1265	-0.1265	0.0693				
6	5	C.	-C.1959	-C.1959	0.0645				
6	6	C.	-0.0429	-0.0429	0.0676				
7	0	C.	C.1301	C.1301	0.0579				
7	1	C.	C.2151	0.2151	0.0608				
7	2	C.	C.2220	0.2220	0.0589				
7	3	C.	C.3048	0.3048	0.0585				
7	4	0.	-C.3513	-C.3513	0.0587				
7	5	C.	C.1468	0.1468	0.0537				
7	6	C.	-0.1542	-0.1542	0.0536				
7	7	C.	0.0513	0.0613	0.0544				
8	0	C.	C.1254	0.1254	0.0489				
8	1	C.	-C.0363	-0.0363	0.0492				
8	2	C.	-0.0024	-0.0024	0.0484				
8	3	C.	-C.0315	-0.0315	0.0474				
8	4	0.	-0.0161	-0.0161	0.0466				
8	5	C.	-C.1784	-0.1784	0.0450				
8	6	0.	C.0938	0.0938	0.0421				
8	7	C.	-0.0453	-0.0453	0.0437				
8	8	C.	-0.0290	-0.0290	0.0449				

From the adjusted anomalies, harmonic coefficients have also been computed by summation, using a program written by Uotila. The differences between the coefficients computed by least squares and those computed by summation were not of any significance for our experiment.

The differences between the coefficients computed from the adjusted field and those computed by the generated fields give a measure of the accuracy with which anomaly fields similar to the generated ones can determine the harmonic coefficients. The RMS differences in each degree are given below:

RMS Differences			
Degree	Field No. 1	Field No. 2	Field No. 3
2	0.56	0.53	0.46
3	0.35	0.20	0.20
4	0.20	0.11	0.14
5	0.15	0.10	0.13
6	0.12	0.14	0.12
7	0.08	0.10	0.10
8	0.09	0.09	0.09

The above RMS differences are actually the standard errors of the determined coefficients in each degree and give coefficients for equation (4.2 - 3) of 0.60, 0.52, and 0.51 respectively. The mean of these values is 0.54. Thus we can say that if the assigned standard errors of the presently predicted mean anomalies in  $5^\circ \times 5^\circ$  blocks are correct, then the set of harmonic coefficients that we obtain by least squares solutions has standard errors described by the equation

$$\sigma_{(c,s)} = \frac{0.54 \times 10^{-6}}{n-1}$$

We know, however, that the assigned standard errors are not correct but are only estimates, and this further affects the accuracy of the determined coefficients. The amount of this effect is not known, but a change from 0.54 to 0.68 cannot be ruled out. We decided then to accept equation (4.2 - 3) which has been indicated by the comparison of coefficients computed from gravity anomalies and the SAO 1966 set of harmonic coefficients as expressing the standard errors of the harmonic coefficients computed from terrestrial gravity.

#### 4.23 Determination of Harmonic Coefficients from Astrogeodetic Undulations

Another source for the determination of the spherical harmonic coefficients of the gravity field is the astrogeodetic undulations. The present extent of the astrogeodetic undulations has been published in [Fischer, 1968]. From there and from the Geoid Charts of North and Central America 1967 published by the Army Map Service, we estimated mean astrogeodetic undulations in  $5^\circ \times 5^\circ$  blocks and standard errors applying the procedure described in Section 3.4. The distribution of the observed blocks, with a code for their standard errors, is shown in Fig. 4.2-4.

With this data and using the model (3.4 - 17) for astrogeodetic undulations, observation equations were formed in different combinations of number of coefficients, of approximate values, and of weights. Then normal equations were formed and solved using standard least square techniques. First an  $8 \times 8$  solution from astrogeodetic undulations only was attempted. This attempt resulted in very unrealistic results.

Next we attempted another  $8 \times 8$  solution in which the rotations of the datums were constrained to be zero, but the results were again unreasonable.

