Adjustment Computations

ADJUSTMENT COMPUTATIONS

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Based on former Geodetic Science Courses
GS 650, GS 651, & GS 762
taught at The Ohio State University
by Prof. Burkhard Schaffrin

When you are in a difficult class,
then you learn much, especially
when your teacher is good.

C. ten Boom

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Preface

In its earlier versions of many years ago, Part I of this document was primarily based on notes that I took in Geodetic Science adjustment computations courses GS 650 and GS 651 taught by Burkhard Schaffrin at The Ohio State University in 1997 and 1998. Part II was based on notes that I took in the advanced adjustment computations course GS 762 taught by Burkhard. Gradually, I added content based on my own wishes and requests made by several readers of the notes. In many places, the notes reflect just what Prof. Schaffrin derived on the chalkboard during his lectures. In many other places, I extended them beyond that somewhat.

I would like to thank Dr. Shengjie Ge for his assistance with typesetting an early version with LaTeX. Also, Mr. Xiankun Wang, MSc., is acknowledged for his careful reading of the text and for pointing out several typographical errors that were corrected in the edition distributed in January, 2010. Dr. Dru Smith suggested additional formulas and the inclusion of summary formulas found at the end of most chapters beginning in 2018 and made numerous recommendations to improve Chapter 16 in a late 2020 version. He has continued to make helpful suggestions leading up to the version posted in late 2021. Questions posed by Dr. Jihye Park resulted in improvements to the first edition posted in 2019. A thorough review by Prof. Chris Jekeli in early 2019 resulted in many improvements. Constructive comments from Prof. Clyde Goad were also helpful. Thanks to Dr. Xing Fang for pointing out a few errors before the edition distributed in January 2021. And many thanks to Dr. Tae-Suk Bae for pointing out errors discovered during his Korean translation in 2021.

I would also like to express my thanks to the students of the Adjustment Computations course that I taught at Ohio State in the Fall Semester of 2019, during which time I gradually added 95 pages to these notes. These students had to endure my weekly updates to the notes, and their comments and question, as well as their keen eyes for my typographical errors, helped to improve the notes.

Of course, without the extraordinary teaching of Prof. Burkhard Schaffrin, these notes would not even exist.

Columbus, Ohio, December 2021   *Kyle Snow*

Introduction

This book is a two-part series on adjustment computations, Part 1 being introductory and Part 2 being advanced. Together, both parts should provide the reader with a solid foundation in the theory and application of adjustment computations, especially as they are used in the disciplines of geodetic science, surveying engineering, and allied disciplines. However, it is expected that researchers, data analysts, and practitioners from other science and engineering fields can benefit from the material as well.

The subject of adjustment computations is a rich topic spanning many science and engineering disciplines. The need to adjust observations in some meaningful—or perhaps better yet, *optimal*—way is an old one. The need becomes obvious as soon as one realizes that repeated observations of the same phenomenon or physical quantity usually do not yield the same numerical values, and yet they must be reconciled somehow.

As a young scientist, Carl Gauss was faced with the problem of how to best use redundant data to predict the trajectory of the asteroid Ceres. Gauss settled on a method that came to be known as "least-squares adjustment," some 15 years before it was made known to the public by Legendre, in 1805, who claimed to be its original discoverer. The debate over who first discovered the method of least squares for the adjustment of observations, Gauss or Legendre, is discussed by Stigler (1981), who provides evidence from important geodetic surveys that favor Gauss' claim, though Stigler admits that there still is not conclusive evidence to be absolutely certain about who was first. Perhaps only Gauss ever really knew.

The term least squares is often used adjectivally as in least-squares adjustment, least-squares solution, the method of least squares, etc. These terms are all more or less synonymous. The term comes from the mathematical technique of minimizing the *sum of squares of residuals* (or sum of squared residuals), where residual means the difference between an observation and its adjusted value. When observational weights are involved, the descriptive phrase should be modified to "the sum of squares of weighted residuals." Furthermore, the use of weight matrices in a linear algebra formulation adds another level of detail (correlations between random observation errors) that is often omitted from the descriptive phrase to avoid overburdening it. But that is jumping too far ahead for this introduction. In the chapters that follow, the term *adjustment computations* will apply to the adjustment of observational data by the method of least squares, unless otherwise noted.

Calculus is used to setup the problem for minimizing the sum of squared residuals, but the resulting solution can be found equivalently through both geometrical (projection of vector spaces) and statistical methods. While most derivations in this book involve minimization of an objective (target) function that includes Lagrange multipliers (the Lagrangian approach using calculus), equivalent statistical derivations and geometric relationships are also given in some places.

Introduction to Part I   The topics covered in Part I include classification of errors, measures of dispersion, variance and covariance, propagation of errors, observation and normal equations, representation of residuals, estimation of a single variance component, datum parameters, condition equations with and without parameters, various algorithms and examples, model constraints, statistical tests, and error ellipses.

Chapter 1 contains a discussion about observations, model parameters, and random variables, each of which are important quantities for the chapters that follow. A review of some essential material from linear algebra is also presented as a refresher.

Chapter 2 covers the model of direct observations and shows how the least-squares estimate of the unknown model parameter is derived. Equivalencies to the arithmetic and weighted means are discussed, and the concept of observation weighting is introduced, with a few examples from geodesy provided.

Chapter 3 introduces the Gauss-Markov Model for the case of multiple unknown parameters. This model forms a foundation for its extensions in chapters 5–7. The least-squares estimate within the model is fully derived, along with the prediction of residuals and an estimate for the unknown variance component of the model. The concept of datum information is briefly discussed, leaving a more thorough development for Part II.

Chapter 4 treats the model of condition equations, which is suitable for problems requiring observations to be adjusted without the estimation of parameters.

Chapters 5 and 6 extend the Gauss-Markov Model to account for constraints on the model parameters; the constraints are given without variance in Chapter 5, while Chapter 6 treats stochastic constraints on the parameters.

Chapter 7 treats the topic of sequential adjustments, which is important in certain real-time applications and when new data are provided that must be combined with the results of a previous adjustment. It takes a step in the direction of Kalman filtering, a topic presented in Part II.

Chapter 8 develops the Gauss-Helmert Model, for which the associated least-squares adjustment permits some problems to be solved more efficiently than the could be with the earlier presented models. It may also allow for "orthogonal regression" or "total least-squares" solutions to be computed in some cases.

Chapter 9 focuses on statistical analysis of least-squares adjustments. Among other things, it develops concepts and formulas for hypothesis testing on estimated parameters and for outlier detection for the observations.

Introduction to Part II   The objectives of the course that Part II is based on (GS 762) were stated by Burkhard Schaffrin in the original course syllabus as follows:

> The course makes students aware of various special adjustment techniques. Relations between the Gauss-Markov Model and traditional least-squares solutions are explored and compared to the collocation technique. Ranks of matrices are discussed, and they are derived for matrices usually encountered in adjustment computations. The introduction to generalized matrices will give the possibility to solve rank-deficient systems. Estimable and non-estimable quantities in adjustment are defined and discussed, as well as the estimation of variance components. The role of prior information is clarified, and it is shown how the least-squares adjustment in a Dynamic Linear Model leads to Kalman filtering. As a result, students should be able to make a prudent choice of a proper model and the corresponding adjustment techniques for a host of overdetermined problems in geodetic science, no matter how complicated.

Chapter 11 begins with a review of the nonlinear Gauss-Markov Model, showing how the least-squares solution for the unknown parameters of the model can be arrived at equivalently via both algebraic-geometric and statistical approaches, resulting in an equivalency between LESS (LEast Squares-Solution) and BLUUE (Best Linearly Uniformly Unbiased Estimate).

Chapter 12 introduces the linear algebra concepts of *vec operator* and *Kronecker product*, which provide essential tools for the derivations in chapters that follow. These concepts deserve attention here, because they typically are not covered in first courses in linear algebra, which may be the extent of background in linear algebra for many students reading this book.

In Chapter 13, the estimation of the unknown variance component appearing in the Gauss-Markov Model is derived from the statistical concept of Best Invariant Quadratic Uniformly Unbiased Estimate, or BIQUUE.

Chapter 14 treats the concept of *expectation-dispersion correspondence*, which allows a quadratic model for the unknown variance component to be transformed into a linear model and shows how BIQUUE for the unknown variance component in the Gauss-Markov Model is equivalent to LESS within the linear model.

Chapter 15 introduces the *rank-deficient Gauss-Markov Model*, providing a useful way to treat many adjustment problems in geodetic science where the observations do not provide enough information to estimate all the unknown parameters of the model. The concept of generalized inverse is presented, which allows for characterizing the solution space for the unknown model parameters, among which the estimators MINOLESS (and its equivalent BLUMBE) and partial MINOLESS are perhaps most useful.

Chapter 16 introduces *variance component estimation* for multiple variance components, which allows one to deal with multiple observational weight matrices for which the relative accuracies among them may be unknown or uncertain, in which

case, estimating a variance component for each one of them might be desired. This chapter shows how to do that.

Chapter 17 introduces the notion of *prior information* for the unknown model parameters. Here, the concept of random, rather than fixed, model parameters is introduced, and a model is presented that contains both types of parameters, leading to least-squares estimates of the fixed parameters and predictions of the random parameters.

Chapter 18 presents the *dynamic linear model* and derives the least-squares solution within it, which is also known as a Kalman filter.

An appendix contains several matrix properties and identities used throughout the book. A bibliography at the end includes referenced material and material for suggested reading.

Notation

A few comments about the notation used in this book may be helpful. Matrices are displayed in uppercase. Vectors are lowercase and are set in bold-face type (bold face is not used for any other symbols). Scalar variables are generally lower-case. Greek letters are used for unknown, non-random parameters, while Latin letters are used for unknown, random variables. Symbols denoting estimates of non-random variables use Greek letters with a hat on top, while predictions of random variables are shown as Latin letters with tildes on top. The following tables list variables, mathematical operators, and abbreviations used herein.

Table 1: Variables and mathematical operators

Symbol	Description
A	coefficient (design) matrix in the Gauss-Markov Model
B	coefficient matrix in the Model of Condition Equations
$\boldsymbol{c}$	right-side vector in the system of normal equations $N\hat{\boldsymbol{\xi}} = \boldsymbol{c}$
$C\{\cdot\}$	covariance operator
$D\{\cdot\}$	dispersion operator
$\operatorname{diag}(\cdot)$	a diagonal matrix with diagonal elements comprised of $(\cdot)$, which represents a list of elements or a column vector.
dim	the dimension of a matrix or of a vector space
$\boldsymbol{e}$	unknown random error vector for the observations
$\tilde{\boldsymbol{e}}$	predicted random error (residual) vector for the observations
$\boldsymbol{e}_0$	unknown random error vector associated with stochastic constraints
$\tilde{\boldsymbol{e}}_0$	predicted random error (residual) vector for $\boldsymbol{e}_0$
$E\{\cdot\}$	expectation operator
H_0	null hypothesis
H_A	alternative hypothesis

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Symbol	Description
K	constraint matrix used in the Gauss-Markov Model with (stochastic) constraints
m	number of unknown parameters
$\text{MSE}\{\cdot\}$	mean squared error operator
n	number of observations
N	normal-equations matrix in the system of normal equations $N\hat{\boldsymbol{\xi}} = \boldsymbol{c}$
$\mathcal{N}(\cdot)$	the nullspace (kernel) of a matrix or the normal distribution, depending on the context
P	weight matrix for the observations
P_0	weight matrix for stochastic constraints
q	rank of the coefficient (design) matrix A
Q	cofactor matrix for the observations
$Q_{\tilde{e}}$	cofactor matrix for the predicted random errors (residuals)
r	redundancy of data model
$\mathbb{R}$	the field of real numbers
$\mathcal{R}(\cdot)$	the range (column) space of a matrix
rk	the rank of a matrix
tr	the trace of a matrix
U	matrix of eigenvectors
$\boldsymbol{w}$	constant vector in the Model of Condition Equations
$\boldsymbol{y}$	vector of observations (possibly in linearized form, resulting in "incremental observations")
$\boldsymbol{z}$	vector of constraints used in the Gauss-Markov Model with stochastic constraints
α	significance level for statistical tests
$\boldsymbol{\alpha}$	observation coefficient vector in the Model of Direct Observations
β	a quantity associated with the power of a statistical test
χ^2	chi-square statistical distribution
δ	a small deviation or non-random error, as in δP denoting a non-random error in matrix P
Φ	Lagrange target function
$\boldsymbol{\eta}$	unit vector used in the Outlier Detection Model
$\boldsymbol{\kappa}_0$	vector of specified constants used in the Gauss-Markov Model with constraints
$\boldsymbol{\lambda}$	unknown vector of Lagrange multipliers
$\hat{\boldsymbol{\lambda}}$	estimated vector of Lagrange multipliers
$\mu, \boldsymbol{\mu}$	the expected value of a non-random variable, either the scalar μ or vector $\boldsymbol{\mu}$
$\hat{\mu}, \hat{\boldsymbol{\mu}}$	the estimate of a non-random variable
$\hat{\boldsymbol{\mu}}_y$	vector of adjusted observations
ν	statistical degrees of freedom

Symbol	Description
θ	the orientation of a confidence ellipse
σ_0^2	unknown variance component
$\hat{\sigma}_0^2$	estimated variance component
Σ	dispersion (or covariance) matrix for the observations
$\boldsymbol{\tau}$	vector of ones (also called "summation vector")
Ω	(weighted) sum of squared residuals (unconstrained case)
$\boldsymbol{\xi}$	vector of unknown parameters (possibly in linearized form, resulting in "incremental parameters")
$\hat{\boldsymbol{\xi}}$	estimated parameter vector
$:=$	a binary operator that denotes that the variable on the left is defined in terms of variables on the right, which have already been introduced

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Table 2: List of abbreviations

Abbrev.	Meaning
BIQUUE	Best Invariant Quadratic Uniformally Unbiased Estimate
BLUUE	Best Linear Uniformly Unbiased Estimate
BLIP	Best LInear Prediction
cdf	cumulative distribution function
GHM	Gauss-Helmert Model
GMM	Gauss-Markov Model
LESS	LEast-Squares Solution
MSE	Mean Squared Error
pdf	probability density function
rms	root mean square

Part I

Adjustment Computations

Chapter 1

Foundations: Observations, Parameters, Random Errors, and Essential Math

1.1 Parameters and Observations, Purpose of Adjustment Computations

In geodetic science, *observations* (measurements) are typically made for the purpose of estimating certain unknown quantities, for example, coordinates of GPS reference stations, or heights of benchmarks. These unknown quantities are often expressed as *parameters of an observational model*. In some cases an unknown parameter might be measured directly (say the length of a bridge), but often parameters are only "measured indirectly," for example by measuring angles and distances to determine coordinates of points. In any case, for our purposes we will consider these unknown quantities to be *fixed parameters*, rather than random parameters, which are treated in Part II. The terms fixed and random parameters refer to the statistical (stochastic) properties of these unknown quantities. Physically, one may think of a fixed parameter as representing a quantity that does not vary in time, or space, at least not over the time span or region of interest.

While some observations might be made with the naked eye, for example by reading the graduations on a steel tape to determine a distance between survey markers, more often they are made by use of an instrument. Traditionally, in surveying most instruments were optical-mechanical, such as a surveyor's level or transiting theodolite ("transit"). These instruments required scales or rods to be read with the aid of telescopes and magnifying eyepieces. Eventually, optical-electronic instruments added electronic distance measuring functionality, while horizontal and vertical angles were still read by optical means. Later, with the advent of the total station, both angles and distances were measured electronically, and perhaps not even recorded manually if a data collector was used in conjunction with the instru-

ment. Nowadays, robotic total stations, digital levels, GPS (or GNSS) receivers, and laser scanners, not to mention drones with GPS receivers, cameras, and LI-DAR, remove most of the traditional elements of human observation. Nevertheless, we still refer to the quantities they measure and record (the data) as observations. *The import thing to know about observations, is that they always contain an element of unknown, random error*, whether they are made and recorded manually by a human, or made and recorded electronically with little or no human involvement.

Errors in observations differ by type (nature). The types we are most concerned with are random, systematic (bias), and blunders (mistakes). Blunders might belong to the statistical category of *outlier*. We will discuss these categories of errors in more detail in a later section. For now we simply assume that 1) all observations contain random errors, 2) that it is often possible to account for systematic errors in some way (if we are aware that they exist and know how to quantify them), and 3) that blunders must be avoided or found and removed.

Let us summarize these concepts with the following brief definitions:

Observation A measured quantity that has a numerical value and unit associated with it. Observations always contain unknown random errors and might also be corrupted by systematic errors (biases) and blunders. Because of its random component, an observation is treated as a (realization of a) random variable. Quoting Koch (1999, p. 82)

> For an experiment, whose result is registered by a real number or by an instrument with a digital display, the mapping of the set of elementary events onto the set of real numbers is achieved by digital recording. The random variable thus defined will be called a *measurement* or an *observation*.

Parameter An unknown quantity of interest that is to be estimated. Unknown parameters are a component of an *observational model*, which models the observations as functions of these unknowns. In Part 1, we treat only fixed parameters, by which we mean they do not vary statistically. Later, in Chapter 17, random parameters (random effects) are introduced.

Purpose of adjustment computations   Let us assume for now that our observations are free of biases and blunders. They still contain unknown random errors. What's more, by design we usually have more observations than the minimum necessary to determine the unknown parameters of our observational model. How then do we deal with these extra observations and their random errors? This is the subject of adjustment computations, the purpose of which is to adjust the observations in some way so that the difference between the given observations and their adjusted values (called *residuals*) is as small as possible according to a stated criterion. One particular method for doing so is the method of *least-squares adjustment*, which is the primary subject of this book. The term "least squares" is due to the requirement that

> *the sum of the squares of the (weighted) residuals is a minimum.*

1.2 Functional Relations and Stochastic Properties

As noted above, observations are typically made to compute the value of some unknown quantities of interest. In order to relate the observations to the unknown quantities, a mathematical function is specified. The function may be linear or nonlinear depending on the complexity of the relation between the observations and the unknowns. In the case where the unknown quantity can be observed directly, a simple linear function might be suitable. In other cases, the chosen function may be highly nonlinear.

As an example, suppose a distance is measured between points p_1 and p_2 whose coordinates in the horizontal plane must be determined. In this case, the measured distance, call it y, is the observation. The unknown quantities are the coordinate pairs of the two points, viz. (x_1, y_1) and (x_2, y_2). The functional relationship between the measured distance and unknown coordinates can be written as

$$y(x_1, y_1, x_2, y_2) \approx \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1.1a}$$

Obviously, the function is nonlinear in the unknown variables x_1, y_1, x_2, and y_2. Note that the observation variable y is the dependent variable; it depends on the unknown coordinates, which are the independent variables of the function. The approximately-equals sign is used because the observation contains random error, and thus the unknown quantities do not fully explain the observation variable. Recall that the unknown quantities are considered to be nonrandom ("fixed").

In order to change the approximately-equals sign to an equals sign, an additional term must be added to the function so that both sides of the equation have a random term (or, equivalently, the random term could be subtracted from the left side). The random term, call it e, accounts for the *unknown random error* in the observation. By introducing e, (1.1a) is then modified to read

$$y = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + e, \tag{1.1b}$$

where the function arguments on the left side are dropped for simplicity.

Equation (1.1b) is in the form of an (nonlinear) *observation equation*, which is what we call an equation that expresses an observation as a random variable that depends on unknown quantities that must be determined. Thus, we say that (1.1b) *models the observation* as a function of unknown variables; we call these unknowns *parameters* of the model. We want to determine (solve for) these parameters in some optimal way. As we will see later, the determination of the values of the parameters cannot be made with absolute certainty. Thus we use the statistical term *estimation* when we speak of determining numerical values for the parameters.

So, we may refer to (1.1b) as an observational model; however, it is not a complete model, because more needs be said about the stochastic nature of the random error e in order to exploit its random properties when estimating the unknown parameters. In the following sections we will discuss in some detail how the stochastic properties of random errors can be characterized. At this stage it is enough to say,

for the sake of least-squares adjustments, that only the *expectation* and the *variance* of the errors need to be specified in the model. Expectation is a statistical term that denotes the value we expect a random variable to take on, at least in an average sense. And in this context, variance is a statistical term that denotes our uncertainty about the expected value of a random variable, i.e., it puts statistical bounds around the expected value. Formally, variance is described as a measure of dispersion.

Unless otherwise noted, we will always specify the expectation of the random errors to be zero. A succinct way to express the expectation and variance of the random error e is $e \sim (0, \sigma^2)$, which reads, "e is distributed with zero expectation and sigma-squared variance." Thus, the observational model (1.1b) is made complete by extending it to

$$y = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + e, \quad e \sim (0, \sigma^2). \qquad (1.1c)$$

The observation equation is sometimes referred to as the functional part of the model (or functional model), while the statement $e \sim (0, \sigma^2)$ is sometimes referred to the stochastic part of the model (or stochastic model). We call the inverse of the variance the *weight* of the observation (weights are discussed in more detail in a later section). Note that the unit of measurement of the random error e is the same as that of the observation y, and the unit of measurement of the variance σ^2 is the square of the observation's unit.