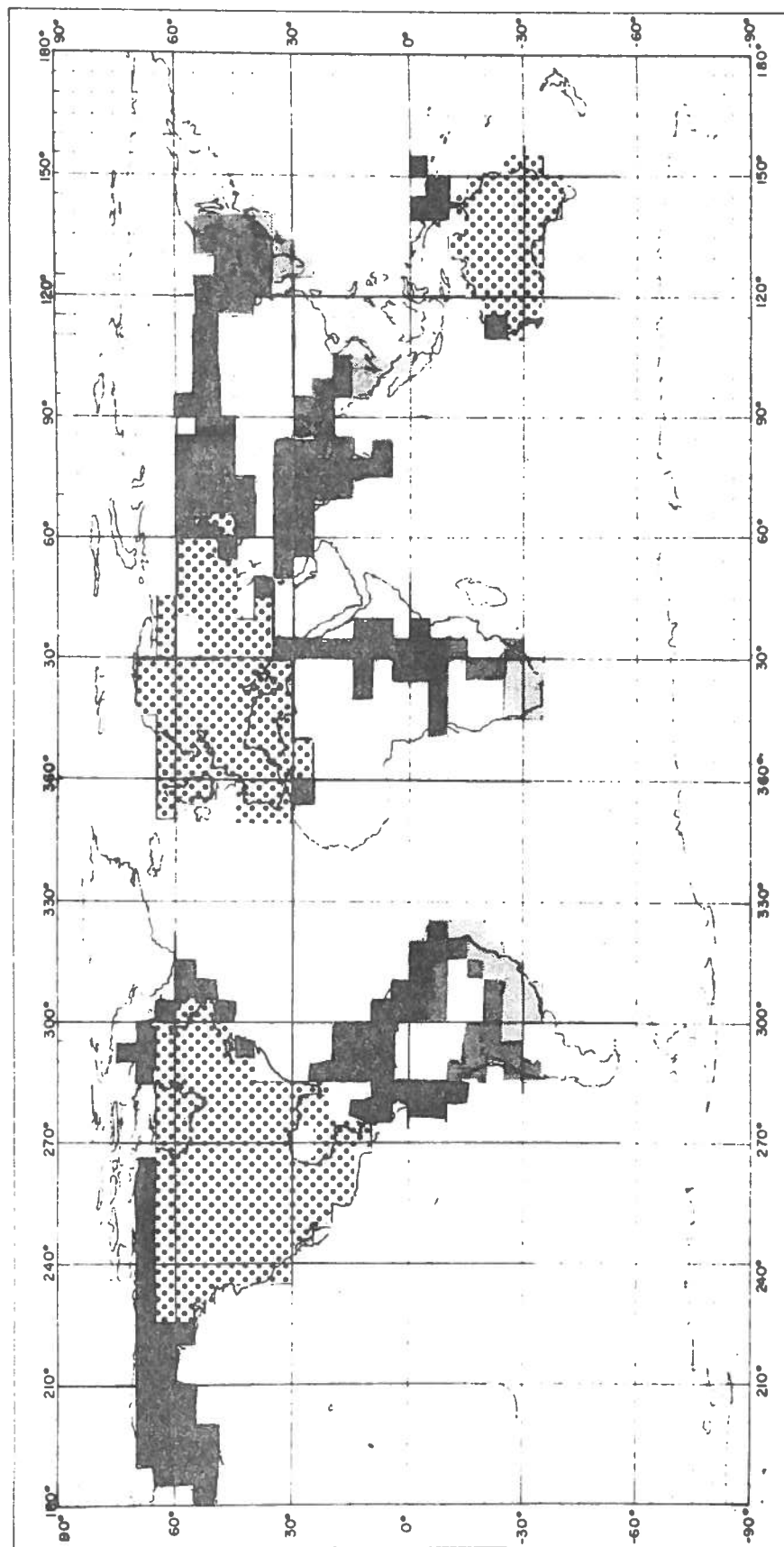


Fig. 4.2-4 Astrogeodetic undulation blocks and their standard errors

From a solution only for the shifts and the semidiameter shown in Table 4.2-12, it is evident that the shifts and the semidiameter were determined with very poor accuracy and consequently the harmonic coefficients were also poorly determined. This is shown by the standard errors given in the last column of Table 4.2-12. The standard errors listed in this table and the following tables are computed with an a priori variance of unit weight of 1. The estimate of the standard error of unit weight obtained from each adjustment is listed at the bottom of each table. We then weighted the shifts, the rotations, and the semidiameter (Table 4.2-13) using as a priori values the latest values for the modified Mercury datum [Fischer, 1968].

We remark that the approximate values introduced in the solution and given in the column "Original Value" are the differences between the values of the shifts for the modified Mercury Datum and the shifts that have already been introduced in the solution by changing the element of the ellipsoid of the datum to an ellipsoid with

$$a = 6378150 \text{ m}$$

$$1/f = 298.3$$

as is explained below:

	Modified Mercury Datum 1968	Shifts Introduced by Changing the Ellipsoid of the Datum	Differences Used As Approximate Value
North American Datum	- 18 m	17.5 m	- 35.5 m
	145	116.7	28.3
	183	208	- 25.0
European Datum	- 81	-176.8	95.8
	-104	- 41.0	- 63.0
	-121	- 85.9	- 35.1
Australian Datum	-105	6.8	-111.8
	- 44	- 6.4	- 37.6
	94	1.9	92.1

Table 4.2 - 12

Determination of Shifts and Semidiameter from  
Astrogeodetic Undulations

		ORIGINAL	CORRECTION	ADJUSTED	Standard Error
		VALUE		VALUE	
North American Datum	$dx_{01}$	-0.	-164.7497	-164.7497	22.3875
	$dy_{01}$	-0.	-79.4507	-79.4507	24.1651
	$dz_{01}$	-0.	-45.9269	-45.9269	19.5702
	dA	0.	0.	0.	0.
	$d\mu$	0.	0.0026	0.0026	0.0316
	$d\nu$	0.	-0.0000	-0.0000	0.0316
	$\epsilon$	0.	0.	0.	0.
European Datum	$dx_{01}$	-0.	136.8063	136.8063	17.2829
	$dy_{01}$	-0.	74.6486	74.6486	39.1229
	$dz_{01}$	-0.	-118.0177	-118.0177	20.3563
	dA	0.	-0.	-0.	0.
	$d\mu$	0.	-0.0010	-0.0010	0.0316
	$d\nu$	0.	0.0000	0.0000	0.0316
	$\epsilon$	0.	0.	0.	0.
Australian Datum	$dx_{01}$	-0.	-138.4206	-138.4206	49.8832
	$dy_{01}$	-0.	5.3865	5.3865	43.7994
	$dz_{01}$	-0.	175.4738	175.4738	37.1469
	dA	0.	-0.	-0.	0.
	$d\mu$	0.	-0.0010	-0.0010	0.0316
	$d\nu$	0.	-0.0000	-0.0000	0.0316
	$\epsilon$	0.	0.	0.	0.
	semidiameter	0.	-69.2604	-69.2604	28.8219
$\bar{C}$	2 0	-484.2029	-0.0000	-484.2029	0.0000
		ORIGINAL		PRESENT	
LPL (VPV)		0.488025E 04		0.153498E 04	
STD ERR (UNIT WEIGHT)		0.356497E 01		0.203164E 01	
UNIT WEIGHT RMS DEV		0.334563E 01		0.190664E 01	
RMS DISCREPANCY		0.356710E 02		0.118887E 02	

Table 4.2 - 13

Astrogeodetic Solution with Weighted Shifts, Rotations, and Semidiameter  
But Not Weighted Harmonic Coefficients

	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	Standard Error	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	N	M
North American Datum	dx <sub>01</sub>	-35.5000	0.0419	-35.4581	3.1550				
	dy <sub>01</sub>	28.3000	-0.2940	28.0060	3.1451				
	dz <sub>01</sub>	-25.0000	0.3974	-24.6026	3.1322				
	dA	0.	0.	0.	0.				
	dμ	0.	-0.0615	-0.0615	0.1783				
European Datum	dν	0.	-0.0636	-0.0636	0.3762				
	ε	0.	0.	0.	0.				
	dx <sub>01</sub>	95.8000	-0.0646	95.7354	3.1278				
	dy <sub>01</sub>	-63.0000	-0.2416	-63.2416	3.1230				
	dz <sub>01</sub>	-35.1000	-0.2906	-35.3906	3.1097				
Australian Datum	dA	0.	0.	0.	0.				
	dμ	0.	-0.1687	-0.1687	0.1845				
	dν	0.	0.0413	0.0413	0.1821				
	ε	0.	0.	0.	0.				
	dx <sub>01</sub>	-111.8000	0.1083	-111.6917	3.1604				
semidiameter	dy <sub>01</sub>	-37.6000	0.1314	-37.4686	3.1582				
	dz <sub>01</sub>	92.1000	-0.0440	92.0560	3.1617				
	dA	0.	0.	0.	0.				
	dμ	0.	0.0549	0.0549	0.1993				
	dν	0.	0.0177	0.0177	0.1982				
	ε	0.	0.	0.	0.				
	dx <sub>01</sub>	-484.0358	-0.7559	-484.7917	5.6583				
	dy <sub>01</sub>	0.	-0.9217	-0.9217	1.8175				
	dz <sub>01</sub>	0.	-1.2232	-1.2232	2.0605				
	dA	0.	3.9201	3.9201	3.9614				
	dμ	0.	-2.8480	-2.8480	2.5945				
	dν	0.	7.8377	7.8377	3.3688				
	ε	0.	1.5850	1.5850	1.6442				
	dx <sub>01</sub>	0.7825	-1.9222	-1.1397	4.6210				
	dy <sub>01</sub>	0.	1.7869	1.7869	5.0606				
	dz <sub>01</sub>	0.	1.7283	1.7283	4.2250				
	dA	0.	-8.1663	-8.1663	2.8821				
	dμ	0.	0.3731	0.3731	2.6857				
	dν	0.	1.6108	1.6108	4.9465				
	ε	0.	7.7503	7.7503	4.9857				
	dx <sub>01</sub>	0.	-2.9228	-2.9228	2.2224				
	dy <sub>01</sub>	0.	1.5285	1.5285	2.7445				
	dz <sub>01</sub>	0.	7.6275	7.6275	2.0603				
	dA	0.	-5.2025	-5.2025	2.8683				
	dμ	0.	7.9773	7.9773	3.9756				
	dν	0.	-6.8935	-6.8935	3.8123				
	ε	0.	-3.5130	-3.5130	1.7421				
	dx <sub>01</sub>	0.	4.0876	4.0876	1.8389				
	dy <sub>01</sub>	0.	-4.6784	-4.6784	1.8161				
	dz <sub>01</sub>	0.	-3.8911	-3.8911	2.0655				
	dA	0.	5.1863	5.1863	1.5953				
	dμ	0.	-7.1040	-7.1040	1.5682				
	dν	0.	-0.0022	-0.0022	2.2634				
	ε	0.	4.9173	4.9173	1.8650				
	dx <sub>01</sub>	0.	-0.9058	-0.9058	1.1904				
	dy <sub>01</sub>	0.	-1.2781	-1.2781	1.7331				
	dz <sub>01</sub>	0.	4.0564	4.0564	0.7903				
	dA	0.	-1.9073	-1.9073	1.1141				
	dμ	0.	-2.4816	-2.4816	0.8204				
	dν	0.	1.4932	1.4932	0.5075				
	ε	0.	1.5674	1.5674	0.7667				
	dx <sub>01</sub>	0.	-1.1182	-1.1182	0.6873				
	dy <sub>01</sub>	0.	-0.8566	-0.8566	0.4790				
	dz <sub>01</sub>	0.	0.9046	0.9046	0.3744				
	dA	0.	-1.0913	-1.0913	0.3490				
	dμ	0.	0.0072	0.0072	0.4043				
	dν	0.	0.3773	0.3773	0.3085				
	ε	0.	0.6596	0.6596	0.4354				
	dx <sub>01</sub>	0.							

	ORIGINAL	PRESENT
LPL (VPV)	0.187797E 05	0.266193E 05
STD LRR (UNIT WEIGHT)	0.144294E 01	0.918819E 00
UNIT WEIGHT RMS LPL	0.656298E 01	0.816189E 00
RMS DISCREPANCY	0.106190E 02	0.607133E 01



In the Mercury datum it is supposed that the existing datums are not rotated, thus we used zero as the a priori value for the rotations. The variances used to weight the shifts and the rotations are the ones found in Section 4.1 to be the most probable for a datum oriented by satellite observations. The variance for the semidiameter is the one found from a combination of satellite survey and astrogeodetic solutions, as will be explained in Section 4.3. The a priori variances assigned to the parameters were as follows:

Parameters	Variances
$dx_{01}, dy_{01}, dz_{01}$	$10 \text{ m}^2$
$d\mu, d\nu$	$0.16 \text{ sec}^2$
$da$	$36 \text{ m}^2$

The new solutions were still unsatisfactory. From all the above we came to the conclusion that the distortions in the solution were primarily due to the incomplete coverage of our astrogeodetic undulations. To help the solution in that respect we decided to use some kind of predictions for the astrogeodetic undulations in the vast unsurveyed areas.

The fact that the astrogeodetic undulations lie on different datums complicates any statistical treatment of them so much that no statistical predictions for undulations in the unobserved areas can be established. On the other hand, we can assign zero undulations to the unobserved blocks and use the RMS variation of the mean astrogeodetic or gravimetric undulation of  $5^\circ \times 5^\circ$  blocks as the standard errors. However, the undulations still must be referred to some reference surface. They can be referred either to the initial datum or to the approximate ellipsoid used. In both cases, in addition to the possible effect on the harmonics, the estimated undulations will invalidate the solution as far as the shifts and the semidiameter are concerned because these values will contribute and take part in that determination, pulling the true solution from the geocenter.

We decided then to use zero as the a priori value for the coefficients and to use the RMS coefficient variations to compute a priori weights. The RMS coefficient variation determined by Rapp [1968a] is

$$\sigma = \frac{6.2 \times 10^{-6}}{n^2}$$

We first tried a solution with no rotations and without approximate values for the shifts and the semidiameter (Table 4.2 - 14). We see that the shifts are still determined poorly and that the solution does not resemble other estimates of the harmonic coefficients. This poor determination of shifts and harmonics is improved slightly if we use a weighted approximate value for the semidiameter only (Table 4.2 - 15).

The solution was improved considerably when we used weighted approximate values for the shifts, the rotations, and the semidiameter. The solution from this data is given in Table 4.2 - 16. This solution approximates the combination of dynamic and geometric determinations of stations and an astrogeodetic solution for shifts and spherical harmonics.