The observational model (1.1c) is relevant to a particular problem, that is, to a particular type of observation (an observed distance) and to particular parameters (coordinates of two points). We would rather generalize it for use in a wide variety of geodetic adjustment problems. For that we collect the unknown parameters in a vector $\boldsymbol{\xi}$, the symbol used to denote a vector of m unknown parameters throughout this book. Furthermore, (1.1c) contains only one observation; it must be generalized to handle any number of observations, possibly all with their own unique variances. For this we make use of matrices and vectors.

Suppose rather than a single observation y we are given an $n \times 1$ vector of observations $\boldsymbol{y} = [y_1, \ldots, y_n]^T$, which has an associated, unknown vector of random errors $\boldsymbol{e} = [e_1, \ldots, e_n]^T$. Our general model should allow each of the random errors to have its own unique variance, and it should allow for covariances between the random errors (covariances are defined in Section 1.5.2). Thus an $n \times n$ *cofactor matrix* Q is introduced, with its inverse $P := Q^{-1}$ called weight matrix. When Q is multiplied by an unknown scalar σ_0^2 called *variance component*, the result is called *covariance matrix*, which is denoted by the symbol Σ, i.e., $\Sigma := \sigma_0^2 Q = \sigma_0^2 P^{-1}$. Note that some authors call the variance matrix "variance-covariance matrix," and some authors call the variance component the "variance of unit weight." Putting these components together results in the following model:

$$\underset{n \times 1}{\boldsymbol{y}} = \underset{\mathbb{R}^m \to \mathbb{R}^n}{\boldsymbol{f}(\boldsymbol{\xi})} + \underset{n \times 1}{\boldsymbol{e}}, \quad \boldsymbol{e} \sim (\underset{n \times 1}{\boldsymbol{0}}, \sigma_0^2 \underset{n \times n}{P^{-1}}). \qquad (1.2a)$$

Note that the vector of functions $\boldsymbol{f}$ maps $\mathbb{R}^m$ into $\mathbb{R}^n$, denoted mathematically by $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^n$.

Now, if the vector of functions $\boldsymbol{f}$ is nonlinear in the unknown parameters $\boldsymbol{\xi}$, it can be linearized by a truncated Taylor series expansion (see Appendix B). Whether we have a linear form $\boldsymbol{f}(\boldsymbol{\xi})$ or a linearized form, we can represent it by an $n \times m$ *coefficient matrix* A, so that the model (1.2a) can be restated as

$$\underset{n \times 1}{\boldsymbol{y}} = \underset{n \times m}{A} \underset{}{\boldsymbol{\xi}} + \underset{n \times 1}{\boldsymbol{e}}, \quad \boldsymbol{e} \sim (\underset{n \times 1}{\boldsymbol{0}}, \sigma_0^2 \underset{n \times n}{P^{-1}}). \tag{1.2b}$$

The development of the model (1.2b) is an important step in understanding the relations between observations, parameters, and random errors. The model is of type *Gauss-Markov*, which is an important model in geodetic science and one that is used extensively in Chapter 3, with particular extensions of it as the focus of Chapters 5 and 6. More details about random errors, covariances, and weights will follow, and the usefulness of model (1.2b) will become more apparent in later chapters. For now, we summarize with a brief description of each element of the model.

$\boldsymbol{y}$ is a *given* $n \times 1$ vector of (incremental) observations.

A is a *given* $n \times m$ coefficient (or design) matrix that has full column rank, i.e, $\operatorname{rk} A = m$.

$\boldsymbol{\xi}$ is an $m \times 1$ vector of *unknown* (incremental) parameters.

$\boldsymbol{e}$ is an $n \times 1$ vector of *unknown* random errors associated with the observations.

σ_0^2 is an *unknown* variance component (scalar quantity). Note that σ_0^2 is unitless.

P is an $n \times n$ weight matrix such that $P^{-1} := Q$ for a *given* cofactor matrix Q, and where the covariance matrix Σ is defined as $\Sigma := \sigma_0^2 P^{-1}$. Note that the diagonal elements of Q have units that are the square of the units of their associated observations.

1.3   Fundamentals of Matrix Algebra

Matrix algebra (or linear algebra) is fundamental to the mathematics of adjustment computations, and it is used extensively in this book. Most of the concepts in matrix algebra used here are covered in a first course in linear algebra at the college or university level. Beyond that, there are many derivations in the chapters that follow that make use of certain matrix relations and identities involving inverses of sums and products of matrices that generally do not appear in a first course on linear algebra. These relations are helpful both for reducing complicated formulas to simpler forms and for showing alternative, but equivalent, solutions to the same problem. (Seeing more than one solution to a problem may help to provide greater insight into it, and we will find that sometimes one formula may be more or less efficient than another equivalent one depending on the problem at hand.)

A list of matrix relations and identities used in this book is provided in Appendix A. The ones involving only a single line should be memorized. While it's

not necessary to memorize the multi-line formulas to read these notes well, being able to recognize them or readily refer to them will make some of the derivations in later chapters easier to follow. To facilitate reading of the derivations, their equation numbers are usually referred to when they are used.

1.3.1 Important Concepts

Below is a list of concepts from linear algebra that the reader should be familiar with. Some are described briefly in the paragraphs that follow. These books are good sources for more complete descriptions: Strang (2006); Strang and Borre (1997).

- Gaussian elimination and back substitution

- Gauss-Jordan elimination

- The column space of a matrix

- The nullspace of a matrix

- The basis and dimension of a vector space

- The rank of a matrix

- Consistent and inconsistent systems of equations

- Eigenvalues and eigenvectors

- The properties of an invertible matrix

- The terms positive definite and positive semidefinite

- The term idempotent

- Choleskey's decomposition

- All other items in Appendix A

Vector spaces   The space $\mathbb{R}^n$ consists of all vectors with n real-valued components. Two important vector spaces in adjustment computations are the column space and the nullspace of a matrix.

A basis of a vector space   *A basis for a vector space is a sequence of vectors that are linearly independent and that span the space.* A vector space may have many different bases, but given a basis, every vector in the space can be expressed as a unique linear combination of the basis vectors. All bases for a vector space contain the same number of vectors. This number is the dimension of the space. The columns of an invertible $m \times m$ matrix provide a basis for $\mathbb{R}^m$.

Column space   *The column space of a matrix A consists of all linear combinations of its columns.* It is denoted by $\mathcal{R}(A)$ and is also called the range of A. Its dimension equals the rank of A, which is also the number of linearly independent columns in the space. We say that the columns of A *span* the column space of A.

   Note that the column space of the matrix product AB is contained in the column space of A, denoted mathematically by

$$\mathcal{R}(AB) \subset \mathcal{R}(A). \tag{1.3}$$

In words, it means that every column of the matrix product AB is a linear combination of the columns of A.

Nullspace   *The nullspace of A consists of all solutions to $A\boldsymbol{x} = \boldsymbol{0}$.* It is denoted by $\mathcal{N}(A)$ and is also called the kernel of A. The dimension of the nullspace of A is the number of nonzero vectors in the space. It can be computed by

$$\dim \mathcal{N}(A) = m - \mathrm{rk}\, A, \ \text{ if } A \text{ has } m \text{ columns.} \tag{1.4}$$

This dimension is also called the nullity. If A is a square, nonsingular matrix, the only vector in its nullspace is $\boldsymbol{x} = \boldsymbol{0}$, and thus the dimension of its nullspace is zero.

   The relationship between the dimensions of the column space and nullspace is given by

$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = \dim \mathbb{R}^m = m, \ \text{ if the size of } A \text{ is } n \times m. \tag{1.5}$$

The rank of a matrix   The rank of a matrix A is the number of its independent rows, which is also the number of its independent columns.

Consistent and inconsistent systems of equations   *A consistent system of equations is one that is solvable.* The equation $A\boldsymbol{x} = \boldsymbol{b}$ is only consistent if $\boldsymbol{b}$ is in the column space of A. For example, the equation in (1.2b) would not be consistent if the random error vector $\boldsymbol{e}$ were removed from it. That is because the observation vector $\boldsymbol{y}$ is not in the column space of the coefficient matrix A, due to the random errors inherent in the observations.

Properties of an invertible matrix   *A matrix A is invertible if there exists a matrix A^{-1} such that*

$$A^{-1}A = I \ \text{ and } \ AA^{-1} = I. \tag{1.6}$$

   Only square matrices are possibly invertible. If matrix A is invertible:

- It is nonsingular (regular).

- Its inverse is unique.

- Its rank is equal to its dimension (size or order), i.e., $\mathrm{rk}\, A = m$ if the size of A is $m \times m$.

- Its rank is equal to the dimension of its column space, i.e., $\operatorname{rk} A = \dim \mathcal{R}(A)$.

- The vector $\boldsymbol{x} = \boldsymbol{0}$ is the only vector in its nullspace. Therefore, $\dim \mathcal{N}(A) = 0$.

- All its eigenvalues are nonzero.

Positive-definite and positive-semidefinite matrices:

Positive definite   *A matrix A is positive definite if $\boldsymbol{x}^T A \boldsymbol{x} > 0$ for all nonzero vectors $\boldsymbol{x}$.* A positive-definite matrix is nonsingular. All of its eigenvalues are greater than zero. If the matrix is also symmetric, it can be factored by the Cholesky decomposition. See page 28 for properties of a positive-definite matrix.

Positive semidefinite   *A matrix A is positive semidefinite if $\boldsymbol{x}^T A \boldsymbol{x} \geq 0$ for all nonzero vectors $\boldsymbol{x}$.* A positive-semidefinite matrix is singular. At least one of its eigenvalues is zero; none are less than zero; thus, a positive-semidefinite matrix is also called a non-negative definite (NND) matrix.

Idempotent matrices   *An idempotent matrix equals its own square.* It is a square matrix, and it is singular unless it is the identity matrix.

$$\text{The } n \times n \text{ matrix } P \text{ is idempotent if } PP = P. \tag{1.7a}$$

$$\text{If the } n \times n \text{ matrix } P \text{ is idempotent, then so is } I_n - P. \tag{1.7b}$$

$$\text{If } P \text{ is idempotent, } \operatorname{tr} P = \operatorname{rk} P. \tag{1.7c}$$

$$\text{The eigenvalues of an idempotent matrix are 0 or 1.} \tag{1.7d}$$

$$\text{Projection matrices are idempotent.} \tag{1.7e}$$

1.3.2   Practice Problems

The reader should know how to solve the following problems:

1. Solve the following system of equations, for the variables $x_1, \ldots, x_6$, by Gaussian elimination and back substitution:

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \quad\quad + 2x_5 \quad\quad\quad &= \quad 0 \\
2x_1 + 6x_2 - 5x_3 - \ 2x_4 + 4x_5 - \ 3x_6 &= -1 \\
5x_3 + 10x_4 \quad\quad + 15x_6 &= \quad 5 \\
2x_1 + 6x_2 \quad\quad + \ 8x_4 + 4x_5 + 18x_6 &= \quad 6
\end{aligned}
$$

2. Solve the preceding system of equations by Gauss-Jordan elimination.

3. Find a basis for the column space of

$$
A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}.
$$

4. Find a basis for the row space of

$$B = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}.$$

5. Find a basis for the nullspace of matrix A^T and a basis for the nullspace of matrix B above. Confirm that the basis vectors in these nullspaces are orthogonal to the column space of A and the row space of B, respectively (see (A.45a)–(A.45d)).

6. What are the ranks of matrices A and B above.

7. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

For each matrix, check that the sum of its eigenvalues equals its trace and that the product of its eigenvalues equals its determinant.

8. Compute the Cholesky factor of

$$N = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{bmatrix}$$

and then compute the inverse of N using the Cholesky factor.

9. Assuming the partitioned matrix

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

is nonsingular and that its sub-matrices N_{11} and N_{22} are also nonsingular, without referring to Appendix A, derive its inverse using elementary row operations.

10. If N is an $m \times m$ nonsingular matrix and K is an $l \times m$ matrix with $l < m$ such that $\mathrm{rk}[N \,|\, K^T] = m$, then the matrix

$$\begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}$$

is nonsingular and so is the matrix $(N + K^T K)$. Note that 0 denotes an $l \times l$ matrix of zeros. Without reference to Chapter 5, derive its inverse using elementary row operations. Hint: start by multiplying the bottom row on the left by K^T and add the result to the top row.

11. With reference to Appendix A, derive (A.6b) from (A.6a).

12. If N is a nonsingular matrix defined by $N = A^T P A$, show that the quantity $I - A N^{-1} A^T P$ is idempotent (see (1.7a)), where I is the identity matrix.

13. If the matrix P is idempotent, show that $I - P$ is too.

14. Can the dimension of the nullspace of a rectangular matrix ever be zero? Why or why not?

1.4   Random Variables

From here to the beginning of Section 1.5 we use notation consistent with textbooks in statistics for easy comparison to them. Accordingly, we use X to denote a random variable and x to denote a numerical value that the random variable could take on. After these sections, we resume the use of notation long used by Burkhard Schaffrin for adjustment computations.

1.4.1   Review From Statistics

According to Mikhail and Ackermann (1982), probabilities are associated with statistical events, which are the outcomes of statistical experiments. If an event has several possible outcomes, we associate with it a *stochastic or random variable X*, which can take on different numerical values x for different outcomes. The total of all possible outcomes of a statistical event associated with a random variable is called the *population*. Because of its large size, it is not practical, or even possible, to evaluate all the elements of a population. For this reason, we only select a small number of them (by making observations), the set of which is called a *sample* of the population.

   Let's associate these abstract statistical terms with a concrete example from geodetic science. Suppose the coordinates of a geodetic network are to be determined from data collected by GPS receivers. The act of collecting and processing those data is the experiment. The outcome is a set of coordinate differences between points in the network, which we take to be observations in this example. These coordinate differences could take on different values (i.e., no two experiments are likely to produce the same set of values). Therefore, each observed coordinate difference is considered to be a realization of a random variable. Obviously, we cannot obtain the entire population of observed coordinate differences among the network points, because there are an infinite number of them. Rather we must settle for a finite number of observations obtained from the experiment, which constitutes a sample.

Quoting Mikhail and Ackermann, "the total set of possible values of a random variable, X, together with their probabilities, constitute what is termed a *probability distribution* associated with the random variable." A probability distribution involves a function that assigns a probability to all possible values of the random variable it is associated with. The two types of probability distribution functions are *cumulative distribution function* (or simply distribution function) and *probability density function* (or density function or probability function). These two distribution functions are defined in the following two sections for a single random variable (univariate distribution).

In general, the properties of cumulative distribution functions hold for both continuous and discrete random variables. However, probability density functions pertain only to continuous functions. Their discrete analog is the probability mass function. In the following, we will limit our discussion to continuous random variables. An important property of a continuous random variable is that the probability that it will take a particular value is zero. That is, if X is a continuous random variable, then

$$P\{X = x\} = 0 \ \text{ for all } x. \tag{1.8}$$

1.4.1.1 Cumulative Distribution Function

The cumulative distribution function, $F(x)$[1], gives the probability of the event $\{X \le x\}$ for every number x. It is written as

$$F(x) = P\{X \le x\} = P\{-\infty < X \le x\}. \tag{1.9}$$

In words, (1.9) says that the probability that the random variable X will take on a numerical value less than or equal to x is given by the function $F(x)$. By definition, the probabilities are limited to values between 0 and 1, i.e,

$$0 \le P \le 1, \ \text{ implying that } \ \lim_{x \to -\infty} F(x) = 0 \ \text{ and } \ \lim_{x \to \infty} F(x) = 1. \tag{1.10}$$

Finally,

$$P\{x < X\} = 1 - F(x) \ \text{ for all } x. \tag{1.11}$$

1.4.1.2 Probability Density Function

A probability density function $f(x)$ of a continuous random variable X provides a means to calculate the probability that X lies within a specified *interval*. It does not, however, give the probability that X equals a specific value x, because $P\{X = x\} = 0$ for all numbers x as stated in (1.8). The probability that X belongs to an interval $[a, b]$ is given by the integral

$$P\{a \le X \le b\} = \int_a^b f(x)\, dx, \tag{1.12}$$

---

[1]Here we omit the subscript X, as in $F_X(x)$ and $f_X(x)$, which is often used in statistics textbooks.

which is the area under the curve $f(x)$ between a and b as shown in Figure 1.1.

If the lower limit a is replaced by $-\infty$, then a relationship between the cumulative distribution function $F(x)$ and the probability density function $f(x)$ can be written as

$$F(x) = \int_{-\infty}^{x} f(t)\, dt, \tag{1.13}$$

which, considering the Fundamental Theorem of Calculus, leads to the relation

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_{-\infty}^{x} f(t)\, dt = f(x), \tag{1.14}$$

for all values of x. It is important to note that a probability density function must satisfy the following two properties:

$$f(x) \geq 0 \ \text{ for all numbers } x. \tag{1.15a}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1. \tag{1.15b}$$

Any integrable function that satisfies these two properties is the probability density function of some random variable X.

It is also noted that, as a consequence of (1.8), $P\{X = a\} = 0$ and $P\{X = b\} = 0$, and therefore

$$P\{a \leq X \leq b\} = P\{a < X \leq b\} = P\{a \leq X < b\} = P\{a < X < b\}. \tag{1.16}$$
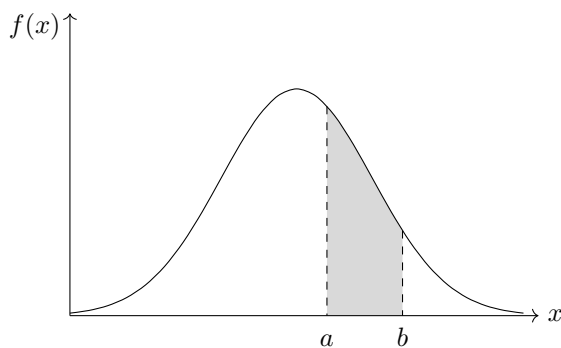
Figure 1.1: The shaded area under the curve of the probability density function $f(x)$ is the probability that a random variable takes on values in the interval $[a, b]$

1.4.2   Distributions for Adjustment Computations

The four distributions discussed in this book are

1. The Gaussian or normal distribution.

2. The t (Student's t) distribution.

3. The χ^2 distribution.

4. The F (Fisher) distribution.

For our purposes, these distributions are primarily used for hypothesis testing to validate statistically the results of various adjustment computations. Standard texts in statistics can be consulted for obtaining critical values of the distributions from tables, some of which can also be found in the appendices herein. More details about these distributions can be found in Chapter 9. Here, we briefly describe the normal and standard normal distributions.

1.4.2.1   The Normal Distribution

The probability density function $f(x)$ of the normal distribution is defined by the equation

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty. \tag{1.17}$$

The parameters of the function are the mean μ and the variance σ^2 ($\sigma^2 > 0$). The graph of $f(x)$ is a bell-shaped curve that is symmetric about μ and that extends over the entire horizontal axis. The shorthand notation for indicating that a random variable X has a normal distribution with mean μ and variance σ^2 is

$$X \sim \mathcal{N}(\mu, \sigma^2). \tag{1.18}$$

Because $f(x)$ is symmetric about μ and reaches its maximum at $x = \mu$, the mean of the normal distribution is equal to its median and mode.

1.4.2.2   The Standard Normal Distribution

To avoid having to generate statistical tables for many values of μ and σ, the random variable X is transformed to a standardized form by the equation

$$Z = \frac{X - \mu}{\sigma}. \tag{1.19}$$

The resulting *standardized random variable*[2] Z has mean $\mu_Z = 0$ and variance $\sigma_Z^2 = 1$. It indicates the distance of X from its mean μ in units of its standard deviation σ, as shown in Figure 1.2. Since $Z \sim \mathcal{N}(0, 1)$, its probability density function $f(z)$ is defined by the equation

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}. \tag{1.20}$$

---

[2]Snedecor and Cochran (1980, p. 40) state that "The quantity Z goes by various names— *standard normal variate; standard normal deviate; normal variate in standard measure...*" Zar (1996, p. 73) states that "carrying out the calculation of Equation [(1.19)] is known as *normalizing*, or *standardizing* [X] ..."