We now compare this solution with the harmonic coefficients and uncertainties of Table 4.2 - 4. We first notice the remarkable agreement of  $\bar{C}_{20}$  of this solution with that determined from satellite data. We again form the absolute difference of the coefficients and the confidence interval  $1.96 \sqrt{\sigma_s^2 + \sigma_u^2}$ , as was done for Table 4.2 - 7. In Table 4.2 - 17 we give the difference and the confidence intervals for the coefficients for which the hypothesis of equality of means is rejected. The factors by which the present standard errors must be multiplied so that the above hypothesis will not be rejected are given in parentheses. The degree variances computed with the coefficients of this solution are also given in the same table. From Table 4.2 - 17 it appears that the estimated standard errors should be multiplied by approximately 1.5 — 1.6 to give better accuracy estimates.

We have to remember, however, that the shifts, the rotations, and the semidiameter used, probably do not have the assigned accuracy of 3 m, 0".4,

Table 4.2 - 14

**Astrogeodetic Solution with Weighted Coefficients, But not Weighted  
Approximate Values for Shifts, Rotations, and Semidiameter**

C				S			
	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	Standard Error	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE
North American Datum	dx <sub>01</sub>	-0.	-77.3781	18.0906			
	dy <sub>01</sub>	-0.	-40.6074	19.6817			
	dz <sub>01</sub>	-0.	-40.3659	14.1668			
	dA	0.	-0.	0.			
	dμ	0.	0.0021	0.0316			
	dν	0.	0.0000	0.0316			
European Datum	ε	0.	-0.	0.			
	dx <sub>01</sub>	-0.	98.4382	11.8357			
	dy <sub>01</sub>	-0.	39.1686	26.5228			
	dz <sub>01</sub>	-0.	-72.2819	13.9909			
	dA	0.	0.	0.			
	dμ	0.	-0.0000	0.0316			
Australian Datum	dν	0.	-0.0000	0.0316			
	ε	0.	-0.	0.			
	dx <sub>01</sub>	-0.	-36.0165	28.8691			
	dy <sub>01</sub>	-0.	25.6859	28.7255			
	dz <sub>01</sub>	-0.	89.7897	27.5534			
	dA	0.	0.	0.			
semidiameter	dμ	0.	0.0003	0.0316			
	dν	0.	-0.0000	0.0316			
	ε	0.	-0.	0.			
	semidiameter	0.	-54.7184	18.9756			
	2 0	-484.0358	-0.7103	-484.7461	0.9094		
	2 2	0.	-1.5447	-1.5447	1.0168		
2 0	0.	0.	-0.5611	-0.5611	0.4716		
	3 1	0.	0.7171	0.7171	0.4090		
	3 2	0.	1.0410	1.0410	0.3813		
	3 3	0.	0.1951	0.1951	0.3958		
	4 0	0.7825	-0.1945	0.5880	0.2786		
	4 1	0.	-0.3513	-0.3513	0.2747		
4 2	0.	0.	0.5745	0.5745	0.2512		
	4 3	0.	0.1558	0.1558	0.2653		
	4 4	0.	0.0668	0.0668	0.2430		
	5 0	0.	0.0295	0.0295	0.1823		
	5 1	0.	-0.2411	-0.2411	0.1923		
	5 2	0.	0.4559	0.4559	0.1806		
5 3	0.	0.	-0.3504	-0.3504	0.1633		
	5 4	0.	-0.1637	-0.1637	0.1714		
	5 5	0.	-0.2200	-0.2200	0.1771		
	6 0	0.	0.0973	0.0973	0.1311		
	6 1	0.	-0.0141	-0.0141	0.1324		
	6 2	0.	0.1828	0.1828	0.1308		
6 3	0.	0.	0.0859	0.0859	0.1288		
	6 4	0.	-0.0335	-0.0335	0.1221		
	6 5	0.	-0.3745	-0.3745	0.1264		
	6 6	0.	0.1029	0.1029	0.1337		
	7 0	0.	0.0912	0.0912	0.1000		
	7 1	0.	0.0585	0.0585	0.1006		
7 2	0.	0.	-0.0522	-0.0522	0.0971		
	7 3	0.	0.2060	0.2060	0.0982		
	7 4	0.	-0.2126	-0.2126	0.0958		
	7 5	0.	-0.1094	-0.1094	0.0933		
	7 6	0.	-0.0245	-0.0245	0.0963		
	7 7	0.	0.0954	0.0954	0.0996		
8 0	0.	0.	0.0567	0.0567	0.0762		
	8 1	0.	0.1020	0.1020	0.0754		
	8 2	0.	-0.0079	-0.0079	0.0701		
	8 3	0.	-0.0756	-0.0756	0.0704		
	8 4	0.	-0.0745	-0.0745	0.0700		
	8 5	0.	-0.1812	-0.1812	0.0676		
8 6	0.	0.	0.0122	0.0122	0.0722		
	8 7	0.	0.0263	0.0263	0.0753		
	8 8	0.	-0.0558	-0.0558	0.0815		

	ORIGINAL	PRESENT
LPL (VPV)	0.491421E 04	0.390538E 03
STD ERR (UNIT WEIGHT)	0.380739E 01	0.107333E 01
UNIT WEIGHT RMS DEV	0.335725E 01	0.946430E 00
RMS DISCREPANCY	0.359525E 02	0.711881E 01

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## Astrogeodetic Solution with Weighted Harmonic Coefficients and Semidiameter

C					S				
	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	Standard Error	ORIGINAL VALUE	CORRECTION	ADJUSTED VALUE	N	M
North American Datum	$dx_{01}$	-0.	-54.9011	-54.9011	16.1320				
	$dy_{01}$	-0.	1.4593	1.4593	12.3752				
	$dz_{01}$	-0.	-17.8403	-17.8403	11.5548				
	$dA$	0.	-0.	-0.	0.				
	$d\mu$	0.	0.0005	0.0005	0.0316				
	$d\nu$	0.	0.0000	0.0000	0.0316				
European Datum	$dx_{01}$	0.	-0.	-0.	0.				
	$dy_{01}$	-0.	93.3151	93.3151	11.6886				
	$dz_{01}$	-0.	-25.4766	-25.4766	12.2552				
	$dA$	0.	-49.7903	-49.7903	11.3428				
	$d\mu$	0.	0.	0.	0.				
	$d\nu$	0.	-0.0005	-0.0005	0.0316				
Australian Datum	$dx_{01}$	0.	0.0000	0.0000	0.0316				
	$dy_{01}$	-0.	-0.	-0.	0.				
	$dz_{01}$	-0.	-54.0469	-54.0469	28.1233				
	$dA$	0.	4.7412	4.7412	27.7104				
	$d\mu$	0.	74.4476	74.4476	26.9773				
	$d\nu$	0.	0.	0.	0.				
semidiameter	0.	-0.	-0.	-0.	0.				
2 0	-484.0358	-4.9718	-4.9718	5.7209	0.				
2 2	0.	0.2657	-483.7701	0.8171	0.				
3 0	0.	0.0812	0.0812	0.8278	0.	-0.9675	-0.9675	0.8039	2 2
3 1	0.	-0.0924	-0.0924	0.4398	0.				
3 2	0.	0.5028	0.5028	0.4015	0.	-0.2621	-0.2621	0.3497	3 1
3 3	0.	1.2103	1.2103	0.3761	0.	-0.3134	-0.3134	0.3725	3 2
4 0	0.7825	0.2292	0.2292	0.3956	0.	1.8172	1.8172	0.3970	3 3
4 1	0.	-0.2088	0.5737	0.2786	0.				
4 2	0.	-0.2877	-0.2877	0.2737	0.	-0.4534	-0.4534	0.2575	4 1
4 3	0.	0.7284	0.7284	0.2450	0.	0.2338	0.2338	0.2617	4 2
4 4	0.	0.2162	0.2162	0.2644	0.	-0.1555	-0.1555	0.2552	4 3
5 0	0.	0.0614	0.0614	0.2430	0.	0.1729	0.1729	0.2638	4 4
5 1	0.	0.0605	0.0605	0.1870	0.				
5 2	0.	-0.2942	-0.2942	0.1913	0.	0.0342	0.0342	0.1777	5 1
5 3	0.	0.5444	0.5444	0.1778	0.	-0.1500	-0.1500	0.1728	5 2
5 4	0.	-0.3870	-0.3870	0.1628	0.	-0.0529	-0.0529	0.1697	5 3
5 5	0.	-0.1986	-0.1986	0.1710	0.	-0.4603	-0.4603	0.1855	5 4
6 0	0.	-0.2949	-0.2949	0.1750	0.	-0.4777	-0.4777	0.1815	5 5
6 1	0.	0.1284	0.1284	0.1306	0.				
6 2	0.	-0.0044	-0.0044	0.1324	0.	0.2935	0.2935	0.1241	6 1
6 3	0.	0.2127	0.2127	0.1303	0.	-0.0876	-0.0876	0.1251	6 2
6 4	0.	0.0680	0.0680	0.1286	0.	-0.1151	-0.1151	0.1228	6 3
6 5	0.	-0.0420	-0.0420	0.1220	0.	-0.1788	-0.1788	0.1267	6 4
7 0	0.	-0.3395	-0.3395	0.1257	0.	-0.3905	-0.3905	0.1266	6 5
7 1	0.	0.0754	0.0754	0.1333	0.	-0.3552	-0.3552	0.1289	6 6
7 2	0.	0.0950	0.0950	0.1000	0.				
7 3	0.	0.0768	0.0768	0.1004	0.	0.4634	0.4634	0.0943	7 1
7 4	0.	-0.0372	-0.0372	0.0969	0.	0.0326	0.0326	0.0963	7 2
7 5	0.	0.1952	0.1952	0.0981	0.	-0.1567	-0.1567	0.0948	7 3
7 6	0.	-0.2161	-0.2161	0.0958	0.	-0.0503	-0.0503	0.0930	7 4
7 7	0.	-0.1226	-0.1226	0.0932	0.	0.1318	0.1318	0.0951	7 5
8 0	0.	-0.0297	-0.0297	0.0962	0.	0.0574	0.0574	0.0973	7 6
8 1	0.	0.0948	0.0948	0.0996	0.	-0.0806	-0.0806	0.1019	7 7
8 2	0.	0.0643	0.0643	0.0761	0.				
8 3	0.	0.1000	0.1000	0.0754	0.	0.3427	0.3427	0.0712	8 1
8 4	0.	0.0025	0.0025	0.0700	0.	-0.0425	-0.0425	0.0721	8 2
8 5	0.	-0.0873	-0.0873	0.0702	0.	-0.2624	-0.2624	0.0694	8 3
8 6	0.	-0.0700	-0.0700	0.0699	0.	0.0004	0.0004	0.0676	8 4
8 7	0.	-0.1962	-0.1962	0.0674	0.	0.1401	0.1401	0.0696	8 5
8 8	0.	-0.0021	-0.0021	0.0720	0.	0.0205	0.0205	0.0711	8 6
9 0	0.	0.0077	0.0077	0.0749	0.	-0.0141	-0.0141	0.0746	8 7
9 1	0.	-0.0626	-0.0626	0.0814	0.	0.1304	0.1304	0.0806	8 8

	ORIGINAL	PRESENT
LPL (VPV)	0.491421E 04	0.393746E 03
STD ERR (UNIT WEIGHT)	0.380739E 01	0.107773E 01
UNIT WEIGHT RMS DEV	0.335725E 01	0.950309E 00
RMS DISCREPANCY	0.359525E 02	0.712358E 01

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and Harmonic Coefficients157

Table 4.2 - 17

Differences of Satellite and Astrogeodetic Coefficients  
and Their Confidence Intervals

$\bar{C}$				$\bar{S}$			
Order	Difference	$1.96 \sqrt{\sigma_T^2 + \sigma_S^2}$		Order	Difference	$1.96 \sqrt{\sigma_T^2 + \sigma_S^2}$	
30	0.78	0.71	(1.1 )				
31	0.74	0.62	(1.2 )				
33	0.67	0.58	(1.17)				
43	0.64	0.56	(1.40)				
				54	0.46	0.366	(1.28)
60	0.31	0.27	(1.8 )	61	0.32	0.25	(1.29)
				71	0.30	0.26	(1.26)
72	0.36	0.21	(1.8 )	81	0.285	0.17	(1.90)
				83	0.26	0.18	(1.68)
				86	0.29	0.23	(1.63)

Anomaly Degree Variances from Coefficients Computed from Astrogeodetic Undulations	
Degree	Degree Variance
2	7.0
3	22.0
4	8.4
5	18.3
6	17.4
7	14.2
8	14.1

and 6 m which will be the case in the actual combination solution. The solution that we have now performed is of poorer accuracy than the one that the accuracy shows. We estimated then that the present standard errors must be multiplied by 1.3 to yield more reliable accuracy figures.