In summary, we state that

> If the random variable X has a normal distribution with mean μ and variance σ^2, the standardization $Z = (X - \mu)/\sigma$ of X has the standard normal distribution; i.e., $Z \sim \mathcal{N}(0, 1)$.
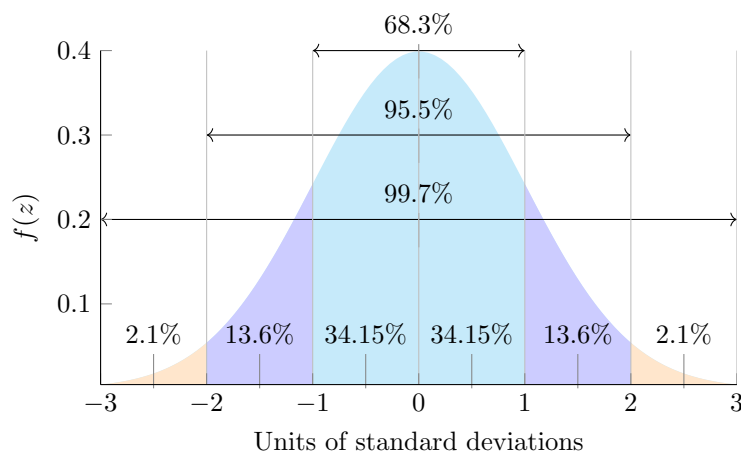
Figure 1.2: Normal distribution curve (shown only over the domain $[-3, 3]$), with percent of areas under curve denoting probabilities. Image derived from TikZ code by John Canning, Senior Lecturer at the University of Brighton (http://johncanning.net/wp/?p=1202).

The reason that the curve in Figure 1.2 appears to peak near 0.4 is because $f(z = 0) = 1/\sqrt{2\pi} \approx 0.4$. The probabilities shown in the figure (as percentages) are due to the probability statements

$$P(-1 < z < 1) = P(\mu - \ \sigma < x < \mu + \ \sigma) = 0.683, \tag{1.21a}$$
$$P(-2 < z < 2) = P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.955, \tag{1.21b}$$
$$P(-3 < z < 3) = P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.997. \tag{1.21c}$$

The intervals associated with these probability statements are commonly referred to as the "1-sigma," "2-sigma," and "3-sigma" confidence intervals, respectively. Other commonly used intervals are the so-called 50%, 90%, 95%, and 99% confidence intervals. Their respective probability statements are given by

$$0.5 = P(-0.674 < z < 0.674), \tag{1.22a}$$
$$0.9 = P(-1.645 < z < 1.645), \tag{1.22b}$$
$$0.95 = P(-1.960 < z < 1.960), \tag{1.22c}$$
$$0.99 = P(-2.576 < z < 2.576). \tag{1.22d}$$

The probabilities associated with these statements can be obtained from Table C.1. For example, (1.21a) is obtained by subtracting $F(-1) = 0.1587$ from $F(1) = 0.8413$, which results in 0.6827. They can also be obtained in MATLAB with the expression `normcdf(1)-normcdf(-1)`.

For statements (1.22a)–(1.22d), the probabilities shown on the left sides must be found within columns 2–11 of the table, and then the corresponding z-values can be read (interpolated) from the first column and the heading. Because the limits are centered around z, but the table lists $P[Z \leq z]$, one should determine the value to find in the table as follows: if P denotes the probability, the value $1 - (1 - P)/2 = (1 + P)/2$ is the value to find in the table to obtain the upper limit of z. For the lower limit of z, which only differs in sign from the upper one, use $(1 - P)/2$. For example, for (1.22a), find $(1 + 0.5)/2 = 0.75$ and $(1 - 0.5)/2 = 0.25$ in the table. These limits can also be found by using the MATLAB function `norminv`. For example `norminv(0.25)` returns -0.6745, and `norminv(0.75)` returns 0.6745.

Further discussions about the standard normal distribution can be found in Section 9.3.1.

1.5 Random Variables in Adjustment Computations

In the following, we present some properties of random variables, which are also called stochastic variables by some authors (e.g., Bjerhammar, 1973). More particularly, we focus on variables that represent random observation errors. Such errors also have been called accidental errors *ibid*, pg. 5. Though we cannot know what values random errors will take on, we may state what we expect their values to be, and we may also specify their level of deviation or variance about their expected values.

In the following sections, the notions of expectation and dispersion are defined mathematically. We first start with the univariate case, where only one random error is considered. Then we proceed to the multivariate case, where a vector of n random errors is considered.

1.5.1 Univariate Case

The univariate case deals with one-dimensional random variables, i.e., it treats scalar quantities rather than vector quantities. Let us introduce the continuous random variable e with a given probability density function (pdf) $f(e_t)$, where e_t represents a realization of e, i.e., a possible value that e might take on.

Expectation The probabilistic mean of e is the value that we *expect* e to take on. We denote the *expectation* of e as μ_e and define it as follows:

$$\mu_e := E\{e\} = \int_{-\infty}^{\infty} e_t \, f(e_t) \, de_t, \tag{1.23}$$

where E is called expectation operator. Equation (1.23) is also called the first moment of e. If the random variable e represents measurement error, then, ideally, $E\{e\} = 0$. If $E\{e\} \neq 0$, we say that the measurement error is biased.

Dispersion   The *dispersion*, or *variance*, of e is denoted by σ_e^2 and is defined by

$$\sigma_e^2 := E\{(e - E\{e\})^2\} = \int\limits_{-\infty}^{\infty} (e_t - \mu_e)^2 f(e_t)\, de_t\,. \tag{1.24a}$$

If $E\{e\} = 0$, then obviously

$$\sigma_e^2 = \int\limits_{-\infty}^{\infty} e_t^2 f(e_t)\, de_t\,. \tag{1.24b}$$

Equation (1.24a) is also called the second centralized moment of e. In addition to σ_e^2, $D\{e\}$ is also used to denote the dispersion (variance) of e, where D is the dispersion operator, but usually we reserve this notation for the multivariate case, where it refers to an $n \times n$ covariance matrix. The terms dispersion and variance are used interchangeably throughout this book, though formally variance is described as a measure of dispersion. The positive square root of variance is called *standard deviation*.

Variance is an indicator of how closely the values taken on by a random variable are to its expected value. It is reflective of measurement precision and is inversely proportional to it. Thus, a small variance indicates high precision, and a large variance indicates low precision. A succinct expression for the expectation and variance of the random variable e, when e is assumed to be unbiased random measurement error, is

$$e \sim (0, \sigma_e^2). \tag{1.25}$$

The expression (1.25) is said in words as "*e is distributed with zero mean and sigma-sub-e-squared variance.*" Note that (1.25) does not specify a pdf for e but only its expectation and dispersion (or variance), or its first and second moments.

1.5.1.1   Expectation and Variance Propagation

Consider the *observation equation*
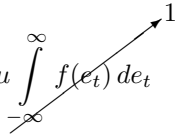
$$y = \mu + e, \quad e \sim (0, \sigma_e^2), \tag{1.26}$$

where y is an observation (measurement), μ is an unknown *observable*, and e accounts for the random error inherent in the observation y. We want to find the expectation and variance of y. In other words, we want to know how the expectation and variance propagate from the random variable e to the random variable y. Note that μ is a constant, or *non-random*, variable. The expectation of a constant is the constant itself; i.e., $E\{\mu\} = \mu$.

Using (1.23), we can write the expectation of $y = \mu + e$ as

$$E\{y\} = \int_{-\infty}^{\infty} (\mu + e_t)\, f(e_t)\, de_t, \tag{1.27a}$$

where e_t was defined in the preceding section as a value that the random variable e can take on. The expectation operator is linear. Thus, the expectation of the sum of random variables is the sum of their individual expectations. And, as noted already, μ is a constant variable. Therefore

$$E\{y\} = \mu \int_{-\infty}^{\infty} f(e_t)\, de_t \quad + \int_{-\infty}^{\infty} e_t\, f(e_t)\, de_t = \mu + E\{e\} = \mu + 0 = \mu. \tag{1.27b}$$

The first integral evaluates to one according to (1.15b); the second integral was defined as expectation in (1.23).

The following rules are useful when working with expectations, given random variables x and y and constant c:

$$E\{E\{x\}\} = E\{x\}, \tag{1.28a}$$

$$E\{x + y\} = E\{x\} + E\{y\}, \tag{1.28b}$$

$$E\{c\} = c, \tag{1.28c}$$

$$E\{cx\} = c \cdot E\{x\}, \tag{1.28d}$$

$$E\{x \cdot y\} = E\{x\} \cdot E\{y\} \tag{1.28e}$$

if and only if x and y are independent random variables,

$$E\{x^2\} \neq E\{x\}^2 \ \text{ in general.} \tag{1.28f}$$

These rules can be extended to the multivariate case by replacing random variables x and y with random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, and by replacing the constant c with a constant matrix A.

After introducing y_t as a variable of integration, as was done for e_t above, the dispersion (variance) of y is defined by

$$D\{y\} = \int_{-\infty}^{\infty} (y_t - E\{y\})^2 f(y_t)\, dy_t =$$

$$= \int_{-\infty}^{\infty} (\mu + e_t - \mu)^2 f(e_t)\, de_t = \int_{-\infty}^{\infty} e_t^2 f(e_t)\, de_t = \sigma_e^2. \tag{1.29}$$

Summarizing, the first and second moments (i.e., the mean and variance) of y can be written succinctly as $y \sim (\mu, \sigma_e^2)$.

Another useful formula for the dispersion of a random variable y expresses it as the difference of the expectation of the square of the variable and the square of the

variable's expected value. It is derived as follows:

$$D\{y\} = E\{(y - E\{y\})^2\} = \tag{1.30a}$$
$$= E\{y^2 - 2yE\{y\} + E\{y\}^2\} =$$
$$= E\{y^2 - 2y\mu + \mu^2\} =$$
$$= E\{y^2\} - 2\mu E\{y\} + E\{\mu^2\} =$$
$$= E\{y^2\} - 2\mu^2 + \mu^2 =$$
$$= E\{y^2\} - \mu^2 \Rightarrow$$

$$\boxed{D\{y\} = E\{y^2\} - E\{y\}^2 = \sigma_y^2.} \tag{1.30b}$$

Given constants α and γ, and in light of the above formulas, properties for expectation and dispersion can be summarized as follows:

$$\boxed{\begin{aligned} E\{\alpha y + \gamma\} &= \alpha E\{y\} + \gamma, \\ D\{\alpha y + \gamma\} &= \alpha^2 D\{y\}. \end{aligned}}$$

$$\tag{1.31a}$$
$$\tag{1.31b}$$

Equation (1.31b) represents *the law of error propagation* (covariance propagation) in its simplest form. It shows that, in contrast to the expectation, the dispersion operator is not linear. Furthermore, it shows that dispersion is not affected by a constant offset.

1.5.1.2 Mean Squared Error

The mean squared error, or MSE, of y is the expectation of the square of the difference of y and its *true value* μ. It is defined mathematically as

$$\text{MSE}\{y\} = E\{(y - \mu)^2\} \tag{1.32}$$

(compare to (1.30a)). It is useful to express the MSE as a combination of the dispersion and a (squared) *bias* term. This is done via the following derivation:

$$\text{MSE}\{y\} = E\{(y - \mu)^2\} = E\{[(y - E\{y\}) - (\mu - E\{y\})]^2\} =$$
$$= E\{(y - E\{y\})^2 - 2(y - E\{y\})(\mu - E\{y\}) + (\mu - E\{y\})^2\} =$$
$$= E\{(y - E\{y\})^2\} - 2E\{(y - E\{y\})(\mu - E\{y\})\} + E\{(\mu - E\{y\})^2\}. \tag{1.33}$$

Note that while y is a random variable, $E\{y\}$ is not. So, in the middle term, the expectation operator only applies to y. Therefore, we may continue with

$$\text{MSE}\{y\} = D\{y\} - 2(\underbrace{E\{y\} - E\{y\}}_{0})(\mu - E\{y\}) + (\mu - E\{y\})^2 \Rightarrow$$
$$\text{MSE}\{y\} = D\{y\} + \beta^2, \tag{1.34}$$

where bias is defined formally as

$$\beta := E\{y - \mu\} = E\{y\} - \mu. \tag{1.35}$$

Thus, we see that the dispersion of y and the MSE of y are only equal in the absence of bias, or in other words, only if indeed $\mu = E\{y\}$.

We noted previously that dispersion (variance) is an indicator of precision. In contrast, MSE is a measure of accuracy; it includes both dispersion and bias terms. In general, it is harder to meet accuracy standards than precision standards. We can typically increase precision by making more observations (though this may come with additional costs in time and resources); however it might not be possible to reduce bias by making more observations, and it may be very difficult to determine the origin of bias.

Finally, we note that the square root of MSE is commonly called rms (root mean square). Thus, strictly speaking, standard deviation and rms are only equivalent in the absence of bias.

The reader may also enjoy reading the article *Useful Statistics for Land Surveyors* by Urho Uotila (2006), who was Burkhard Schaffrin's predecessor as professor of adjustment computations at The Ohio State University.

1.5.2 Multivariate Case

The multivariate case deals with multidimensional random variables represented by column vectors. For example, multiple observations of the observable μ in (1.26) can be expressed in the following system of equations:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{\tau}\mu + \boldsymbol{e} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \mu + \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}, \tag{1.36}$$

where $\boldsymbol{\tau}$ is a "summation vector"[3] defined as $\boldsymbol{\tau} := [1, \ldots, 1]^T$. In the case of unbiased observations, i.e. $E\{\boldsymbol{e}\} = \boldsymbol{0}$, the expectation of the random error vector $\boldsymbol{e}$ is written as

$$E\{\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}\} = \begin{bmatrix} E\{e_1\} \\ \vdots \\ E\{e_n\} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{1.37}$$

showing that the expectation of a vector can be written component-wise. Likewise, for the dispersion of each element e_j of $\boldsymbol{e}$, we have

$$D\{e_j\} = E\{\big(e_j - \underbrace{E\{e_j\}}_{0}\big)^2\} = E\{e_j^2\}. \tag{1.38}$$

---

[3]The phrase "summing vector" has also been used. The name comes from the fact that if the dot product is taken between a vector of ones and another vector, the result is the sum of the elements of the other vector.

For the multivariate case, we must introduce the concept of *covariance*, which is a measure of similar behavior between random variables, e.g., between elements e_j and e_k of $\boldsymbol{e}$. Formally, the definition of covariance is

$$C\{e_j, e_k\} = \sigma_{jk} := E\{(e_j - E\{e_j\})(e_k - E\{e_k\})\}. \tag{1.39}$$

Obviously,

$$C\{e_j, e_k\} = C\{e_k, e_j\}. \tag{1.40}$$

Moreover, when $E\{\boldsymbol{e}\} = \boldsymbol{0}$, the covariance between two of its elements reduces to

$$C\{e_j, e_k\} = E\{e_j e_k\}, \tag{1.41}$$

since $E\{e_j\} = E\{e_k\} = 0$. Even though we see from the definition of the covariance (1.39) that it does not depend on bias, in practice we often find that bias appears as positive correlation (see (1.51) for the definition of correlation coefficient).

Two random variables are said to be independent if their joint probability distribution is equal to the product of their individual probability distributions. Mathematically, this is written as

$$f\{e_j, e_k\} = f(e_j) \cdot f(e_k) \Leftrightarrow e_j \text{ and } e_k \text{ are independent.} \tag{1.42}$$

If two random variables are independent, their covariance is zero. The converse is not true unless the random variables are jointly normally distributed.

In light of the concept of covariance, the dispersion of a vector of random variables is represented by a matrix. The jth diagonal element of the matrix is denoted by σ_j^2 (or σ_{jj}^2) and the j, k off-diagonal term is written as σ_{jk}. The matrix is called a *covariance matrix* and is represented by Σ. Due to (1.40), the covariance matrix is symmetric. An explicit representation of the covariance matrix Σ is given by

$$D\{\underset{n\times1}{\boldsymbol{e}}\} = \begin{bmatrix} D\{e_1\} & C\{e_1, e_2\} & \dots & C\{e_1, e_n\} \\ C\{e_2, e_1\} & D\{e_2\} & \dots & C\{e_2, e_n\} \\ \vdots & \vdots & \ddots & \vdots \\ C\{e_n, e_1\} & C\{e_n, e_2\} & \dots & D\{e_n\} \end{bmatrix} =: \underset{n\times n}{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{bmatrix}. \tag{1.43}$$

Obviously, if the random variables are uncorrelated, the covariance matrix is diagonal.

An important property of a covariance matrix is that it must be at least *positive semidefinite* (or, equivalently, non-negative definite (Searle and Khuri, 2017, p. 202)). A matrix is positive semidefinite if, and only if, all of its eigenvalues are non-negative. For many applications in geodetic science, the covariance matrix is *positive definite*, which means that all its eigenvalues are greater than zero. The following statements hold for every positive-definite matrix Σ:

- $\boldsymbol{\alpha}^T \Sigma \boldsymbol{\alpha} = 0 \Rightarrow \boldsymbol{\alpha} = \boldsymbol{0}$.

- Σ is a nonsingular matrix (also called a *regular* matrix).

- All eigenvalues of Σ are positive and non-zero.

- All principle submatrices of Σ are also positive definite.

In the following chapters, where we present observational models, we factor out of the covariance matrix Σ a scalar term denoted by σ_0^2, called a *variance component*, with the resulting matrix called the *cofactor* matrix. We label the cofactor matrix as Q; its inverse is labeled P and is called *weight matrix*. The relations between these terms are written mathematically as

$$\Sigma = \sigma_0^2 Q = \sigma_0^2 P^{-1}. \tag{1.44}$$

The simplest form of a covariance matrix Σ is the case where the cofactor matrix Q is equal to the identity matrix I_n. Indeed, if Q is a multiple of the identity matrix, the data are said to be *homogeneously* distributed. Another term for that case is *independent and identically distributed*, abbreviated iid. If the covariance matrix is diagonal, but its diagonal elements are not all the same, the data are said to have a *heteroscedastic* distribution. These cases are illustrated as follows:

- Homogeneous case

$$D\{e\} = \sigma_0^2 Q = \sigma_0^2 \begin{bmatrix} q & 0 & \cdots & 0 \\ 0 & q & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & q \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1/q & 0 & \cdots & 0 \\ 0 & 1/q & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & 1/q \end{bmatrix} = \frac{1}{q} \cdot I_n \tag{1.45a}$$

- Heteroscedastic case

$$D\{e\} = \sigma_0^2 Q = \sigma_0^2 \begin{bmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & q_{nn} \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1/q_{11} & 0 & \cdots & 0 \\ 0 & 1/q_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & 1/q_{nn} \end{bmatrix} \tag{1.45b}$$

- General case

$$D\{e\} = \sigma_0^2 Q = \sigma_0^2 \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & q_{23} & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & \cdots & \cdots & q_{nn} \end{bmatrix}, \text{ with } q_{ij} = q_{ji} \Rightarrow P = Q^{-1} \tag{1.45c}$$

Note that for $P = [p_{ij}]$, $p_{ii} \neq 1/q_{ii}$. Thus, the inverse of the diagonal components of Q are not weights in this case!

1.5.2.1   Error Propagation with Matrices

The derivations of (1.31a), (1.31b) and (1.34) can easily be extended to the multivariate case. Here we show their matrix analogs without deriving them (though some are derived in the example problems that follow).

If $\boldsymbol{y}$ is a random vector, A a constant matrix, and $\boldsymbol{\gamma}$ a constant vector, then the formulas for propagation of expectation and dispersion (error or covariance propagation) are summarized as follows:

Expectation:

$$\boxed{E\{A\boldsymbol{y} + \boldsymbol{\gamma}\} = A{\cdot}E\{\boldsymbol{y}\} + \boldsymbol{\gamma}} \tag{1.46a}$$

Dispersion (law of error propagation):

$$\boxed{D\{A\boldsymbol{y} + \boldsymbol{\gamma}\} = A{\cdot}D\{\boldsymbol{y}\}{\cdot}A^T} \tag{1.46b}$$

Also, analogous to (1.30a) and (1.30b) we have

$$\boxed{D\{\boldsymbol{y}\} = E\{(\boldsymbol{y} - E\{\boldsymbol{y}\})(\boldsymbol{y} - E\{\boldsymbol{y}\})^T\} = E\{\boldsymbol{y}\boldsymbol{y}^T\} - E\{\boldsymbol{y}\}E\{\boldsymbol{y}\}^T.} \tag{1.47}$$

Covariance: Given two random vectors, $\boldsymbol{y}$ and $\boldsymbol{z}$, their covariance is written as

$$\boxed{C\{\boldsymbol{z}, \boldsymbol{y}\} = E\{(\boldsymbol{z} - \boldsymbol{\mu}_z)(\boldsymbol{y} - \boldsymbol{\mu}_y)^T\} = E\{\boldsymbol{z}\boldsymbol{y}^T\} - \boldsymbol{\mu}_z\boldsymbol{\mu}_y^T.} \tag{1.48}$$

Mean Squared Error: If $\boldsymbol{y}$ is a random vector with true value $\boldsymbol{\mu}$, the MSE of $\boldsymbol{y}$ is written as

$$\mathrm{MSE}\{\boldsymbol{y}\} = D\{\boldsymbol{y}\} + \boldsymbol{\beta}\boldsymbol{\beta}^T, \tag{1.49a}$$

where the bias vector $\boldsymbol{\beta}$ is defined formally as

$$\boldsymbol{\beta} := E\{\boldsymbol{y} - \boldsymbol{\mu}\} = E\{\boldsymbol{y}\} - \boldsymbol{\mu}. \tag{1.49b}$$

Once again, we see that the mean squared error matrix of a random vector is only equal to the dispersion matrix of the random vector in the absence of bias, i.e., when $\boldsymbol{\mu} = E\{\boldsymbol{y}\} \Rightarrow \boldsymbol{\beta} = \boldsymbol{0}$.