With the above considerations, it appears that the accuracies that can be obtained for harmonic coefficients from the present astrogeodetic data, properly supported by dynamic, geometric, and survey determinations which will assure the above accuracies in shifts, rotations, and semidiameter, can be represented by the following model:

$$\sigma_{c,s} = \frac{0.75 \times 10^{-6}}{n-1} \quad (4.2-6)$$

where

$\sigma$  is the standard deviation of harmonic coefficient of  $n$  degree

#### 4.24 Determination of Harmonic Coefficients from a Combination of Satellite, Terrestrial, and Astrogeodetic Data

We will try to see now what the approximate results of the combination of the above data and methods for determination of spherical harmonic coefficients will be.

According to the discussion in Section 4.21, the accuracy of a combination solution of terrestrial gravity and satellite data can be approximated by a quadratic propagation of the variances of the individual solutions. This means that the accuracy of higher-order harmonics are not significantly increased when we combine a truncated satellite set with terrestrial data. The accuracy of the higher harmonics remains almost the same as in the terrestrial solution. Thus the degree up to which we solve in a combination solution like this is usually determined by the accuracy of the terrestrial or the astrogeodetic solutions.

The accuracy of the higher-order harmonics is increased, however, when combining the terrestrial gravity and the astrogeodetic undulations since

the quadratic propagation of variances will yield a standard deviation of

$$\sigma = \frac{0.50 \times 10^{-6}}{n-1} \quad (4.2-5)$$

As an accuracy criterion, we require that the harmonic coefficients obtained have a standard error smaller than or at least equal to the root mean square of coefficient variation. The use of harmonics beyond that limit would decrease the accuracy of the determined quantities for ex-geoid height computations. We can then determine the degree up to which we can solve by equating the accuracy of the coefficients for terrestrial gravity (equation(4.2-3)) and for astrogeodetic undulations (equation (4.2-4)) with the root mean square coefficient variation  $\sigma$ . The RMS coefficient variation is given by Rapp [1968a] as

$$\sigma = \frac{6.2 \times 10^{-5}}{n^2} \quad (4.2-6)$$

and by Kaula [1967] as

$$\sigma = \frac{10^{-5}}{n^2}$$

This result was also verified by Anderle [et al., 1968]. Pellinen has recently estimated the degree variance of gravity anomalies as [Pellinen, 1969]

$$\sigma_{\Delta g_n}^2 = \frac{A}{n^\alpha} \quad (4.2-7)$$

with  $A = 120 \text{ mgal}$  and  $\alpha = 1.13$ . As the degree variance for gravity anomalies is

$$\sigma_{\Delta g_n}^2 = \gamma^2 (n-1)^2 \sum_{m=0}^n (\overline{C}_{nm}^2 + \overline{S}_{nm}^2) \quad (4.2-8)$$

and as there are  $2n+1$  terms in each degree, the summation can be expressed by the root mean square coefficient variation as

$$\sigma_{\Delta g_n}^2 = \gamma^2 (n-1)^2 (2n+1) \sigma^2 \quad (4.2-9)$$



then

$$\sigma^2 = \frac{A}{\gamma^2 (n-1)^2 (2n+1) n^{1.13}} \quad (4.2-10)$$

or

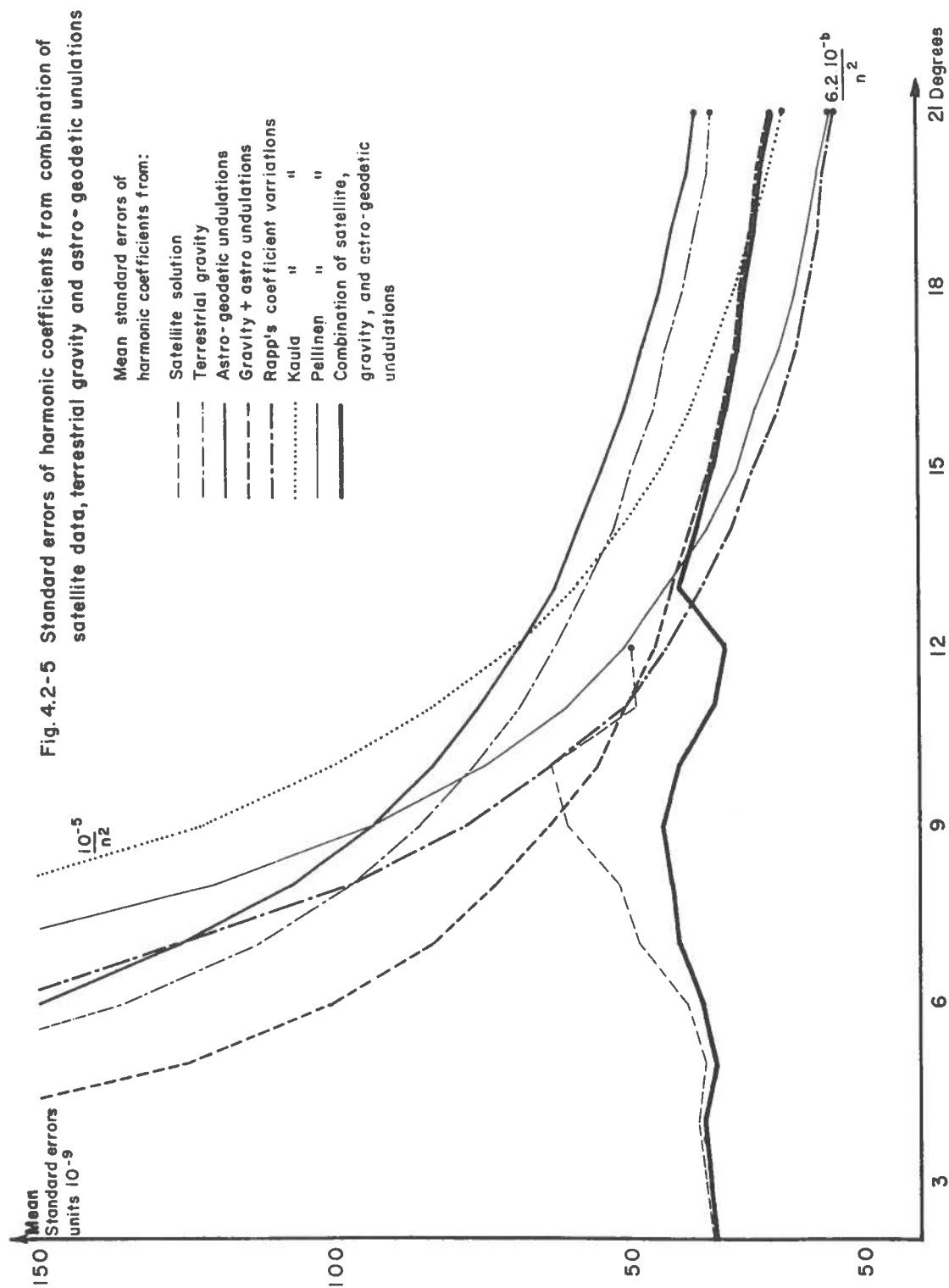
$$\sigma = \frac{11.2}{(n-1) \sqrt{(2n+1) n^{1.13}}} \quad (4.2-11)$$