1.5.2.2   Correlation Matrix

A measure of *correlation* can be derived from the Cauchy-Schwartz inequality, which is given by

$$C\{e_j, e_k\} = \iint (e_t)_j \cdot (e_t)_k \cdot f((e_t)_j, (e_t)_k)\, d(e_t)_j\, d(e_t)_k = \sigma_{jk} \leq$$

$$\leq \sqrt{\int (e_t)_j^2 \cdot f\big((e_t)_j\big)\, d(e_t)_j \cdot \int (e_t)_k^2 \cdot f\big((e_t)_k\big)\, d(e_t)_k} = \sqrt{\sigma_j^2 \sigma_k^2}. \tag{1.50}$$

Since σ_{jk} can take on a positive or a negative value, the above inequality leads to the notion of a *correlation coefficient*, defined as

$$\boxed{\rho_{jk} := \frac{\sigma_{jk}}{\sigma_j \sigma_k}, \quad \text{with} \ -1 \le \rho_{jk} \le 1.} \tag{1.51}$$

Analogous to the covariance matrix, we may form a matrix of correlation coefficients. Such a matrix is called a *correlation matrix* and is defined as

$$\underset{n \times n}{R} := \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix} = R^T. \tag{1.52}$$

Given a covariance matrix Σ, the correlation matrix can be generated easily by

$$R = \operatorname{diag}\left(1/\sigma_1, \ldots, 1/\sigma_n\right) \cdot \Sigma \cdot \operatorname{diag}\left(1/\sigma_1, \ldots, 1/\sigma_n\right). \tag{1.53}$$

A note on units: Units must be properly accounted for in covariance matrices. The following list clarifies the units of relevant terms.

σ_0^2 unitless

ρ_{jk} unitless

σ_j^2 has squared units of observation y_j

σ_{jk} has units of observation y_j multiplied by the units of observation y_k

A further discussion on observations and random errors is given in Section 2.1.1 in the context of data models and least-squares adjustments.

1.5.2.3 Examples of Covariance Propagation

1. Given $\boldsymbol{y}$ as an $n \times 1$ observation vector and $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{y})$ as an $m \times 1$ vector of nonlinear functions of $\boldsymbol{y}$.

   Find the $m \times n$ covariance matrix $C\{\boldsymbol{z}, \boldsymbol{y}\}$.

   Solution: Let $\boldsymbol{\mu}$ be the true value of $\boldsymbol{y}$ and linearize about expansion point $\boldsymbol{\mu}_0$ to get $\boldsymbol{z} = \boldsymbol{\alpha}_0 + A(\boldsymbol{y} - \boldsymbol{\mu}_0)$, with $\boldsymbol{\alpha}_0 := f(\boldsymbol{\mu}_0)$ and A as the Jacobian matrix of $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{y})$.

   Law of covariance propagation:

$$\underset{m \times n}{C\{\boldsymbol{z}, \boldsymbol{y}\}} = E\{\boldsymbol{z}\boldsymbol{y}^T\} - E\{\boldsymbol{z}\}E\{\boldsymbol{y}\}^T =$$

$$= E\{\left[\boldsymbol{\alpha}_0 + A(\boldsymbol{y} - \boldsymbol{\mu}_0)\right]\boldsymbol{y}^T\} - E\{\boldsymbol{\alpha}_0 + A(\boldsymbol{y} - \boldsymbol{\mu}_0)\} \cdot E\{\boldsymbol{y}\}^T =$$

$$= \boldsymbol{\alpha}_0 \cdot E\{\boldsymbol{y}\}^T + A \cdot E\{\boldsymbol{y}\boldsymbol{y}^T\} - A\boldsymbol{\mu}_0 \cdot E\{\boldsymbol{y}\}^T -$$

$$- \boldsymbol{\alpha}_0 \cdot E\{\boldsymbol{y}\}^T - A \cdot E\{\boldsymbol{y}\} \cdot E\{\boldsymbol{y}\}^T + A\boldsymbol{\mu}_0 \cdot E\{\boldsymbol{y}\}^T =$$

$$= A\left[E\{\boldsymbol{y}\boldsymbol{y}^T\} - E\{\boldsymbol{y}\}E\{\boldsymbol{y}\}^T\right] = A \cdot D\{\boldsymbol{y}\} \Rightarrow$$

$$C\{\boldsymbol{z}, \boldsymbol{y}\} = A \cdot D\{\boldsymbol{y}\}$$

2. Rather than one variable $\boldsymbol{z}$ as above, suppose we have $\boldsymbol{z}_1$ of size $m_1 \times 1$ and $\boldsymbol{z}_2$ of size $m_2 \times 1$. Find the $m_1 \times m_2$ covariance matrix $C\{\boldsymbol{z}_1, \boldsymbol{z}_2\}$.

Solution: After linearization

$$C\{\boldsymbol{z}_1 = \boldsymbol{\alpha}_1^0 + A_1\boldsymbol{y}, \boldsymbol{z}_2 = \boldsymbol{\alpha}_2^0 + A_2\boldsymbol{y}\} = \underset{m_1 \times n}{A_1} \cdot \underset{n \times n}{D\{\boldsymbol{y}\}} \cdot \underset{n \times m_2}{A_2^T}$$

3. Given the $m_1 \times 1$ random vector $\boldsymbol{z}_1$, the $m_2 \times 1$ random vector $\boldsymbol{z}_2$, constant vectors $\boldsymbol{\beta}_1(l_1 \times 1)$ and $\boldsymbol{\beta}_2(l_2 \times 1)$ and constant matrices $B_1(l_1 \times m_1)$ and $B_2(l_2 \times m_2)$.

Find the covariance matrix of $\boldsymbol{x}_1 = \boldsymbol{\beta}_1 + B_1\boldsymbol{z}_1$ and $\boldsymbol{x}_2 = \boldsymbol{\beta}_2 + B_2\boldsymbol{z}_2$.

Solution:

$$C\{\boldsymbol{x}_1 = \boldsymbol{\beta}_1 + B_1\boldsymbol{z}_1, \boldsymbol{x}_2 = \boldsymbol{\beta}_2 + B_2\boldsymbol{z}_2\} = \underset{l_1 \times m_1}{B_1} \underset{m_1 \times m_2}{C\{\boldsymbol{z}_1, \boldsymbol{z}_2\}} \underset{m_2 \times l_2}{B_2^T}$$

Note that the matrix $C\{\boldsymbol{z}_1, \boldsymbol{z}_2\}$ is not necessarily symmetric.

4. What is the covariance of the random variable $\boldsymbol{y}$ with itself?

Solution:
$$C\{\boldsymbol{y}, \boldsymbol{y}\} = E\{\boldsymbol{y}\boldsymbol{y}^T\} - E\{\boldsymbol{y}\}E\{\boldsymbol{y}\}^T = D\{\boldsymbol{y}\}$$

5. Given $n \times 1$ vectors $\boldsymbol{y} = \boldsymbol{\mu} + \boldsymbol{e}$ with $E\{\boldsymbol{e}\} = \boldsymbol{0}$, which implies that $E\{\boldsymbol{y}\} = \boldsymbol{\mu}$ and $D\{\boldsymbol{e}\} = E\{\boldsymbol{e}\boldsymbol{e}^T\}$.

Find: The dispersion matrix $D\{\boldsymbol{y}\}$.

Solution:

$$D\{\boldsymbol{y}\} = E\{(\boldsymbol{y} - E\{\boldsymbol{y}\})(\boldsymbol{y} - E\{\boldsymbol{y}\})^T\} = E\{(\boldsymbol{y} - \boldsymbol{\mu})(\boldsymbol{y} - \boldsymbol{\mu})^T\} =$$
$$= E\{\boldsymbol{y}\boldsymbol{y}^T - \boldsymbol{y}\boldsymbol{\mu}^T - \boldsymbol{\mu}\boldsymbol{y}^T + \boldsymbol{\mu}\boldsymbol{\mu}^T\} = E\{\boldsymbol{y}\boldsymbol{y}^T\} - \boldsymbol{\mu}\boldsymbol{\mu}^T - \boldsymbol{\mu}\boldsymbol{\mu}^T + \boldsymbol{\mu}\boldsymbol{\mu}^T \Rightarrow$$
$$D\{\boldsymbol{y}\} = E\{\boldsymbol{y}\boldsymbol{y}^T\} - \boldsymbol{\mu}\boldsymbol{\mu}^T$$

6. Given random vectors $\boldsymbol{y}$ and $\boldsymbol{z}$, with corresponding expectations $E\{\boldsymbol{y}\} = \boldsymbol{\mu}_y$ and $E\{\boldsymbol{z}\} = \boldsymbol{\mu}_z$, find the covariance matrix $C\{\boldsymbol{z}, \boldsymbol{y}\}$.

Solution:

$$C\{\boldsymbol{z}, \boldsymbol{y}\} = E\{(\boldsymbol{z} - \boldsymbol{\mu}_z)(\boldsymbol{y} - \boldsymbol{\mu}_y)^T\} = E\{\boldsymbol{z}\boldsymbol{y}^T - \boldsymbol{z}\boldsymbol{\mu}_y^T - \boldsymbol{\mu}_z\boldsymbol{y}^T + \boldsymbol{\mu}_z\boldsymbol{\mu}_y^T\} =$$
$$= E\{\boldsymbol{z}\boldsymbol{y}^T\} - \boldsymbol{\mu}_z\boldsymbol{\mu}_y^T - \boldsymbol{\mu}_z\boldsymbol{\mu}_y^T + \boldsymbol{\mu}_z\boldsymbol{\mu}_y^T = E\{\boldsymbol{z}\boldsymbol{y}^T\} - \boldsymbol{\mu}_z\boldsymbol{\mu}_y^T$$

7. Suppose y_1, y_2, and y_3 are independent measurements with standard deviations $\sqrt{2}$ cm, 2 cm, and 1 cm, respectively. The quantities x_1 and x_2 are computed from the measurements as follows

$$x_1 = 2y_1 + y_2 + 2y_3,$$
$$x_2 = y_1 - 2y_2 \quad .$$

Evaluate the covariance matrix for the random vector $\boldsymbol{x} = [x_1, \ x_2]^T$.

Solution: The given equations can be written in matrix form as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{y} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \ D\{\boldsymbol{y}\} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{cm}^2 = \Sigma_{yy}$$

Now apply the law of error propagation (1.46b):

$$D\{\boldsymbol{x}\} = A{\cdot}D\{\boldsymbol{y}\}{\cdot}A^T = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 2 & 0 \end{bmatrix} \text{cm}^2 =$$

$$= \begin{bmatrix} 16 & -4 \\ -4 & 18 \end{bmatrix} \text{cm}^2 = \Sigma_{xx}$$

$$\Rightarrow \sigma_{x_1} = 4\,\text{cm}, \ \sigma_{x_2} = 3\sqrt{2}\,\text{cm}, \ \sigma_{x_1 x_2} = -4\,\text{cm}^2$$

$$\Rightarrow \rho_{x_1 x_2} = \frac{\sigma_{x_1 x_2}}{\sigma_{x_1}\sigma_{x_2}} = \frac{-4\,\text{cm}^2}{4\,\text{cm} \cdot 3\sqrt{2}\,\text{cm}} = -0.2357.$$

Correlation matrix:

$$R = \begin{bmatrix} \frac{1}{4\,\text{cm}} & 0 \\ 0 & \frac{1}{3\sqrt{2}\,\text{cm}} \end{bmatrix} \begin{bmatrix} 16 & -4 \\ -4 & 18 \end{bmatrix} \text{cm}^2 \begin{bmatrix} \frac{1}{4\,\text{cm}} & 0 \\ 0 & \frac{1}{3\sqrt{2}\,\text{cm}} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -0.2357 \\ -0.2357 & 1 \end{bmatrix}$$

8. An azimuth α and distance s were measured from known point C to point D to determine the coordinates of D (see Figure 1.3). Compute the coordinates of D and their covariance matrix, along with the correlation matrix, based on the following data.

$$
\begin{array}{ll}
x_c = 2000.0\,\text{m}, & \sigma_{x_c} = 1\,\text{cm} \\
y_c = 3000.0\,\text{m}, & \sigma_{y_c} = 1\,\text{cm} \\
\alpha = 120°00'00'', & \sigma_\alpha = 10'' \\
s = 1600.00\,\text{m}, & \sigma_s = 5\,\text{cm}
\end{array}
$$

Principle: covariance propagation $D\{A\boldsymbol{y} + \boldsymbol{\gamma}\} = A{\cdot}D\{\boldsymbol{y}\}{\cdot}A^T$

Let the random variable $\boldsymbol{y} := [x_c, y_c, \alpha, s]^T$ and the random variable $\boldsymbol{x} := [x_D, y_D]^T$.

Functional relations:

$$x_D = x_C + s \cdot \sin \alpha$$
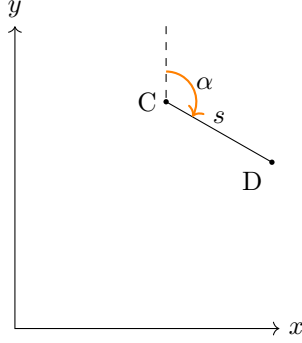
Figure 1.3: Azimuth α and distance s measured from point C to point D to determined the coordinates of point D.

$$y_D = y_C + s \cdot \cos \alpha$$

$x = f(y)$, x is a nonlinear function of y. Under linearization

$$x \approx f(y_0) + \left.\frac{\partial f(y)}{\partial y^T}\right|_{y_0} (y - y_0)$$

Use values of observations for y_0.

$\boxed{f(y_0) \text{ gives: } x_D = 3385.64\,\text{m}, \quad y_D = 2200.00\,\text{m}}$

$$\frac{\partial x_D}{\partial x_C} = 1, \ \frac{\partial x_D}{\partial y_C} = 0, \ \frac{\partial x_D}{\partial \alpha} = s\,[\text{m}] \cdot \cos \alpha, \ \frac{\partial x_D}{\partial s} = \sin \alpha,$$

$$\frac{\partial y_D}{\partial x_C} = 0, \ \frac{\partial y_D}{\partial y_C} = 1, \ \frac{\partial y_D}{\partial \alpha} = -s\,[\text{m}] \cdot \sin \alpha, \ \frac{\partial y_D}{\partial s} = \cos \alpha$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & s\,[\text{m}] \cdot \cos \alpha & \sin \alpha \\ 0 & 1 & -s\,[\text{m}] \cdot \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & -800.0\,\text{m} & 0.866 \\ 0 & 1 & -1385.64\,\text{m} & -0.5 \end{bmatrix}$$

Note that the distance s is in units of meters, and these units must be carried into matrix A to ensure the units of the resulting covariance matrix are correct.

Covariance matrix for given data:

$$\Sigma_{yy} = \begin{bmatrix} (0.01\,\text{m})^2 & 0 & 0 & 0 \\ 0 & (0.01\,\text{m})^2 & 0 & 0 \\ 0 & 0 & \left(\frac{10''}{3600''/1°} \frac{\pi}{180°}\right)^2 & 0 \\ 0 & 0 & 0 & (0.05\,\text{m})^2 \end{bmatrix}$$

Covariance matrix for coordinates of point D:

$$D\left\{\begin{bmatrix} x_D \\ y_D \end{bmatrix}\right\} = A \cdot \Sigma_{yy} \cdot A^T = \begin{bmatrix} 0.0035 & 0.0015 \\ 0.0015 & 0.0052 \end{bmatrix} m^2 = \Sigma_{xx}$$

Standard deviations for coordinates of point D:

$$\boxed{\Rightarrow \sigma_{x_D} = 6\,\text{cm}, \quad \sigma_{y_D} = 7\,\text{cm}}$$

Correlation matrix:

$$R = \begin{bmatrix} 1/\sigma_{x_D} & 0 \\ 0 & 1/\sigma_{y_D} \end{bmatrix} \cdot \Sigma_{xx} \cdot \begin{bmatrix} 1/\sigma_{x_D} & 0 \\ 0 & 1/\sigma_{y_D} \end{bmatrix} = \begin{bmatrix} 1 & 0.3568 \\ 0.3568 & 1 \end{bmatrix}$$

1.5.3 Practice Problems

1. Let X be a random variable with the following probability density function:

$$f(x) = \begin{cases} \frac{1}{8}(x-1) & \text{for } 1 < x < 5, \\ 0 & \text{otherwise.} \end{cases}$$

Derive the cumulative distribution function of X and evaluate $P[X < 2]$, $P[X > 4]$, and $P[1.5 < X < 4.5]$.

2. Let X be a random variable with the following probability density function:

$$f(x) = \frac{\sin x}{2} \text{ for } 0 < x < \pi.$$

Derive the cumulative distribution function of X and evaluate $P[X < \pi/4]$, $P[X > \pi/2]$, and $P[\pi/4 < X < \pi/2]$. Sum the three probabilities and comment on the result.

3. Evaluate the mean and variance of the random variable in the preceding problem (hint: integration by parts).

4. Two measurements are normally distributed with standard deviations of 0.55 m and 0.35 m, respectively. Compute the standard deviation of the sum and difference of the two measurements if the correlation coefficient of the two measurements is: (a) 0.5, (b) 0, (c) -0.5, (d) 1.0.

5. The X and Y coordinates of a survey point have standard deviations of $\sigma_x = 0.045$ m and $\sigma_y = 0.025$ m, respectively. (a) Compute the correlation coefficient of X and Y if the covariance of X and Y is $0.000\,12$ m^2. (b) Compute the covariance of X and Y if the correlation coefficient is 0.333.

6. Consider a linear equation $Y = a + bX$, where X is a random variable having a normal distribution, i.e., $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$.

   Show that $E\{Y\} = a + b \cdot \mu_X$ and $\sigma_Y^2 = b^2 \cdot \sigma_X^2$. Then show that

   $$Z = \frac{X - \mu_X}{\sqrt{\sigma_X^2}}$$

   has zero mean and unit variance.

7. Consider the following system of equations

   $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 2 \\ -1 & 3 & 2 & -1 \\ 1 & -1 & 6 & 7 \\ 2 & -2 & 14 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{y} = A\boldsymbol{x},$$

   where y_1, y_2, y_3, and y_4 are independent and identically distributed (iid) each with the mean 0 and variance σ^2.

   (a) Express x_1, x_2, x_3, and x_4 in terms of y_1, y_2, y_3, and y_4.

   (b) Compute the covariance matrix for $\boldsymbol{x}$.

   (c) Suppose now that instead of being iid, the dispersion of $\boldsymbol{y}$ is given by the matrix

   $$\begin{bmatrix} \sigma^2 & \rho\sigma^2 & 0 & 0 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & 0 \\ 0 & \rho\sigma^2 & \sigma^2 & \rho\sigma^2 \\ 0 & 0 & \rho\sigma^2 & \sigma^2 \end{bmatrix}.$$

   Answer question (b) in this case.

8. Suppose three points A, B, and C are sequentially located on a straight line (Figure 1.4). A total station was used to measure distances between them, so that the total distance between A and C could be estimated. The data are listed in Table 1.1.
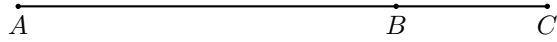
Figure 1.4: Points A, B, and C on a straight line

   The variance of each observation is given by $\sigma^2 = (9\,\mathrm{mm})^2 + (d/100)^2 mm^2$, where d is distance in meters. Unbeknownst to the surveyor, a constant bias $\beta = 2.5\,\mathrm{cm}$ affected every observation, which is reflected as $+0.025\,\mathrm{m}$ in Table 1.1.

   Suppose the surveyor estimated the total distance $\overline{AC}$ by the formula

   $$\overline{AC} = z = \frac{1}{6}(y_1 + y_2 + y_3 + y_6 + y_7 + y_8) + \frac{1}{4}(y_4 + y_5).$$

Table 1.1: Observations of line segments

y_i	Segment	Observation [m]
y_1	$\overline{AB}$	$52.154 + 0.025 = 52.179$
y_2	$\overline{AB}$	$52.157 + 0.025 = 52.182$
y_3	$\overline{AB}$	$52.155 + 0.025 = 52.180$
y_4	$\overline{AC}$	$70.180 + 0.025 = 70.205$
y_5	$\overline{AC}$	$70.178 + 0.025 = 70.203$
y_6	$\overline{BC}$	$18.022 + 0.025 = 18.047$
y_7	$\overline{BC}$	$18.021 + 0.025 = 18.046$
y_8	$\overline{BC}$	$18.025 + 0.025 = 18.050$

(a) Compute the standard deviation and rms (square root of MSE) of z using $\mu_{AC} = 70.179\,\text{m}$ as a hypothetical "true value" of the total distance $\overline{AC}$.

(b) Now use the same formula for z with the unbiased observations (i.e., remove the $2.5\,\text{cm}$ bias from each observation). Compute its variance and compare to the variance of part (a). Do you expect the variances to be the same? Why or why not?

(c) Find a different combination of the measurements that would provide an estimate for the total distance that is not affected by the bias in the measurements. Compute the standard deviation and rms for this estimate (again using $\mu_{AC} = 70.179\,\text{m}$ as a hypothetical "true value"). Compare these results to those of parts (a) and (b) and comment on your findings.

9. Given a random variable y with expectation $E\{y\} = \mu_y$ and variance σ_y^2, suppose f and g are functions of y defined by $f = e^y$ and $g = y^3$, respectively.

(a) Using a Taylor series expansion, express the expectations and dispersions of f and g in terms of μ_y, σ_y^2, and $\delta = (\mu - \mu_0)$, where μ_0 is an approximation of μ.

(b) Assume that $E\{y\}$ coincides with the true value of μ_y of y, so that biases are due to the truncation of the Taylor series. What are the biases in the f and g due to the series truncation? Which bias is larger?

(c) Assume that the approximate value μ_0 coincides with the expectation μ_y of y. What are the expectations and dispersions now?

10. Sides a and b of the right-angled plane triangle in Figure 1.5 were measured. The values obtained are $a = 399.902\,\text{m}$ and $b = 300.098\,\text{m}$, with variances $\sigma_a^2 = (0.015\,\text{m})^2$ and $\sigma_b^2 = (0.020\,\text{m})^2$, respectively. The correlation coefficient is $\rho_{ab} = 0.2$. Compute side c and angle β and their standard deviations. Also determine the correlation, if any, between computed side c and angle β.
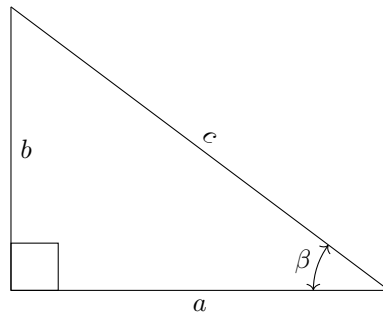
Figure 1.5: Right-angled plane triangle with measured sides a and b

11. The area of a trapezoidal parcel of land is computed by

$$A = \left(\frac{a_1 + a_2}{2} \right) b,$$

where a_1, a_2, and b were measured independently. The measurements and their standard deviations are $a_1 = 301.257\,\mathrm{m}, 0.025\,\mathrm{m}$, $a_2 = 478.391\,,0.045\,\mathrm{m}$, and $b = 503.782\,\mathrm{m}, 0.030\,\mathrm{m}$. Compute the area of the parcel and the standard deviation of the computed area.

Chapter 2

The Model of Direct Observations

2.1   Model Definition

When an unknown parameter μ can be observed directly, the *model of direct observations* can be formed for the data by

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \mu + e_1 \\ \vdots \\ \mu + e_n \end{bmatrix} = \boldsymbol{\tau}\mu + \boldsymbol{e}, \tag{2.1a}$$

$$\boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 Q \right), \quad Q := P^{-1}. \tag{2.1b}$$

The terms in the data model are defined as follows:

$\boldsymbol{y}$ is a *given* $n \times 1$ vector of observations with random properties.

μ is an unknown, non-random parameter to be *estimated*.

$\boldsymbol{\tau}$ is an $n \times 1$ vector of ones ("summation vector"), i.e., $\boldsymbol{\tau} := [1, \ldots, 1]^T$.

$\boldsymbol{e}$ is an $n \times 1$ vector of unknown, random errors to be *predicted*.

Q is a *given* $n \times n$ cofactor matrix associated with $\boldsymbol{e}$. It is symmetric, positive definite, and non-random.

P is an $n \times n$ positive-definite weight matrix, being the inverse of Q.

σ_0^2 is an unknown, non-random variance component that can be estimated.

Equation (2.1a) is called *observation equations*, while (2.1b) provides a *stochastic model* for the random observational errors. Together, these two equations comprise a complete *data model*.

2.1.1   Terminology: Observations, Redundancy, Residuals and Their Minimization

The observation vector $\boldsymbol{y}$ was described above as a given quantity. It is given in the sense that it consists of measurements (observations) that are typically made and recorded in the field. The measurements are considered to be a physical realization of an *observable* — the quantity ("the thing") being observed. An observable could be a dimension of an element of a physical object or a relationship between its elements, such as an angle between two connected edges of a geodetic network, the end points of which being accessible monuments in the ground. Or, and observable could be a property of an immaterial object, such as the phase of an electromagnetic wave. Another example of an observable is the length of a bridge from a mark scribed in concrete at its beginning to another at its end; then, an associated observation could be a distance measured and recorded with a surveyor's total station between those two marks. Being a measurement of an observable, an observation is a numerical value with an associated unit of measurement.

Even though the vector of observations $\boldsymbol{y}$ is given, it has *random* properties due to unavoidable *random errors* inherent both in making observations and in the instruments used to make them. These random errors are *unknown* quantities, and they are accounted for in the observation equations (2.1a) by the random error vector $\boldsymbol{e}$. Thus, we can say that we know the value of an observation, but we do not know the value of its random error constituent. However, we have already seen that we can say something about the expected values of the random errors (i.e., $E\{\boldsymbol{e}\} = \boldsymbol{0}$). Likewise, a statement can be made about the expectation of the observations, viz.

$$\boldsymbol{\mu}_y := E\{\boldsymbol{y}\} = E\{\boldsymbol{\tau}\mu + \boldsymbol{e}\} = \boldsymbol{\tau}\mu. \tag{2.2a}$$

We may think of the vector $\boldsymbol{\mu}_y$ as the vector of *true observations*, the values of which are unknown, though they can be estimated via

$$\widehat{E\{\boldsymbol{y}\}} =: \hat{\boldsymbol{\mu}}_y = \boldsymbol{\tau}\hat{\mu}, \tag{2.2b}$$

where $\hat{\mu}$ is an estimate of the unknown parameter μ. The vector $\hat{\boldsymbol{\mu}}_y$ is called the vector of *adjusted observations*.

Because the given observations, $\boldsymbol{y}$, contain unknown random errors represented by $\boldsymbol{e}$, we cannot possibly expect that $\boldsymbol{y}$ will equal $\boldsymbol{\tau}\mu$, though we may usually hope that at least $\boldsymbol{y} \approx \boldsymbol{\tau}\mu$. The inequality $\boldsymbol{y} \neq \boldsymbol{\tau}\mu$ should be immediately evident from the symbols, since they imply that $\boldsymbol{y}$ is random and $\boldsymbol{\tau}\mu$ is not. (Recall the use of Latin characters for random variables and Greek characters for non-random variables as discussed on page 4.) The rule eluded to here is that when one side of an equation results in a random quantity, so must the other side. The incongruency reflected in $\boldsymbol{y} \neq \boldsymbol{\tau}\mu$ is rectified in (2.1a) by the addition of $\boldsymbol{e}$ on the right side. But practically speaking, $\boldsymbol{e}$ is not much help, since it is unknown. This is where least-squares adjustment theory and techniques can come to the rescue. For if there are more observations than parameters in the model (i.e., more than one observation for model (2.1)), we can use these *redundant* observations to *predict* values for $\boldsymbol{e}$

using a *predictor* derived from the principle of least-squares adjustment (see below for a brief discussion on predictors and estimators).

The number of independent, redundant observations is called the *redundancy* of the model. Another term for it is *degrees of freedom*, sometimes abbreviated df in the statistical literature. The vector of predicted random errors is denoted by $\tilde{e}$, and it is also called the vector of *residuals*.

The idea behind least-squares adjustment is to predict the residuals so that the (weighted) sum of their squares is minimized, while still satisfying the relation $\boldsymbol{y} = \boldsymbol{\tau}\mu + \boldsymbol{e}$ shown in (2.1a), but now with the predicted random errors $\tilde{e}$ and the estimated parameter $\hat{\mu}$ rather than their corresponding "true," but unknown, quantities. That is, the relation

$$\boldsymbol{y} = \boldsymbol{\tau}\hat{\mu} + \tilde{e} \tag{2.3}$$

must hold after the adjustment, and the (weighted) sum of squared residuals $\Omega := \tilde{e}^T P \tilde{e}$ must be as small as possible. That is both the objective and the outcome of least-squares adjustments.

It is the data that are being adjusted in least-squares adjustments. They are adjusted so that the inconsistent equation $\boldsymbol{y} \neq \boldsymbol{\tau}\mu$ is replaced by the consistent equation $\hat{\boldsymbol{\mu}}_y = \boldsymbol{\tau}\hat{\mu}$. So, we speak of adjusted data, predicted residuals, and estimated parameters as the outcomes of a least-squares solution, which is derived in the next section.

Estimate vs. estimator   In these notes we have hardly distinguished between the terms estimate and estimator. This is partly because sometimes the same symbol works for both terms depending on the context, though in some places we might have used estimate when we could have used estimator, and the same can be said for prediction and predictor. The distinction between these terms made by Tukey (1987, p. 633) is quoted as follows:

> An estimator is a function of the observations, a specific way of putting them together. It may be specified by an arithmetic formula, like $\bar{y} = \Sigma x_i/n$, or by words alone, as in directions for finding a sample median by ordering and counting. We distinguish between the estimator and its value, an estimate, obtained from the specific set of data. The variance estimator, $s^2 = \Sigma(x_i - \bar{x})^2/(n-1)$, yields the estimate 7 from the three observations 2, 3, 7. We say s^2 is an estimator for σ^2, and we call σ^2 the estimated. In the numerical example, 7 estimates σ^2.

2.2   The Least-Squares Solution

In order to minimize $\Omega := \tilde{e}^T P \tilde{e}$ while satisfying (2.3) we form the Lagrange target function

$$\Phi(\boldsymbol{e}, \boldsymbol{\lambda}, \mu) := \boldsymbol{e}^T P \boldsymbol{e} + 2\boldsymbol{\lambda}^T(\boldsymbol{y} - \boldsymbol{\tau}\mu - \boldsymbol{e}), \tag{2.4}$$

where $\boldsymbol{\lambda}$ is an unknown $m \times 1$ vector of Lagrange multipliers. The target function is made stationary with respect to the unknown terms $\boldsymbol{e}$, $\boldsymbol{\lambda}$, and μ when its first partial derivatives are set equivalent to zero, which is reflected in the following *Euler-Lagrange necessary conditions*:

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}} = \frac{1}{2}\left[\frac{\partial \Phi}{\partial e_j}\right]_{n \times 1} = P\tilde{\boldsymbol{e}} - \hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \tag{2.5a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = \frac{1}{2}\left[\frac{\partial \Phi}{\partial \lambda_j}\right]_{n \times 1} = \boldsymbol{y} - \boldsymbol{\tau}\hat{\mu} - \tilde{\boldsymbol{e}} \doteq \boldsymbol{0}, \tag{2.5b}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \mu} = \boldsymbol{\tau}^T\hat{\boldsymbol{\lambda}} \doteq 0. \tag{2.5c}$$

These necessary conditions are sometimes called first-order conditions due to the involvement of first partial derivatives. The sufficient condition for minimization is satisfied by the fact that the second partial derivative of Φ is $\partial^2\Phi/(\partial \boldsymbol{e}\partial \boldsymbol{e}^T) = 2P$, where the weight matrix P is positive definite according to (2.1). Therefore, the solution to the system of equations (2.5) yields the minimum of Φ, and thus the weighted sum of squared residuals (weighted SSR) $\Omega = \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}}$ is minimized. A proof that Ω is minimum is shown by Koch (1999, Eq. (3.25) and Theorem (3.26)) for the multivariate case (i.e., a vector of unknown parameters), which is treated in the next chapter. Also, see Appendix A for comments on derivatives of quadratic functions with respect to column vectors.

Throughout these notes, we use a hat to denote an *estimate* of a non-random variable, whereas a tilde denotes a *prediction* of a random variable. The hat and tilde marks were introduced into (2.5) to distinguish between the unknown variables of the target function (2.4) and those particular quantities that satisfy the necessary conditions. This reflects that $\tilde{\boldsymbol{e}}$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\mu}$ cannot take on just any values but rather only those that result from setting the first partial derivatives of the target function to zero (denoted by the $\doteq$ sign), which explains why it would not be logical to introduce the hat and tilde symbols in (2.4). Also note that, for the vector $\tilde{\boldsymbol{e}}$, we use the terms *residual* and *predicted random error* synonymously.

Now we must solve the system of equations (2.5) to obtain the least-squares solution (LESS) as follows:

$$\hat{\boldsymbol{\lambda}} = P\tilde{\boldsymbol{e}} = P(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}) \qquad \text{using (2.5a) and (2.5b)} \tag{2.6a}$$

$$\boldsymbol{\tau}^T\hat{\boldsymbol{\lambda}} = \boldsymbol{\tau}^T P\boldsymbol{y} - (\boldsymbol{\tau}^T P\boldsymbol{\tau})\hat{\mu} = 0 \qquad \text{using (2.6a) and (2.5c)} \tag{2.6b}$$

Equation (2.6b) leads to

$$\boxed{\hat{\mu} = \frac{\boldsymbol{\tau}^T P\boldsymbol{y}}{\boldsymbol{\tau}^T P\boldsymbol{\tau}}} \tag{2.7}$$

for the estimate of the unknown parameter μ. And, from (2.5b), we have

$$\boxed{\tilde{\boldsymbol{e}} = \boldsymbol{y} - \boldsymbol{\tau}\hat{\mu} \Rightarrow} \tag{2.8a}$$

$$\tilde{e} = \left[I_n - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T P\right]\boldsymbol{y} \tag{2.8b}$$

for the prediction of the random error vector $\boldsymbol{e}$. As stated already, the prediction $\tilde{e}$ is also called residual vector. We say that the quantities $\hat{\mu}$, $\tilde{e}$, and $\hat{\boldsymbol{\lambda}}$ belong to a LEast-Squares Solution (LESS) within the model of direct observations (2.1).

It turns out that $\hat{\mu}$ is an unbiased estimator of μ, since

$$E\{\hat{\mu}\} = E\{(\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1}\boldsymbol{\tau}^T P \boldsymbol{y}\} = (\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1}\boldsymbol{\tau}^T P {\cdot} E\{\boldsymbol{y}\} = (\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1}\boldsymbol{\tau}^T P \boldsymbol{\tau}\mu = \mu. \tag{2.9}$$

Likewise, the residual vector $\tilde{e}$ is an unbiased predictor of the random error vector $\boldsymbol{e}$, since

$$E\{\tilde{e}\} = \left[I_n - \boldsymbol{\tau}(\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1}\boldsymbol{\tau}^T P\right]{\cdot}E\{\boldsymbol{y}\} =$$
$$= \left[I_n - \boldsymbol{\tau}(\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1}\boldsymbol{\tau}^T P\right]\boldsymbol{\tau}\mu = \boldsymbol{\tau}\mu - \boldsymbol{\tau}\mu = \boldsymbol{0}. \tag{2.10}$$

The vectors $\boldsymbol{\tau}$ and $\tilde{e}$ are said to be *P-orthogonal* since

$$\boldsymbol{\tau}^T P \tilde{e} = \boldsymbol{\tau}^T P(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}) = \boldsymbol{\tau}^T P\left[I_n - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T P\right]\boldsymbol{y} =$$
$$= \boldsymbol{\tau}^T P \boldsymbol{y} - \boldsymbol{\tau}^T P \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T P \boldsymbol{y} = 0. \tag{2.11}$$

This result reveals that the sum of the P-weighted residual vector within the model of direct observations is zero.

The adjusted observations, $\boldsymbol{\tau}\hat{\mu}$, on the right side of (2.8a) can also be expressed as

$$\boxed{\hat{\boldsymbol{\mu}}_y := \widehat{E\{\boldsymbol{y}\}} = \boldsymbol{\tau}\hat{\mu} = \boldsymbol{y} - \tilde{e}.} \tag{2.12}$$

Obviously, since $\boldsymbol{\tau}^T P \tilde{e} = 0$, we also have

$$\left(\boldsymbol{\tau}\hat{\mu}\right)^T P \tilde{e} = \hat{\boldsymbol{\mu}}_y^T P \tilde{e} = 0. \tag{2.13}$$

Equation (2.13) reveals an important characteristic of LESS; viz., the vector of adjusted observations and the vector of P-weighted residuals are orthogonal to one another. From a geometric point of view (illustrated in Figure 2.1), the orthogonal relationship between these vectors means that the vector of observations $\boldsymbol{y}$ and the vector of adjusted observations $\hat{\boldsymbol{\mu}}_y$ are as close as possible to each other (considering the weights in P), which is exactly what we require from a least-squares adjustment: a minimum adjustment of the data that will satisfy the given observational model.

In addition to solving for the estimated parameter $\hat{\mu}$ and the predicted random error vector $\tilde{e}$, we are typically interested in their dispersions (variances), which are an indicator of their precisions. To compute their dispersions, we apply the law of covariance propagation. First, for the dispersion of the estimated parameter $\hat{\mu}$ we have

$$D\{\hat{\mu}\} = \frac{\boldsymbol{\tau}^T P}{\boldsymbol{\tau}^T P \boldsymbol{\tau}} D\{\boldsymbol{y}\} \frac{P \boldsymbol{\tau}}{\boldsymbol{\tau}^T P \boldsymbol{\tau}} = \frac{\boldsymbol{\tau}^T P\left(\sigma_0^2 P^{-1}\right)P \boldsymbol{\tau}}{\boldsymbol{\tau}^T P \boldsymbol{\tau}\,\boldsymbol{\tau}^T P \boldsymbol{\tau}} \Rightarrow$$

$$\bar{P} := \left[I_n - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T P\right]$$
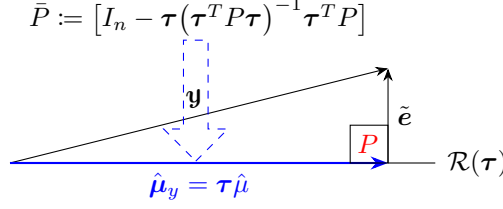
Figure 2.1: Depiction of P-orthogonality between residual vector $\tilde{\boldsymbol{e}}$ and vector of adjusted observations $\boldsymbol{\tau}\hat{\mu}$. The appearance of the weight matrix P in the box represents its roll in the orthogonality relationship. The effect of the projection matrix $\bar{P}$ is depicted by the dashed arrow as projecting the observation vector $\boldsymbol{y}$ onto the (one-dimensional) range space of $\boldsymbol{\tau}$. The vectors sum together as $\boldsymbol{y} = \boldsymbol{\tau}\hat{\mu} + \tilde{\boldsymbol{e}}$, just as they should.

$$D\{\hat{\mu}\} = \frac{\sigma_0^2}{\boldsymbol{\tau}^T P \boldsymbol{\tau}}. \tag{2.14}$$

The $n \times n$ dispersion matrix for the residual vector $\tilde{\boldsymbol{e}}$ is derived by

$$
\begin{aligned}
D\{\tilde{\boldsymbol{e}}\} &= D\left\{\left[I_n - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T P\right]\boldsymbol{y}\right\} = \\
&= \left[I_n - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T P\right] D\{\boldsymbol{y}\}\left[I_n - P\boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T\right] = \\
&= \sigma_0^2\left[P^{-1} - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T\right]\left[I_n - P\boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T\right] = \\
&= \sigma_0^2\left[P^{-1} - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T\right] - \sigma_0^2\boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T + \\
&\qquad + \sigma_0^2\boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T P\boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T \Rightarrow
\end{aligned}
$$

$$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2\left[P^{-1} - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T\right]. \tag{2.15}$$

It turns out that the last matrix in (2.15) involves the dispersion of the adjusted observations, since

$$D\{\hat{\boldsymbol{\mu}}_y\} = \boldsymbol{\tau} D\{\hat{\mu}\}\boldsymbol{\tau}^T = \sigma_0^2\boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T. \tag{2.16}$$

Formally, neither (2.14) nor (2.15) nor (2.16) can be computed, since the variance component σ_0^2 is unknown, though it can be replaced by its estimate shown in (2.38). From (2.15) we see that the dispersion (variance) of the jth element of $\tilde{\boldsymbol{e}}$ is

$$\sigma_{\tilde{e}_j}^2 = \sigma_0^2\left(\sigma_{jj}^2 - \frac{1}{\boldsymbol{\tau}^T P \boldsymbol{\tau}}\right), \tag{2.17}$$

where σ_{jj}^2 is the jth diagonal element of P^{-1}, and σ_0^2 is the variance component from the model (2.1). Thus it is apparent that the variance of the jth element of the residual vector $\tilde{\boldsymbol{e}}$ is smaller than the variance of the corresponding jth element of the true, but unknown, random error vector $\boldsymbol{e}$.