We then have the following results:

Source of Information	Highest Degree Harmonic to Be Solved		
	Rapp	Kaula	Pellinen
Terrestrial gravity	8	13	10
Astrogeodetic undulations	7	12	9
Terrestrial gravity and astro- geodetic undulations	11	19	13-14

The above results are also readily available from Fig. 4.2-5. The highest degree harmonic up to which we can solve from every group of data or from a combination of them with the restriction that the standard deviation is equal to the coefficient variation, is represented by the intersection of the particular coefficient variation curve and the one representing the accuracy of a group or a combination of groups of data. We see that we have a considerable improvement, as far as the highest degree up to which we can solve, by combining terrestrial gravity data and astrogeodetic undulations. In Fig. 4.2-5, we plotted the line which approximately represents the expected accuracies from a combination of satellite, terrestrial gravity and astrogeodetic undulations.

The mean standard error of harmonic coefficients of all degrees from a combination of satellite, terrestrial gravity, and astrogeodetic undulations is approximately  $0.035 \times 10^{-6}$



It is obvious that it makes quite a difference whether we accept Rapp's or Kaula's model for the coefficient variation. It appears that Pellinen's model, which gives values very close to the mean of those of Rapp and Kaula, can be used as a reasonable base for such computations. According to Pellinen's model a combination of terrestrial gravimetric data, astrogeodetic undulations, and satellite solutions could go up to degree and order 14. The standard deviations of the coefficients up to degree 14 from a combination solution are given in Fig. 4.2-5. With these the mean standard deviation of geoid undulations can be determined

$$\sigma_N^2 = \sigma_C^2 + \sigma_T^2 \quad (4.2-12)$$

where

- $\sigma_N$  is the mean standard error fo geoid undulation
- $\sigma_C$  is the standard error of undulations due to the errors of the coefficients
- $\sigma_T$  is the standard error of undulations due to the neglected higher-order terms

If we consider the coefficients independent, the mean standard error of undulation due to the errors of the coefficients is

$$\sigma_C^2 = R^2 \sum_{n=2}^{14} \sum_{m=0}^n (\sigma_{C_{nm}}^2 + \sigma_{S_{nm}}^2) \quad (4.2-13)$$

where

$\sigma_{C_{nm}}, \sigma_{S_{nm}}$  are the standard deviations of the  $\bar{C}_{nm}$  and  $\bar{S}_{nm}$  coefficients

With the uncertainties of Fig. 4.2-5, we get

$$\sigma_C^2 = 13.2 \text{ m}^2$$

The truncation error for geoid undulations is given in [Rapp, 1967b]. For sets of spherical harmonics truncated at degree and order 14, the truncation error is

$$\sigma_T^2 = 11.6 \text{ m}^2$$

Thus the total mean standard deviation of a geoid undulation is

$$\sigma_N \cong 5.0 \text{ m}$$

#### 4.3 Accuracy of a Determination of the Components of the Mean Earth Ellipsoid

The last component of the world geodetic system is composed of the four parameters defining the general terrestrial ellipsoid; in our system these parameters are the rate of rotation  $\omega$ , the second-degree harmonic  $\bar{C}_{20}$ , the semidiameter  $a$ , and the gravitational constant times the mass of the earth  $GM$ . From these the rate of rotation of the earth is known with such a high accuracy that it can be considered constant. The determination of  $\bar{C}_{20}$  has been examined before, and its accuracy has been found to be approximately  $0.03 \times 10^{-6}$ . From this and the equation

$$df = \frac{3}{2} \sqrt{5} d\bar{C}_{20}$$

the accuracy of the flattening can be determined. It is

$$\sigma_f \cong 0.1 \times 10^{-6}$$

and that of the inverse flattening is

$$\sigma_{1/f} = 0.01$$

The gravitational constant times the mass of the earth  $GM$  is determined from observations of the moon which are not considered here, range or Doppler observations of satellites, and radar observations of lunar probes. From optical satellite observations and triangulation,  $GM$  can be found with an accuracy of  $1.2 \text{ km}^3 \text{ sec}^{-2}$  [Kaula, 1963]. If we consider the analysis made by Guier [1964] and the value given in the recent Doppler solution,  $GM = 398600 \text{ km}^3 \text{ sec}^{-2}$  [Anderle et al., 1967], it appears that the accuracy of the determination of  $GM$  from Doppler satellite observations is better than

$1.0 \text{ km}^3 \text{ sec}^{-2}$ . The best determination of GM at present is from radar observation of lunar probes. A recent solution where data from several missions was combined gives an accuracy of  $0.7 \text{ km}^3 \text{ sec}^{-2}$  [Vegos et al., 1967].