2.2.1 Equivalency to Arithmetic Mean and Weighted Arithmetic Mean

In the special case where the random errors are iid (i.e., the case of (1.45a)), the LESS (2.7) reduces to $\hat{\mu} = \boldsymbol{\tau}^T \boldsymbol{y}/(\boldsymbol{\tau}^T \boldsymbol{\tau})$, which is equivalent to the arithmetic mean. This is easily seen by noting that $\boldsymbol{\tau}^T \boldsymbol{y} = \sum_{i=1}^n y_i$ and $\boldsymbol{\tau}^T \boldsymbol{\tau} = n$. Therefore

$$\hat{\mu} = \frac{\sum y}{n}, \text{ if } \boldsymbol{e} \sim (\boldsymbol{0}, \text{iid}), \tag{2.18}$$

which, obviously, is the formula for the arithmetic mean.

In the case where the random errors have a heteroscedastic distribution (i.e., the case of (1.45b) where the weight matrix P is diagonal), the LESS (2.7) is equivalent to the weighted arithmetic mean, since

$$\hat{\mu} = \frac{\boldsymbol{\tau}^T \text{diag}(p_i, \ldots, p_n) \boldsymbol{y}}{\boldsymbol{\tau}^T \text{diag}(p_i, \ldots, p_n) \boldsymbol{\tau}} = \frac{\sum_{i=1}^n p_i y_i}{\sum_{i=1}^n p_i}, \text{ if } \boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 \text{diag}(1/p_i, \ldots, 1/p_n)). \tag{2.19}$$

2.3 Observation Weighting and Weight Propagation

We start our discussion of observation weighting and weight propagation by showing examples of it. Following that, we give some definitions and rules for general cases.

Assume two measurements y_1 and y_2 with the *same (unknown) expectation* μ and given *variance* (precision) σ^2, i.e.

$$y_i \sim (\mu, \sigma^2) \text{ for } i = 1, 2. \tag{2.20}$$

One of the "most plausible" values for μ as derived from the measurements seems to the the *arithmetic mean*

$$\hat{\mu} := \frac{y_1 + y_2}{2}, \tag{2.21a}$$

which is unbiased since

$$E\{\hat{\mu}\} = \frac{1}{2}\mu + \frac{1}{2}\mu = \mu. \tag{2.21b}$$

Its variance (dispersion) is given by

$$D\{\hat{\mu}\} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{\sigma_1^2}{4} + \frac{\sigma_{12}}{2} + \frac{\sigma_2^2}{4} \tag{2.21c}$$

in general, or

$$D\{\hat{\mu}\} = \frac{\sigma^2}{2}, \text{ assuming } \sigma_{12} := 0 \text{ and } \sigma_1^2 = \sigma_2^2 =: \sigma^2. \tag{2.21d}$$

Now, if the result turns out to be insufficiently precise, i.e. the variance $\sigma^2/2$ is still too large, we are forced to perform a *third measurement* y_3. Assuming *independence* (implying $\sigma_{13} = 0 = \sigma_{23}$) and *identical variance* (implying $\sigma_3^2 = \sigma^2$), we are in the position to from *another arithmetic mean* via

(i) Simply averaging the first result $\hat{\mu}$ with the new observation y_3, i.e.

$$\bar{\hat{\mu}} := \frac{\hat{\mu} + y_3}{2}, \tag{2.22a}$$

which results in an unbiased estimate, since

$$E\{\bar{\hat{\mu}}\} = \frac{1}{2}\mu + \frac{1}{2}\mu = \mu. \tag{2.22b}$$

Its variance is given by

$$D\{\bar{\hat{\mu}}\} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \sigma^2 \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{4} \right) = \frac{3\sigma^2}{8}. \tag{2.22c}$$

(ii) Or we may use the arithmetic mean of all three observations via:

$$\hat{\hat{\mu}} := \frac{y_1 + y_2 + y_3}{3}, \tag{2.23a}$$

which is unbiased since

$$E\{\hat{\hat{\mu}}\} = \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{3}\mu = \mu. \tag{2.23b}$$

Its variance is given by

$$D\{\hat{\hat{\mu}}\} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \sigma^2 \left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9} \right) = \frac{\sigma^2}{3}. \tag{2.23c}$$

We see that

$$D\{\hat{\hat{\mu}}\} = \frac{\sigma^2}{3} < \frac{3\sigma^2}{8} = D\{\bar{\hat{\mu}}\}, \tag{2.24}$$

and thus we claim that the estimate $\hat{\hat{\mu}}$ is to be preferred over (is "better than") $\bar{\hat{\mu}}$, since it is more precise, i.e. has smaller variance.

However, we can form a different linear combination of $\hat{\mu}$ and y_3 that will result in $\hat{\hat{\mu}}$, viz.

$$\hat{\hat{\mu}} = \frac{2 \cdot \hat{\mu} + 1 \cdot y_3}{2 + 1}. \tag{2.25a}$$

But, since

$$D\{\hat{\mu}\} = \frac{\sigma^2}{2} \text{ and } D\{y_3\} = \frac{\sigma^2}{1}, \tag{2.25b}$$

we can also write

$$\hat{\hat{\mu}} = \frac{D\{\hat{\mu}\}^{-1}{\cdot}\hat{\mu} + D\{y_3\}^{-1}{\cdot}y_3}{D\{\hat{\mu}\}^{-1} + D\{y_3\}^{-1}}, \tag{2.25c}$$

which is a properly *weighted (arithmetic) mean* of $\hat{\mu}$ and y_3.

Let's take our example one step further by assuming that the third measurement y_3 was performed *twice as precise* as the previous ones, i.e. $\sigma_{y_3} = \sigma/2 \Rightarrow y_3 \sim (\mu, \sigma^2/4)$. The corresponding "most plausible" value of μ would then be the weighted arithmetic mean according to

$$\hat{\hat{\mu}} := \frac{2{\cdot}\hat{\mu} + 4{\cdot}y_3}{2 + 4} = \frac{y_1 + y_2 + 4y_3}{6}, \tag{2.26a}$$

with

$$E\{\hat{\hat{\mu}}\} = \frac{1}{6}\mu + \frac{1}{6}\mu + \frac{4}{6}\mu = \mu, \tag{2.26b}$$

implying that $\hat{\hat{\mu}}$ is an unbiased estimate of μ. Its dispersion is provided by

$$D\{\hat{\hat{\mu}}\} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2/4 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{bmatrix} = \sigma^2\left(\frac{1}{36} + \frac{1}{36} + \frac{4}{9}{\cdot}\frac{1}{4}\right) = \frac{\sigma^2}{6}. \tag{2.26c}$$

Definition: For a set of *uncorrelated random variables* $y_1, \ldots, y_n$, with variances $\sigma_1^2, \ldots, \sigma_n^2$, we define a set of corresponding weights by

$$\boxed{p_j := \frac{\text{const}}{\sigma_j^2} \;\text{ for all } j = 1, \ldots, n,} \tag{2.27}$$

where the constant is to be chosen arbitrarily, but fixed. In this case we obtain the *weight matrix* to be *diagonal* with

$$\boxed{P := \text{diag}(p_1, \ldots, p_n) = \text{const}{\cdot}\,\text{diag}(\sigma_1^{-2}, \ldots, \sigma_n^{-2}) = \text{const}{\cdot}\Sigma^{-1}.} \tag{2.28}$$

Definition: The arbitrarily chosen *constant* σ_0^2 is called *variance component* (or *variance of unit weight* by some authors), yielding the identities

$$\boxed{P := \sigma_0^2{\cdot}\Sigma^{-1} =: Q^{-1} \Leftrightarrow \Sigma = \sigma_0^2 Q = \sigma_0^2 P^{-1},} \tag{2.29}$$

with Q as $n \times n$ *cofactor matrix*.

Remarks:

(i) The variance component σ_0^2 is unitless by definition.

(ii) The preceding definition (2.29) is general enough for non-diagonal matrices $\Sigma = D\{\boldsymbol{y}\}$, or *correlated random variables* $y_1, \ldots, y_n$, respectively.

2.3.1   Choice of Best Weights

If we choose weights according to the rule (2.29), is that the *best* we can do? By
best we mean a choice of weights that lead to a minimum variance for the estimate
of μ. We also want to ensure that $\hat{\mu}$ remains unbiased. With these objectives in
mind, consider the following:

(i) The *weighted (or general) arithmetic mean*

$$\bar{\mu} := \sum_{j=1}^{n} \gamma_j y_j \quad \text{with} \quad \sum_{j=1}^{n} \gamma_j = 1, \tag{2.30a}$$

for $y_j \sim (\mu, \sigma_j^2)$ being *mutually uncorrelated, i.e., $C\{y_i, y_j\} = 0$, is unbiased
since*

$$E\{\bar{\mu}\} = \sum_{j=1}^{n} \gamma_j E\{y_j\} = \mu \cdot \sum_{j=1}^{n} \gamma_j = \mu. \tag{2.30b}$$

> This shows that *all* weighted averages are unbiased, implying that over
> infinitely many measurements they would provide the *true* value for μ.

(ii) The "best variance" of a weighted mean is determined by solving the following
minimization problem.

$$D\{\bar{\mu}\} = \sum_{j=1}^{n} \gamma_j^2 \sigma_j^2 = \min_{\gamma_j} \{\sum_{j=1}^{n} \gamma_j = 1\}. \tag{2.31a}$$

The *Lagrange function*

$$\Phi(\gamma_j, \lambda) := \sum_{j=1}^{n} \gamma_j^2 \sigma_j^2 - 2\lambda \cdot (\sum_{j=1}^{n} \gamma_j - 1) = \underset{\gamma_j, \lambda}{\text{stationary}} \tag{2.31b}$$

is formed for minimization of Φ, with λ introduced as a Lagrange multiplier.
The Euler-Lagrange necessary conditions

$$\frac{1}{2} \frac{\partial \Phi}{\partial \gamma_j} = \sigma_j^2 \gamma_j - \lambda \doteq 0, \quad \text{for all } j \tag{2.31c}$$

$$\frac{1}{2} \frac{\partial \Phi}{\partial \lambda} = -\sum_{j=1}^{n} \gamma_j + 1 \doteq 0 \tag{2.31d}$$

lead to a minimum of Φ, since the sufficient condition

$$\frac{1}{2} \frac{\partial^2 \Phi}{\partial \gamma_j^2} = \sigma_j^2 > 0 \tag{2.31e}$$

is fulfilled.

Equation (2.31c) implies that

$$\lambda = \sigma_j^2 \gamma_j \ \text{ for } all \ j = 1, \dots, n \tag{2.31f}$$

further implying that

$$\sigma_j^2 \gamma_j = \text{const} \Rightarrow \gamma_j = \frac{\text{const}}{\sigma_j^2}. \tag{2.31g}$$

From (2.31d), we have

$$1 = \sum_{j=1}^{n} \gamma_j = \text{const} \cdot \sum_{j=1}^{n} \sigma_j^{-2} \Rightarrow \text{const} = \Big(\sum_{j=1}^{n} \sigma_j^{-2}\Big)^{-1}, \tag{2.31h}$$

further implying that

$$\gamma_j = \frac{\sigma_j^{-2}}{\sum_{i=1}^{n} \sigma_i^{-2}}, \tag{2.31i}$$

which leads to

$$\boxed{\gamma_j = \frac{p_j}{\sum p_i} \ \text{ for } p_j := \frac{1}{\sigma_j^2}.} \tag{2.31j}$$

as an expression for the jth weight γ_j.

Therefore, we can answer the question at the beginning of this section by saying

> If we choose the weights according to rule (2.29), we obtain that *weighted average* having a *minimum variance*, i.e. that which extracts the information out of the measurements in the "best" way.

2.3.2 Examples for Weighting

The following examples illustrate how weights are chosen as the reciprocals of variances when working with quantities that have been derived from observations and that might be combined with other data in an adjustment problem.

1. Leveling:

   Let σ^2 be the *variance* of one leveling setup between consecutive turning points. Then, assuming n different setups for the entire leveling run, we find the *height difference*

   $$H_n - H_0 := (H_n - H_{n-1}) + \dots + (H_{j+1} - H_j) + \dots + (H_1 - H_0) = \sum_{j=1}^{n} h_j,$$

   $$\text{for } h_j := H_j - H_{j-1}.$$
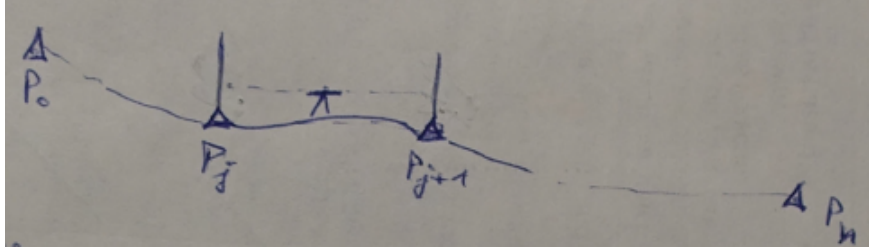
Figure 2.2: A single leveling run from point P_0 to point P_n

Further assuming *uncorrelated* observations, with variance $D\{h_j\} := \sigma^2$ for all $j = 1, \ldots, n$, the *law of error propagation* gives

$$D\{H_n - H_0\} = \sigma^2 \cdot n.$$

Assuming *equal intervals* of length s_0 between consecutive turning points, we find the equivalent expression

$$D\{H_n - H_0\} = (\sigma^2 s_0^{-1}) \cdot S,$$

if S is the *distance* between point P_0 and P_n along the leveling run, implying that the weights are defined by one over the overall distance S, i.e.

$$p := S^{-1}.$$

Here we assume that the interval s_0 is constant among all other leveling runs that may be combined in an adjustment, which is a common case when surveying standards are being adhered to. Thus the term $\sigma^2 s_0^{-1}$ is taken to be a constant "reference variance," and the weighting depends only on the length of the leveling run, which agrees with experience and intuition that suggests longer runs are less precise (thus lower weight) than shorter ones.

2. Horizontal directions: Let φ_j be the average of a set of measured directions to target j and n be the *number of rounds* (or sets) of directions measured. Further assume that the individual directions are uncorrelated and have variance σ^2. Then we find

$$D\{\varphi_j\} = \sigma^2 \cdot n^{-1}$$

as the variance of the averaged direction and

$$p_j := n$$

as its corresponding weight. This agrees with experience and intuition that suggests that the more rounds that are measured, the greater the weight to be assigned when combined with other data in an adjustment.

Note, however, that angles from the same round are *correlated*, since they are essentially differences between two directions.

3. Electronic distance measurements: If S_j is a measured distance and ρ_1 and ρ_2 are coefficients from a calibration of the instrument ($\rho_1 > 0$, $\rho_2 > 0$), then the variance of S_j is

$$D\{S_j\} = \sigma_0^2(\rho_1 + \rho_2 \cdot S_j^2),$$

which implies that the corresponding weight is defined by

$$p_j := \frac{1}{\rho_1 + \rho_2 \cdot S_j^2}.$$

2.4 Estimated Variance Component

The variance component σ_0^2 is an unknown quantity in model (2.1). However, it can be estimated as a function of the P-weighted norm of the residual vector $\tilde{e}$ and can be used as a "goodness of fit statistic," a concept discussed in Section 9.4. The estimated variance component is derived as follows:

The LEast-Squares Solution (LESS) within the model of direct observations is shown in (2.7) as

$$\hat{\mu} = \frac{\boldsymbol{\tau}^T P \boldsymbol{y}}{\boldsymbol{\tau}^T P \boldsymbol{\tau}} = \frac{\boldsymbol{\tau}^T \Sigma^{-1} \boldsymbol{y}}{\boldsymbol{\tau}^T \Sigma^{-1} \boldsymbol{\tau}}, \qquad (2.32a)$$

so that the P-weighted norm of the residual vector

$$\tilde{e}^T P \tilde{e} = \|\boldsymbol{y} - \boldsymbol{\tau} \cdot \hat{\mu}\|_P^2 = (\boldsymbol{y} - \boldsymbol{\tau} \cdot \hat{\mu})^T P(\boldsymbol{y} - \boldsymbol{\tau} \cdot \hat{\mu}) \qquad (2.32b)$$

is a *random variable* with expectation

$$
\begin{aligned}
E\{(\boldsymbol{y} - \boldsymbol{\tau} \cdot \hat{\mu})^T P(\boldsymbol{y} - \boldsymbol{\tau} \cdot \hat{\mu})\} = & \qquad\qquad\qquad (2.33a)\\
= E\{\boldsymbol{y}^T P \boldsymbol{y}\} - E\{\boldsymbol{y}^T P \boldsymbol{\tau} \cdot \hat{\mu}\} - \cancel{E\{\hat{\mu} \cdot \boldsymbol{\tau}^T P \boldsymbol{y}\}} & + \cancel{E\{\hat{\mu}^2 \cdot \boldsymbol{\tau}^T P \boldsymbol{\tau}\}} = \\
= \operatorname{tr} E\{P \boldsymbol{y} \boldsymbol{y}^T\} - (\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1} \cdot \operatorname{tr} & E\{\boldsymbol{\tau}^T P \boldsymbol{y} \boldsymbol{y}^T P \boldsymbol{\tau}\} = \\
= \operatorname{tr}\big[P \cdot E\{\boldsymbol{y} \boldsymbol{y}^T\}\big] - (\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1} \cdot \operatorname{tr} & \big[P \boldsymbol{\tau} \boldsymbol{\tau}^T P \cdot E\{\boldsymbol{y} \boldsymbol{y}^T\}\big] = \\
= \operatorname{tr}\big[P \cdot D\{\boldsymbol{y}\}\big] + \operatorname{tr}\big[P \cdot E\{\boldsymbol{y}\} & E\{\boldsymbol{y}\}^T\big] - \\
- \operatorname{tr}\big[P \boldsymbol{\tau}(\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1} \boldsymbol{\tau}^T P(\sigma_0^2 P^{-1})\big] & - \operatorname{tr}\big[P \boldsymbol{\tau}(\boldsymbol{\tau}^T P \boldsymbol{\tau})^{-1} \boldsymbol{\tau}^T P \boldsymbol{\tau} \mu^2 \boldsymbol{\tau}^T\big] = \\
= \sigma_0^2 \operatorname{tr} I_n + \cancel{\mu^2 \cdot \boldsymbol{\tau}^T P \boldsymbol{\tau}} - \sigma_0^2 & \cdot \frac{\boldsymbol{\tau}^T \cancel{P} \cancel{\boldsymbol{\tau}}}{\cancel{\boldsymbol{\tau}^T} \cancel{P} \cancel{\boldsymbol{\tau}}} - \cancel{\mu^2 \cdot \boldsymbol{\tau}^T P \boldsymbol{\tau}} = \\
= \sigma_0^2(n - 1) \Rightarrow & \\
E\{(n - 1)^{-1}(\boldsymbol{y} - \boldsymbol{\tau} \cdot \hat{\mu})^T P(\boldsymbol{y} - \boldsymbol{\tau} \cdot \hat{\mu})\} = & \sigma_0^2 \qquad\qquad (2.33b)
\end{aligned}
$$

The quantity $n - 1$ is called the *redundancy* or *degrees of freedom* of the model.

Now, we may take the argument of the expectation shown in (2.33b) and assign it the symbol $\hat{\sigma}_0^2$, implying that

$$\hat{\sigma}_0^2 = (n-1)^{-1}(\boldsymbol{y} - \boldsymbol{\tau}\cdot\hat{\mu})^T P(\boldsymbol{y} - \boldsymbol{\tau}\cdot\hat{\mu}) = \tag{2.34a}$$

$$= (n-1)^{-1}(\boldsymbol{y}^T P \boldsymbol{y} - \hat{\mu}\cdot\boldsymbol{\tau}^T P \boldsymbol{y}) = \tag{2.34b}$$

$$= \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{n-1}, \tag{2.34c}$$

which is an *unbiased estimate* of σ_0^2, since $E\{\hat{\sigma}_0^2\} = \sigma_0^2$.

Remark   In fact, $\hat{\sigma}_0^2$ is the "best" in a certain class of quadratic unbiased estimates of σ_0^2 (being invariant with respect to translations in μ) and has — under normality assumptions — dispersion

$$D\{\hat{\sigma}_0^2\} = 2(\sigma_0^2)^2(n-1)^{-1} = \text{MSE}\{\hat{\sigma}_0^2\}. \tag{2.35}$$

In summary, we can write the so-called *sum of squared residuals (SSR)* as

$$\Omega := \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}, \tag{2.36}$$

which, together with the redundancy of the model

$$r := n - 1, \tag{2.37}$$

comprises the formula

$$\hat{\sigma}_0^2 := \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{r} \tag{2.38}$$

for the estimated variance component.

2.5   Computation Checks and an Example

2.5.1   Checks on Residuals

A statistical analysis of the results of various adjustment computations is the subject of Chapter 9, where tests for goodness of fit, detection of outliers, and for particular values of the estimated parameters are presented. But even before statistical analysis is employed, certain checks should be made on the residuals to confirm that they look reasonable and to assure that the computations were made correctly. Below is a minimal list of checks that should be made after an adjustment computation.

1. Inspect the elements of the residual vector $\tilde{\boldsymbol{e}}$ to make sure they look reasonable. As a general rule, if a residual is much greater than three times the square root of the variance of the corresponding observation, one might ought to question the accuracy of that observation or its variance. In that case,

the corresponding observation could be temporarily removed and the adjustment computed again. Then, a residual could be *predicted* for the removed observation and the results inspected to decide if the observation should be retained or not. The method of outlier detection described in Section 9.7 is meant to help lead to decisions about the accuracy of a suspect observation and whether or not to admit it in the final adjustment.

2. Consider the magnitude of the estimated variance component $\hat{\sigma}_0^2$. Is it close to the value you expect it to take on (perhaps 1)? If it varies largely from the value you expect it to take on, it will generally indicate that either 1) the observational model is inaccurate, or 2) the weights (or variances) have not been accurately specified, or both.

   In the case of 1, the model may need to be revised to include more parameters so that the parameters of the model more accurately explain the observations. (Of course, then we would no longer have a model of direct observations with a single parameter μ.) Or, it could be that the observations contain some systematic errors that need to be removed so that the assumption that $E\{\boldsymbol{y}\} = \boldsymbol{\tau}\mu$ is made valid.

   In the case of 2, a relative small value of $\hat{\sigma}_0^2$ suggests that the specified observational variances (reflected in the cofactor matrix $Q = P^{-1}$) were too large (i.e. the observations are more precise than reflected in the cofactor matrix). Conversely, if $\hat{\sigma}_0^2$ turns out to be relatively large, the specified variances in Q might be too small (i.e. the observations are less precise than reflected in the cofactor matrix).

3. Provided the model redundancy is large enough, say greater than 10 or 20, we might expect that approximately half the residuals will be negative and about half positive. Certainly this would be the case if the random observation errors turned out to be normally distributed. So, it is prudent to check the ratio of negative to positive residuals and make sure the ratio is not greatly different than 1. Note that this check might not apply for adjustments within the model of condition equations discussed in Chapter 4.

4. Going beyond the previous item, if the redundancy is large enough, say greater than 10 or 20, a histogram of the residuals should be plotted to check how closely its shape resembles the pdf curve of a normal distribution, if it is assumed that the random observation errors are approximately normally distributed.

5. Compute the estimated variance component both by (2.34b) and (2.34c) and make sure they are equivalent up to the precision of the computations.

6. Compute the trace of the matrix of redundancy numbers as defined in (9.85a) and (9.85b) and confirm that the result is an integer that equals the redundancy of the model r.

2.5.2   Example of LESS Within the Model of Direct Observations

Given the following vector of observations $\boldsymbol{y}$ and its associated dispersion matrix $D\{\boldsymbol{y}\}$, compute the LESS for

1. The estimated parameter $\hat{\mu}$ and its estimated dispersion.

2. The estimated variance component $\hat{\sigma}_0^2$.

3. The vector of predicted residuals $\tilde{\boldsymbol{e}}$ and its estimated dispersion matrix.

$$\boldsymbol{y} = \begin{bmatrix} 100.02\,\text{m} \\ 100.04\,\text{m} \\ 99.97\,\text{m} \end{bmatrix}, \quad D\{\boldsymbol{y}\} = \sigma_0^2 \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{cm}^2.$$

Solution: To simplify the problem somewhat, we may subtract $100\,\text{m}$ from the observations and solve for $\delta\hat{\mu}$ as an intermediate step, working with cm instead of meters. Then, the modified observation vector reads $\boldsymbol{y} \to \boldsymbol{y} = [2\,\text{cm},\ 4\,\text{cm},\ -3\,\text{cm}]^T$. Weight matrix:

$$\Rightarrow P = \begin{bmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} \text{cm}^{-2} = \frac{1}{9} \cdot \begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{cm}^{-2}$$

Estimated parameter and its estimated variance:

$$\delta\hat{\mu} = \frac{\boldsymbol{\tau}^T P \boldsymbol{y}}{\boldsymbol{\tau}^T P \boldsymbol{\tau}} = \frac{\left[2/3 \mid 2/3 \mid 1/9 \right] \boldsymbol{y}}{\left[2/3 \mid 2/3 \mid 1/9 \right] \boldsymbol{\tau}} = \frac{(11/3)\,\text{cm}^{-1}}{(13/9)\,\text{cm}^{-2}} = \frac{33}{13}\text{cm}$$

$$\Rightarrow \boxed{\hat{\mu} = 100\,\text{m} + \delta\hat{\mu} = 100.0254\,\text{m}}$$

$$\text{with} \quad \boxed{D\{\hat{\mu}\} = \frac{\sigma_0^2}{\boldsymbol{\tau}^T P \boldsymbol{\tau}} = \frac{9}{13}\sigma_0^2\,\text{cm}^2}$$

$$\text{Also} \quad \boldsymbol{y}^T P \boldsymbol{y} = \left[0 \mid 12/3 \mid -1/3 \right] \boldsymbol{y} = 17 = 221/13$$

$$\text{and} \quad \delta\hat{\mu} \cdot \boldsymbol{\tau}^T P \boldsymbol{y} = \left(\frac{33}{13}\right) \cdot \left(\frac{11}{3}\right) = \frac{121}{13} \Rightarrow \boxed{\hat{\sigma}_0^2 = \frac{100/13}{2} = \frac{50}{13}}$$

$$\Rightarrow \hat{D}\{\hat{\mu}\} = \left(\frac{9}{13}\right) \cdot \left(\frac{50}{13}\right) = \frac{450}{169} = 2.66\,\text{cm}^2$$

$$\boxed{\hat{\mu} = (100.025 \pm 0.016)\ \text{m}}$$

Predicted residuals and their estimated covariance matrix:

$$\boxed{\tilde{\boldsymbol{e}} = \boldsymbol{y} - \boldsymbol{\tau}\hat{\mu} = \left[-0.5385 \mid +1.4615 \mid -5.5385 \right]^T \text{cm}}$$

$$\text{with} \quad D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \left(\begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 9/13 & 9/13 & 9/13 \\ 9/13 & 9/13 & 9/13 \\ 9/13 & 9/13 & 9/13 \end{bmatrix} \right) \cdot \text{cm}^2 \Rightarrow$$

$$\boxed{D\{\tilde{\boldsymbol{e}}\} = \frac{\sigma_0^2}{13} \begin{bmatrix} 4 & -5/2 & -9 \\ -5/2 & 4 & -9 \\ -9 & -9 & 108 \end{bmatrix} \cdot \text{cm}^2}$$

Checks:

$$P\tilde{\boldsymbol{e}} = \begin{bmatrix} -1.6923 \\ +2.3077 \\ -0.6154 \end{bmatrix} \Rightarrow \boldsymbol{\tau}^T P\tilde{\boldsymbol{e}} = -0.00006 \checkmark \text{ and } \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} = 7.69 \approx 100/13 = \hat{\sigma}_0^2 \cdot 2 \checkmark$$

"Redundancy numbers" (see (9.85a) and (9.85b) for definition of redundancy numbers)

$$D\{\tilde{\boldsymbol{e}}\}\cdot P = \frac{\sigma_0^2}{13} \begin{bmatrix} 7 & -6 & -1 \\ -6 & 7 & -1 \\ -6 & -6 & 12 \end{bmatrix} \Rightarrow \text{tr}(D\{\tilde{\boldsymbol{e}}\}\cdot P)/\sigma_0^2 = \frac{26}{13} = 2 = 3 - 1 = r \checkmark$$

2.6 Best Linear Uniformly Unbiased Estimate

Here we take a statistical approach to estimating the unknown parameter μ. We want to find an estimate for μ, expressed as a linear combination of the observations $\boldsymbol{y}$, that extracts the "best" information from the data. The estimate is denoted by $\hat{\mu}$ and is characterized as the *Best Linear Uniformly Unbiased Estimate* (BLUUE) of μ. The three criteria used to derive the BLUUE are described as follows:

1. Linear criterion: The linear criterion states that the estimated parameter must be a linear combination of the data contained in $\boldsymbol{y}$, i.e.

$$\hat{\mu} = \boldsymbol{\alpha}^T \boldsymbol{y}, \tag{2.39a}$$

   where $\boldsymbol{\alpha}$ is an unknown vector to be determined.

2. Uniformly Unbiased criteria: An unbiased estimator is one for which its expectation is equal to the true, but unknown, quantity it estimates. Stated mathematically,

$$\mu = E\{\hat{\mu}\} = E\{\boldsymbol{\alpha}^T \boldsymbol{y}\} = \boldsymbol{\alpha}^T E\{\boldsymbol{y}\} = \boldsymbol{\alpha}^T E\{\boldsymbol{\tau}\mu + \boldsymbol{e}\} = \boldsymbol{\alpha}^T \boldsymbol{\tau}\mu, \text{ for } all \ \mu \in \mathbb{R},$$

   which implies

$$\boldsymbol{\alpha}^T \boldsymbol{\tau} = 1. \tag{2.39b}$$

   Requiring this condition to hold for *all* $\mu \in \mathbb{R}$ satisfies the "uniform" criterion, whereas the requirement that $\boldsymbol{\alpha}^T \boldsymbol{\tau} = 1$ satisfies the "unbiased" criterion.

3. Best criterion: The best criterion requires minimum MSE($\hat{\mu}$), or, equivalently, minimum dispersion, *since $\hat{\mu}$ is unbiased.* Mathematically, the criterion reads

$$\min D\{\hat{\mu}\}, \;\; \text{where } D\{\hat{\mu}\} = D\{\boldsymbol{\alpha}^T \boldsymbol{y}\} = \boldsymbol{\alpha}^T D\{\boldsymbol{y}\}\boldsymbol{\alpha} \Rightarrow$$

$$\min D\{\hat{\mu}\} = \sigma_0^2 \boldsymbol{\alpha}^T Q \boldsymbol{\alpha}, \;\; \text{subject to } \boldsymbol{\tau}^T \boldsymbol{\alpha} = 1. \tag{2.39c}$$

Accordingly, a Lagrange target function is formed by

$$\Phi(\boldsymbol{\alpha}, \lambda) := \boldsymbol{\alpha}^T Q \boldsymbol{\alpha} + 2\lambda(\boldsymbol{\tau}^T \boldsymbol{\alpha} - 1). \tag{2.40}$$

The necessary conditions for stationarity are provided by the Euler-Lagrange equations, which are written as

$$\frac{1}{2} \frac{\partial \Phi}{\partial \boldsymbol{\alpha}} = Q\hat{\boldsymbol{\alpha}} + \boldsymbol{\tau}\hat{\lambda} \doteq 0, \tag{2.41a}$$

$$\frac{1}{2} \frac{\partial \Phi}{\partial \lambda} = \boldsymbol{\tau}^T \hat{\boldsymbol{\alpha}} - 1 \doteq 0. \tag{2.41b}$$

The sufficient condition for minimization is satisfied by $\partial^2 \Phi/(\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}^T) = 2Q$, which is a positive-definite matrix according to (2.1). Solving (2.41a) and (2.41b) simultaneously yields

$$\hat{\boldsymbol{\alpha}} = -Q^{-1}\boldsymbol{\tau}\hat{\lambda} = -P\boldsymbol{\tau}\hat{\lambda} \;\; \text{using (2.41a)}, \tag{2.42a}$$

$$1 = \boldsymbol{\tau}^T \hat{\boldsymbol{\alpha}} = -\boldsymbol{\tau}^T P\boldsymbol{\tau}\hat{\lambda} \Rightarrow \hat{\lambda} = \frac{-1}{\boldsymbol{\tau}^T P\boldsymbol{\tau}} \;\; \text{using (2.41b) and (2.42a).} \tag{2.42b}$$

Substituting (2.42b) into (2.42a) we get

$$\hat{\boldsymbol{\alpha}} = (\boldsymbol{\tau}^T P\boldsymbol{\tau})^{-1} P\boldsymbol{\tau}. \tag{2.42c}$$

Finally, substituting the transpose of (2.42c) into the linear requirement $\hat{\mu} = \boldsymbol{\alpha}^T \boldsymbol{y}$ yields the BLUUE of μ as

$$\boxed{\hat{\mu} = \frac{\boldsymbol{\tau}^T P\boldsymbol{y}}{\boldsymbol{\tau}^T P\boldsymbol{\tau}}.} \tag{2.43}$$

Equation (2.43) agrees with (2.7) derived for LESS. Thus we see that the LESS and the BLUUE are equivalent within the model of direct observations.

We may also prove mathematically that (2.43) fulfills the weighted LESS principle by showing that the P-weighted residual norm $\tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}}$ for any other solution is larger than that obtained via BLUUE, which we do in the following: Suppose $\hat{\hat{\mu}}$ is any other estimate for μ, then

$$\tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} = (\boldsymbol{y} - \boldsymbol{\tau}\hat{\hat{\mu}})^T P(\boldsymbol{y} - \boldsymbol{\tau}\hat{\hat{\mu}}) =$$

$$= [(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}) - \boldsymbol{\tau}(\hat{\hat{\mu}} - \hat{\mu})]^T P[(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}) - \boldsymbol{\tau}(\hat{\hat{\mu}} - \hat{\mu})] =$$

$$= (\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu})^T P(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}) - 2(\hat{\hat{\mu}} - \hat{\mu})\underbrace{\boldsymbol{\tau}^T P(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu})}_{0} + (\boldsymbol{\tau}^T P\boldsymbol{\tau})(\hat{\hat{\mu}} - \hat{\mu})^2 =$$

$$= (\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu})^T P(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}) + (\boldsymbol{\tau}^T P\boldsymbol{\tau})(\hat{\hat{\mu}} - \hat{\mu})^2 \geq$$

$$\geq \left(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}\right)^T P\left(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}\right) = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$$
Q.E.D.

We have used the P-orthogonality relation (2.11) in the third line of the proof.

Let us briefly summarize these results by stating three important properties of the least-squares solution (LESS) of the unknown parameter μ within the model of direct observations.

The LESS (equivalently BLUUE) within the model of direct observations provides

1. An unbiased estimate $\hat{\mu}$ of the unknown parameter μ, i.e. $E\{\hat{\mu}\} = \mu$.

2. A minimum P-weighted norm of the residual vector, i.e. $\Omega := \|\tilde{\boldsymbol{e}}\|_P^2$ is minimized.

3. A minimum variance (dispersion) $D\{\hat{\mu}\}$.

2.7 Effects of a Wrongly Chosen Weight Matrix in the Model of Direct Observations

Assume that the weight matrix P has been wrongly chosen by an amount δP, where δP is assumed to be a small, positive-(semi)definite matrix that is *uncorrelated* with P. (Apparently δP itself would not have to be positive-(semi)definite as long as the sum $(P + \delta P)$ is positive definite.) Consequently, we have

$$P \to (P + \delta P) \Rightarrow \hat{\mu} \to (\hat{\mu} + \delta\hat{\mu}),\ D\{\hat{\mu}\} \to D\{\hat{\mu} + \delta\hat{\mu}\},\ \text{and}\ \hat{\sigma}_0^2 \to \hat{\sigma}_0^2 + \delta\hat{\sigma}_0^2.$$
$$(2.44)$$

2.7.1 Effect on the Parameter Estimate

The following shows the effect of a wrongly chosen weight matrix on the estimated parameter $\hat{\mu}$:

$$\left(\hat{\mu} + \delta\hat{\mu}\right) = \frac{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{y}}{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}} \Rightarrow$$

$$\delta\hat{\mu} = \frac{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{y}}{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}} - \hat{\mu} = \frac{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{y}}{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}} \cdot \frac{\boldsymbol{\tau}^T P\boldsymbol{\tau}}{\boldsymbol{\tau}^T P\boldsymbol{\tau}} - \left(\frac{\boldsymbol{\tau}^T P\boldsymbol{y}}{\boldsymbol{\tau}^T P\boldsymbol{\tau}}\right) \cdot \frac{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}}{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}} =$$

$$= \frac{\boldsymbol{\tau}^T P\boldsymbol{y}\boldsymbol{\tau}^T P\boldsymbol{\tau} + \boldsymbol{\tau}^T \delta P\boldsymbol{y}\boldsymbol{\tau}^T P\boldsymbol{\tau} - \boldsymbol{\tau}^T P\boldsymbol{y}\boldsymbol{\tau}^T P\boldsymbol{\tau} - \boldsymbol{\tau}^T P\boldsymbol{y}\boldsymbol{\tau}^T \delta P\boldsymbol{\tau}}{(\boldsymbol{\tau}^T P\boldsymbol{\tau})\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}} =$$

$$= \frac{\boldsymbol{\tau}^T \delta P\boldsymbol{y}}{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}} - \frac{\boldsymbol{\tau}^T \delta P\boldsymbol{\tau}\hat{\mu}}{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}} = \frac{\boldsymbol{\tau}^T \delta P\left(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}\right)}{\boldsymbol{\tau}^T(P + \delta P)\boldsymbol{\tau}}$$

Finally, we arrive at

$$\delta\hat{\mu} = \frac{\boldsymbol{\tau}^T \delta P}{\boldsymbol{\tau}^T (P + \delta P)\boldsymbol{\tau}} \tilde{e}. \qquad (2.45)$$

2.7.2   Effect on the Cofactor Matrix for the Estimated Parameter

The following shows the effect of a wrongly chosen weight matrix on the cofactor matrix $Q_{\hat{\mu}}$ for the estimated parameter $\hat{\mu}$, where $D\{\hat{\mu}\} = \sigma_0^2 Q_{\hat{\mu}}$ is the dispersion of $\hat{\mu}$:

$$\delta Q_{\hat{\mu}} = \left(Q_{\hat{\mu}} + \delta Q_{\hat{\mu}}\right) - Q_{\hat{\mu}} = \frac{1}{\boldsymbol{\tau}^T (P + \delta P)\boldsymbol{\tau}} - \frac{1}{\boldsymbol{\tau}^T P \boldsymbol{\tau}} =$$

$$= \frac{\boldsymbol{\tau}^T P \boldsymbol{\tau} - \boldsymbol{\tau}^T (P + \delta P)\boldsymbol{\tau}}{\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)\boldsymbol{\tau}^T (P + \delta P)\boldsymbol{\tau}} = \frac{-\boldsymbol{\tau}^T \delta P \boldsymbol{\tau}}{\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)\boldsymbol{\tau}^T (P + \delta P)\boldsymbol{\tau}}.$$

Thus we have

$$\delta Q_{\hat{\mu}} = -\frac{\boldsymbol{\tau}^T \delta P \boldsymbol{\tau}}{\boldsymbol{\tau}^T (P + \delta P)\boldsymbol{\tau}} Q_{\hat{\mu}}. \qquad (2.46)$$

2.7.3   Effect on the Estimated Variance Component

The following shows the effect of a wrongly chosen weight matrix on the estimated variance component: First note that

$$\tilde{e}^T P \tilde{e} = \left(\boldsymbol{y}^T - \hat{\mu}\boldsymbol{\tau}^T\right) P\left(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}\right) =$$

$$= \boldsymbol{y}^T P(\boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}) - \hat{\mu}\left(\boldsymbol{\tau}^T P \boldsymbol{y} - \boldsymbol{\tau}^T P \boldsymbol{\tau} \frac{\boldsymbol{\tau}^T P \boldsymbol{y}}{\boldsymbol{\tau}^T P \boldsymbol{\tau}}\right) =$$

$$= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{y}^T P \boldsymbol{\tau}\hat{\mu} = \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{\tau}^T P \boldsymbol{y}\hat{\mu} = \boldsymbol{y}^T P \boldsymbol{y} - \hat{\mu}^2 \boldsymbol{\tau}^T P \boldsymbol{\tau}.$$

Following the above logic, we have

$$(n-1)\left(\hat{\sigma}_0^2 + \delta\hat{\sigma}_0^2\right) = \boldsymbol{y}^T (P + \delta P)\boldsymbol{y} - \boldsymbol{\tau}^T (P + \delta P)\boldsymbol{y}\left(\hat{\mu} + \delta\hat{\mu}\right) \Rightarrow$$

$$\Rightarrow (n-1)\delta\hat{\sigma}_0^2 = \boldsymbol{y}^T (P + \delta P)\boldsymbol{y} - \boldsymbol{\tau}^T (P + \delta P)\boldsymbol{y}(\hat{\mu} + \delta\hat{\mu}) - \boldsymbol{y}^T P \boldsymbol{y} + (\boldsymbol{\tau}^T P \boldsymbol{y})\hat{\mu} =$$

(Note: the last term will cancel one of the four terms in the binomial product.)