If we approximate the result of the combination of the above observations with their weighted mean, we will find an accuracy for GM of

$$\sigma_{\text{GM}} = 0.52 \text{ km}^3 \text{ sec}^{-2}$$

The semidiameter of the general terrestrial ellipsoid will be recovered in the world geodetic system by the combination of a kind of astrogeodetic method where the astrogeodetic undulations are used, the adjustment of dynamic coordinates with the survey coordinates, and by the constraint (equation (3.7 - 3)) first used for such a solution in [Rapp, 1967c]. The determination of the semidiameter from the constraint is affected by the errors of the various parameters involved as follows

$$\begin{aligned} da &= \frac{1}{2} a \, d f \\ da &= \frac{1}{2} \frac{a}{\gamma_e} \, d\gamma_e \\ da &= \frac{1}{2} \frac{a}{\text{GM}} \, d(\text{GM}) \end{aligned} \tag{4.3 - 1}$$

The uncertainty of  $\gamma_e$  is composed of the uncertainty of  $\gamma_e$  in the Potsdam system plus the uncertainty of the connection of the Potsdam system to the absolute one. The uncertainty of  $\gamma_e$  will be found from equation (3.7 - 2). As  $d\bar{C}_{20}$  is determined with high accuracy, the uncertainty of  $\gamma_e$  is practically equal to the uncertainty of  $\Delta g_0$ .  $\Delta g_0$  and its accuracy are found from the development of gravity anomalies into spherical harmonics. Some preliminary runs with simulated data indicate that  $\Delta g_0$  is determined better than  $\pm 1 \text{ mgal}$ . In [Kaula, 1961]  $\Delta g_0$  was determined with an accuracy of  $1.2 \text{ mgal}$ . A standard deviation of  $1.5 \text{ mgal}$ , also adopted in [Veis, 1967], seems, however, more appropriate in view of the systematic differences that exist between the absolute gravity measurements in Europe and the United States

[Cook, 1965a]. To this we have to add quadratically the uncertainty with which the Potsdam system is connected to the new absolute measurements. The adopted value for the Geodetic Reference System 1967 for the conversion of the Potsdam system to the absolute one is -14 mgal [Garland, 1967, p.146, Resolution No. 22]. This value was based on the measurements of two groups: the mean value from the first group was  $-12.7 \pm 0.6$  mgal; and from the second,  $-13.8 \pm 0.04$  mgal. For the adopted value, an accuracy of 0.5 mgal seems reasonable. Thus the standard deviation of  $\gamma_e$  is  $\pm 1.58$  mgal. With the above accuracies, the effects on the semidiameter from various parameters are

from the flattening	$\pm 0.32$ m
from $\gamma_e$	$\pm 5.15$ m
from GM	$\pm 4.16$ m

Their quadratic sum gives for the accuracy of the semidiameter

$$\sigma_a = \pm 6.6 \text{ m}$$

The astrogeodetic determination alone of the semidiameter gives an accuracy of about  $\pm 20$  m. If supported by weighted shifts, rotations, and semidiameter, it gives a precision of 6.3 m. This is obtained from the solution given in Table 4.2 - where the a priori standard error of 6 m for the semidiameter has been decreased to 4.4 m. We have estimated that the standard error of that adjustment should be multiplied by 1.3 to give accuracy estimates; thus the standard error of 6.3 m becomes 8.2 m which, when combined quadratically with the previous result of 6.6 m, gives the standard deviation for the semidiameter

$$\sigma_a = \pm 5.1 \text{ m}$$

Thus we have obtained error estimates for the present determination for all the components of the world geodetic system.

## 5. SUMMARY AND CONCLUSIONS

Having accepted that the existing geodetic datums cannot adequately serve the present complex geodetic requirements, we sought a new geocentric and world-wide geodetic system. That system is an improvement over the old frames because in addition to the potential of a level ellipsoid, a more detailed gravity field, consistent with the potential of the mean earth ellipsoid, is available and can be used when required. Also a unification of the geometric notion of the datum with the dynamic model of the normal gravity field was introduced. This was achieved by placing at the geocenter the mean earth ellipsoid, for which the rate of rotation, shape, size, and gravitational potential are defined.

For the establishment of this world-wide geodetic system, we tried to find a way to most effectively use various types of data. Mathematical models were developed to represent each of these groups. The world-wide system is to be found from the simultaneous solution of all the observation equations formed with these models. As normal equations are first formed from groups of observations, it is possible to combine the normal equations of all groups instead of combining all observations. The combination of normal equations was discussed, and the existing difficulties were emphasized. If we had the normals from eqch groups of existing observations, we could have combined them to get the values and the present uncertainties for the components of the world geodetic system. As we could have collected observations for approximately 80 stations which could have been included in a dynamical and geometrical satellite network, for five major geodetic datums, and for harmonic coefficients up to  $14 \times 14$ , the combined system would have had

approximately 500 unknowns, that is, of a size which is within the capability of any medium-sized computer. As we did not have the normal equations and as we considered the above task beyond the scope of this study, we tried to analyze the accuracy to be expected of such a combination solution. For this determination the available solutions were used. In cases where no such solutions were available, as in the case of astrogeodetic undulation, we performed our own determinations.

First the accuracy of the seven datum transformation parameters was examined. Related to this was the question of the accuracy of the geocentric positions of the stations and the contribution of triangulation to the geocentric coordinates. We found that by using seven parameters in this adjustment, differences in rotations and scale were well separated, thus the shifts of the datum were better determined so that the determinations from different observing systems were in better agreement. The contribution of the triangulation was significant and from fifteen well-distributed stations, the standard deviation of 10 m of the satellite determination decreased to 4 m for stations close to the datum origin.

In regard to the spherical harmonic coefficients of the geopotential, we showed that there was a very good determination of the lower-degree terms from satellites and that around degree 12 the coefficients were determined with a standard deviation of the order of their size. For the higher degrees even the well-determined resonant harmonics were not better than their size, which did not make them very useful for general use. The accuracies of spherical harmonic coefficients determined from terrestrial gravity, although lower than usually thought, still made their contribution to the determination of higher-degree coefficients.

The astrogeodetic undulations alone are almost useless for determination of spherical harmonic coefficients because the datum transformation parameters cannot be determined very well by astrogeodetic methods. Properly supported by satellite orientation of the datum, however, they provide a determination



very similar in accuracy to that from terrestrial gravity.

The combination of present satellite solutions and terrestrial data for harmonic coefficients does not permit the determination of terms higher than the terrestrial data alone allows. Thus only by combining terrestrial gravity and astrogeodetic undulations with satellites can we have a meaningful determination of the gravitational field up to the 14th degree and order.

From the present accuracies of the components of the world geodetic system and from the manner in which each group contributes to that determination, we can form a better idea of the data that is required to increase the present accuracy of some of these components. This knowledge will help us decide what data must be gathered to increase the accuracies of these parameters to specified levels, a question which is a natural continuation of the present study.

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