$$= \boldsymbol{y}^T (\delta P)\boldsymbol{y} - \boldsymbol{\tau}^T \delta P \boldsymbol{y}\left(\hat{\mu} + \delta\hat{\mu}\right) - \left(\boldsymbol{\tau}^T P \boldsymbol{y}\right)\delta\hat{\mu} =$$

$$= \boldsymbol{y}^T (\delta P)\boldsymbol{y} - \hat{\mu}\boldsymbol{\tau}^T (\delta P)\boldsymbol{y} - \boldsymbol{\tau}^T (P + \delta P)\boldsymbol{y}\delta\hat{\mu} =$$

$$= \left(\boldsymbol{y}^T - \hat{\mu}\boldsymbol{\tau}^T\right)(\delta P)\boldsymbol{y} - \boldsymbol{\tau}^T (P + \delta P)\boldsymbol{y}\delta\hat{\mu} =$$

$$= \tilde{e}^T (\delta P)\boldsymbol{y} - \frac{\boldsymbol{\tau}^T (P + \delta P)\boldsymbol{y}}{\boldsymbol{\tau}^T (P + \delta P)\boldsymbol{\tau}} \boldsymbol{\tau}^T (\delta P)\tilde{e} =$$

(Note that the previous results for $\delta\hat{\mu}$ have been substituted in the line above.)

$$= \boldsymbol{y}^T(\delta P)\tilde{\boldsymbol{e}} - (\hat{\mu} + \delta\hat{\mu})\boldsymbol{\tau}^T(\delta P)\tilde{\boldsymbol{e}} =$$

$$(\text{Using } \boldsymbol{y}^T(\delta P)\tilde{\boldsymbol{e}} = (\hat{\mu}\boldsymbol{\tau}^T + \tilde{\boldsymbol{e}}^T)\delta P\tilde{\boldsymbol{e}} = \tilde{\boldsymbol{e}}^T\delta P\tilde{\boldsymbol{e}} + \hat{\mu}\boldsymbol{\tau}^T\delta P\tilde{\boldsymbol{e}})$$

$$= \tilde{\boldsymbol{e}}^T(\delta P)\tilde{\boldsymbol{e}} - \delta\hat{\mu}\boldsymbol{\tau}^T(\delta P)\tilde{\boldsymbol{e}} \Rightarrow$$

$$(n-1)\delta\hat{\sigma}_0^2 = \tilde{\boldsymbol{e}}^T(\delta P)\tilde{\boldsymbol{e}} - (\delta\hat{\mu})^2\boldsymbol{\tau}^T(P+\delta P)\boldsymbol{\tau}$$

Finally, we arrive at

$$\delta\hat{\sigma}_0^2 = \frac{1}{n-1}\big[\tilde{\boldsymbol{e}}^T(\delta P)\tilde{\boldsymbol{e}} - (\delta\hat{\mu})^2\boldsymbol{\tau}^T(P+\delta P)\boldsymbol{\tau}\big]. \tag{2.47}$$

2.7.4   Effect on the Estimated Dispersion

The the effect of a wrongly chosen weight matrix on the estimated dispersion of $\hat{\mu}$ is obviously given by

$$\hat{D}\{\hat{\mu} + \delta\hat{\mu}\} = (\hat{\sigma}_0^2 + \delta\hat{\sigma}_0^2)D\{\hat{\mu} + \delta\hat{\mu}\} = (\hat{\sigma}_0^2 + \delta\hat{\sigma}_0^2)(Q_{\hat{\mu}} + \delta Q_{\hat{\mu}}). \tag{2.48}$$

2.8   Practice Problems

1. Show that the LESS of (2.7) is an unbiased estimate of μ.

2. Show that the residual vector of (2.8a) is an unbiased prediction of $\boldsymbol{e}$.

3. Consider the problem of repeated measurements where an unknown distance μ between two points was directly observed n times. The observations are collected in the vector $\boldsymbol{y} = [y_1, y_2, \ldots, y_n]^T$. The distribution of their random errors is described by $\boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2\sigma^2 I_n)$; furthermore $E\{\boldsymbol{y}\} = \boldsymbol{\tau}\mu$.

   (a) If the random variable z is defined by $z = (y_1 + y_2 + \ldots + y_n)/n$, show that $E\{z\} = E\{\hat{\mu}\}$ as shown in (2.7) and that $D\{z\} = D\{\hat{\mu}\}$ as shown in (2.14).

   (b) Assuming that $\sigma_0^2 = 1$ and $\sigma^2 = 1\,\text{cm}^2$, graph the dispersion of $\hat{\mu}$ as a function of the number of observations n from $n = 2$ to $n = 100$.

   (c) Now suppose that there is correlation between successive observation errors described by the relations

   $$\rho_{e_i,e_{i+1}} = 0.001/\sigma^2, \ \ \rho_{e_i,e_{i+2}} = 0.0008/\sigma^2,$$
   $$\rho_{e_i,e_{i+3}} = -0.00006/\sigma^2, \ \ \text{for } i = 1, \ldots, n-3.$$

   Using assumed values $\sigma_0^2 = 1$ and $\sigma^2 = 1\,\text{cm}^2$, compute $D\{\hat{\mu}\}$ for $n = 100$.

   (d) Repeat item (b) for the case of item (c).

4. Twelve direct observations of one unknown parameter μ are listed in Table 2.1. The first set of five observations (I) were made at one time and have measurement variance $\sigma_I^2 = (0.05)^2$. The second set of seven observations (II) were made at a later time with measurement variance $\sigma_{II}^2 = (0.10)^2$. All random measurement errors are independent. No units are given.

Table 2.1: Twelve direct observations of one unknown parameter μ

		Set I, $\sigma_I^2 = (0.05)^2$				
	y_1	y_2	y_3	y_4	y_5	
	9.99	10.04	9.93	9.88	9.93	

			Set II, $\sigma_{II}^2 = (0.10)^2$			
y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}
10.03	10.04	10.05	9.99	10.02	9.95	10.09

(a) Using only data set I:
    i. Compute the BLUUE (or LESS) $\hat{\mu}$.
    ii. Compute the dispersion $D\{\hat{\mu}\}$ (no hat on D).
    iii. Compute the residual vector $\tilde{e}$.
    iv. Compute the estimated variance component $\hat{\sigma}_0^2$.

(b) Now using only data set II, repeat items i–iv, this time denoting the computed values as $\hat{\hat{\mu}}$, $D\{\hat{\hat{\mu}}\}$, $\tilde{\tilde{e}}$, and $\hat{\hat{\sigma}}_0^2$, respectively.

(c) Based on the discussion above about the weighted arithmetic mean, try to estimate the unknown parameter based on the linear combination

$$\hat{\hat{\hat{\mu}}} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\hat{\mu}} \end{bmatrix},$$

using the dispersions computed in the previous two items to determine the "weights" α_1 and α_2. Repeat items i and ii for this case, this time denoting the computed values as $\hat{\hat{\hat{\mu}}}$ and $D\{\hat{\hat{\hat{\mu}}}\}$, respectively.

(d) Now compute i–iv using all 12 observation simultaneously and compare your results to those computed in the preceding items. Comment on your findings.

5. Stellar observations were made in order to determine the astronomical azimuth between two geodetic control points. Table 2.2 shows the arc-minute and arc-second parts of each observation. The degrees part is 126° for all observations. The observations are considered to be *uncorrelated*. The first 12 observations were determined from sightings on Polaris with a precision of $\sigma_1 = 5''$. The remaining 18 observations were determined by sightings on the Sun with a less precise instrument than that used for the first 12. The precision of these observations is $\sigma_2 = 10''$.

Table 2.2: Observations of the astronomical azimuth (in minutes and seconds of arc) between two points. Add $126°$ to all values.

No.	Direction	No.	Direction	No.	Direction
1	$11'34''$	11	$11'34''$	21	$11'19''$
2	$11'30''$	12	$11'38''$	22	$11'22''$
3	$11'34''$	13	$11'35''$	23	$11'01''$
4	$11'29''$	14	$11'40''$	24	$11'44''$
5	$11'29''$	15	$11'37''$	25	$11'33''$
6	$11'37''$	16	$11'27''$	26	$11'23''$
7	$11'37''$	17	$11'33''$	27	$11'44''$
8	$11'37''$	18	$11'22''$	28	$11'13''$
9	$11'33''$	19	$11'39''$	29	$11'29''$
10	$11'24''$	20	$11'19''$	30	$10'38''$

(a) Compute the LESS within the model of direct observations for the estimated parameter $\hat{\mu}$, its estimated dispersion $\hat{D}\{\hat{\mu}\}$, and the estimated variance component $\hat{\sigma}_0^2$.

(b) Repeat the previous part 30 times ($i = 1, \ldots, 30$), removing only the ith observation each time so that each ith solution is based on 29 observations. Tabulate your results and include in each line the difference between the removed observation y_{removed} and the estimated azimuth $\hat{\mu}_i$; let's refer to it as $e_{\text{predicted}} = y_{\text{removed}} - \hat{\mu}_i$. Highlight the solution that has the largest magnitude for $e_{\text{predicted}}$. Call it solution k for reference in the next part.

(c) Now repeat part (a) using all 30 observations, but this time modify the weight of the observation with the value for $e_{\text{predicted}}$ found in solution k of part (b). Use $1/(e_{\text{predicted}})_k^2$ for the new weight. Compare your solution to solution k from part (b). Are they close? Do you expect them to be? Why or why not?

Which of the 32 solutions that you computed would you adopt as the final solution? Give a justification for your choice.

6. Consider the weight matrix $P := P_{(5.a)}$ used in problem 5.a to have been "wrongly chosen" and the weight matrix used in 5.c to be legitimate. Let δP be their difference such that $P_{(5.c)} = P + \delta P$.

Compute the effects of the wrongly chosen weight matrix on the estimated parameter $\hat{\mu}$, its estimated dispersion $\hat{D}\{\hat{\mu}\}$, and the estimated variance component $\hat{\sigma}_0^2$.

Note that the root problem with 5.a is that its last observation appears to be an outlier, not that the weights were necessarily "chosen wrongly." However, it seems that the problem can be mitigated by an appropriate "de-weighting"

of the suspect observation, which provides an opportunity to apply equations (2.45), (2.46), and (2.47).

2.9   Summary Formulas for the Least-Squares Solution Within the Model of Direct Observations

The model of direct observations is given by

$$
\underset{n \times 1}{\boldsymbol{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \mu + e_1 \\ \vdots \\ \mu + e_n \end{bmatrix} = \boldsymbol{\tau}\mu + \boldsymbol{e},
$$
$$
\boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 Q\right), \;\; Q := P^{-1}.
$$

Table 2.3: Summary formulas for the LESS within the model of direct observations

Quantity	Formula	Eq.
Model redundancy	$r = n - 1$	(2.37)
Estimated parameter	$\hat{\mu} = (\boldsymbol{\tau}^T P \boldsymbol{y})/(\boldsymbol{\tau}^T P \boldsymbol{\tau})$	(2.7)
Dispersion of estimated parameter	$D\{\hat{\mu}\} = \sigma_0^2/(\boldsymbol{\tau}^T P \boldsymbol{\tau})$	(2.14)
Vector of predicted residuals	$\tilde{\boldsymbol{e}} = \boldsymbol{y} - \boldsymbol{\tau}\hat{\mu}$	(2.8a)
Dispersion matrix for residuals	$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot \left[P^{-1} - \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T\right]$	(2.15)
Sum of squared residuals (SSR)	$\Omega = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$	(2.36)
Estimated variance component	$\hat{\sigma}_0^2 = (\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}})/(n - 1)$	(2.38)
Vector of adjusted observations	$\widehat{E\{\boldsymbol{y}\}} =: \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}$	(2.12)
Dispersion matrix for adjusted observations	$D\{\hat{\boldsymbol{\mu}}_y\} = \sigma_0^2 \cdot \boldsymbol{\tau}\left(\boldsymbol{\tau}^T P \boldsymbol{\tau}\right)^{-1}\boldsymbol{\tau}^T$	(2.16)

Chapter 3

The Gauss-Markov Model

3.1 Model Definition

The Gauss-Markov Model (GMM) is the underlying data model for many of the topics that follow. Koch points out why the model is named for both Gauss and Markov in his excellent textbook *Parameter Estimation and Hypothesis Testing in Linear Models* (Koch, 1999, p. 154). In presentation of the GMM, it is assumed that the observation equations (3.1a) have been linearized, if necessary. The model is written as follows:

$$\boldsymbol{y} = \underset{n \times m}{A}\, \boldsymbol{\xi} + \boldsymbol{e}, \quad \operatorname{rk} A = m, \tag{3.1a}$$

$$\boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 P^{-1}\right). \tag{3.1b}$$

It is sometimes expressed more succinctly as

$$A\boldsymbol{\xi} = E\{\boldsymbol{y}\} \quad \text{with} \quad D\{\boldsymbol{y}\} = \sigma_0^2 P^{-1}, \tag{3.1c}$$

or, equivalently, as

$$\boldsymbol{y} \sim (A\boldsymbol{\xi}, \sigma_0^2 P^{-1}). \tag{3.1d}$$

In the case of linearization, $\boldsymbol{y}$ is a vector of observations minus "zeroth-order" terms (i.e., $\boldsymbol{y}$ represents incremental observations); A is a known $n \times m$ coefficient matrix (also called *design* or *information* matrix, or *Jacobian* matrix if partial derivatives are involved) relating the observations to the unknown parameters; $\boldsymbol{\xi}$ is a vector of unknown parameters to estimate (updates to initial values in the case of linearization), and $\boldsymbol{e}$ is a vector of random observation errors, having zero expectation. Equation (3.1a) requires the $n \times m$ *coefficient matrix* A to have full column rank; thus, the model is also said to be a *Gauss-Markov Model with full rank* (Koch, 1999, Eq. (3.7)).

The $n \times n$ matrix P is symmetric. It contains weights of the observations, which may be correlated. The inverse of P shown in (3.1) implies that P is a positive-definite matrix; this inverse matrix is called the *cofactor* matrix and is denoted by Q.

The symbol σ_0^2 represents a *variance component*, which is considered unknown but can be estimated. The dispersion matrix $D\{e\} = \sigma_0^2 P^{-1}$ is called the variance-covariance matrix, or simply the *covariance* matrix, and is also denoted by Σ. In summary, we have the following relation between the dispersion, weight, and cofactor matrices of the unknown, random error vector e:

$$D\{e\} = \Sigma = \sigma_0^2 Q = \sigma_0^2 P^{-1}. \tag{3.2}$$

Obviously, if $Q = I$ then $D\{e\} = D\{y\} = \sigma_0^2 I$, and therefore some authors call σ_0^2 the variance of unit weight, though we refrain from using that term.

The redundancy r of the model (3.1a) is defined as

$$\boxed{r := n - \operatorname{rk} A = n - m.} \tag{3.3}$$

Redundancy is also called *degrees of freedom* in the context of statistical testing discussed in Chapter 9.

The GMM shown in (3.1) has two main components. The first component, (3.1a), contains the observation equations $y = A\xi + e$, which show the functional relation between the observations, their random errors, and the unknown parameters that are to be estimated. The second component, (3.1b), shows a stochastic model, $e \sim (0, \sigma_0^2 P^{-1})$, which expresses the expectation and dispersion of the random errors. These quantities are also called the first and second moments, respectively, of the random error vector e.

If the rank of matrix A is less than the number of unknown parameters to estimate, we say that the problem is rank deficient. Such a problem cannot be solved based on the observations alone; additional information about the unknown parameters must be provided. The problem of rank deficiency is covered in Section 3.5 and, much more thoroughly, in the Part II.

3.2   The Least-Squares Solution Within the Gauss-Markov Model

We now derive the LEast-Squares Solution (LESS) for the parameter estimate $\hat{\xi}$ and the predicted random error (residual) vector $\tilde{e}$, with their associated dispersion matrices, under the assumption that the coefficient matrix A has full column rank. For convenience, we define the $m \times m$ matrix N and the $m \times 1$ vector c as

$$[N,\, c] := A^T P[A,\, y]. \tag{3.4}$$

The objective of least-squares minimization is to minimize the P-weighted sum of squared residuals, or, equivalently, to minimize the P-weighted random errors in the model (3.1). Thus the Lagrange target function

$$\Phi(\xi) := (y - A\xi)^T P(y - A\xi) = \text{stationary} \tag{3.5}$$

should be minimized. Forming the the Euler-Lagrange necessary conditions (or first-order conditions) leads directly to the least-squares *normal equations*

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\xi}} = (A^TPA)\hat{\boldsymbol{\xi}} - A^TP\boldsymbol{y} = N\hat{\boldsymbol{\xi}} - \boldsymbol{c} \doteq \boldsymbol{0}. \tag{3.6}$$

The sufficient condition is satisfied by $(1/2)\cdot(\partial^2\Phi/\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}^T) = N$, which is positive definite since matrix A has full column rank according to (3.1a). Equation (3.6) leads to the least-squares solution (LESS)

$$\boxed{\hat{\boldsymbol{\xi}} = N^{-1}\boldsymbol{c}} \tag{3.7}$$

for the unknown parameter vector $\boldsymbol{\xi}$, with its expectation computed by

$$E\{\hat{\boldsymbol{\xi}}\} = N^{-1}E\{\boldsymbol{c}\} = N^{-1}A^TPE\{\boldsymbol{y}\} = N^{-1}A^TPA\boldsymbol{\xi} = \boldsymbol{\xi}. \tag{3.8}$$

The predicted random error vector (also called *residual* vector) is then given by

$$\boxed{\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}} = (I_n - AN^{-1}A^TP)\boldsymbol{y},} \tag{3.9}$$

with expectation

$$E\{\tilde{\boldsymbol{e}}\} = (I_n - AN^{-1}A^TP)E\{\boldsymbol{y}\} = (I_n - AN^{-1}A^TP)A\boldsymbol{\xi} = A\boldsymbol{\xi} - A\boldsymbol{\xi} = \boldsymbol{0}. \tag{3.10}$$

A simple proof that the quadratic form $\tilde{\boldsymbol{e}}^TP\tilde{\boldsymbol{e}}$ is a minimum is shown by Koch (1999, Eq. (3.25) and Theorem (3.26)). For the case where $P = I$, the proof means that minimizing (3.5) results in a minimum sum of squared residuals $\tilde{\boldsymbol{e}}^T\tilde{\boldsymbol{e}}$.

The expectation of the given observation vector is expressed as $E\{\boldsymbol{y}\} = \boldsymbol{\mu}_y$, where $\boldsymbol{\mu}_y$ is the true, but unknown, vector of observables. Thus we write the vector of adjusted observations as

$$\boxed{\widehat{E\{\boldsymbol{y}\}} =: \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}} = A\hat{\boldsymbol{\xi}},} \tag{3.11}$$

with expectation

$$E\{\hat{\boldsymbol{\mu}}_y\} = AE\{\hat{\boldsymbol{\xi}}\} = A\boldsymbol{\xi}. \tag{3.12}$$

Equations (3.8), (3.10) and (3.12) show that the estimated parameters, the residuals, and the adjusted observations, respectively, are unbiased.

The corresponding dispersion matrices are computed by using the law of covariance propagation. The dispersion of the estimated parameters is computed by

$$D\{\hat{\boldsymbol{\xi}}\} = D\{N^{-1}A^TP\boldsymbol{y}\} = (N^{-1}A^TP)D\{\boldsymbol{y}\}(PAN^{-1}) =$$
$$= N^{-1}A^TP(\sigma_0^2 P^{-1})PAN^{-1} \Rightarrow$$

$$\boxed{D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 N^{-1}.} \tag{3.13}$$

And, the dispersion of the residual vector $\tilde{\boldsymbol{e}}$ is

$$D\{\tilde{\boldsymbol{e}}\} = \big(I_n - AN^{-1}A^TP\big)D\{\boldsymbol{y}\}\big(I_n - PAN^{-1}A^T\big) =$$
$$= \big(I_n - AN^{-1}A^TP\big)\big(\sigma_0^2 P^{-1}\big)\big(I_n - PAN^{-1}A^T\big) =$$
$$= \sigma_0^2\big(I_n - AN^{-1}A^TP\big)\big(P^{-1} - AN^{-1}A^T\big) \Rightarrow$$

$$\boxed{D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2\big(P^{-1} - AN^{-1}A^T\big) =} \qquad\qquad (3.14\text{a})$$

$$= D\{\boldsymbol{y}\} - D\{A\hat{\boldsymbol{\xi}}\} =: \sigma_0^2 Q_{\tilde{e}}, \qquad\qquad (3.14\text{b})$$

where the matrix

$$Q_{\tilde{e}} := P^{-1} - AN^{-1}A^T \qquad\qquad (3.14\text{c})$$

is the cofactor matrix of the residual vector $\tilde{\boldsymbol{e}}$. Equations (3.14a) to (3.14c) reveal that the variances of the residuals are smaller than the corresponding variances of the observations, since the matrix product $AN^{-1}A^T$ is positive definite. Finally, the dispersion of the vector of adjusted observations is computed by

$$\boxed{D\{\hat{\boldsymbol{\mu}}_y\} = D\{A\hat{\boldsymbol{\xi}}\} = AD\{\hat{\boldsymbol{\xi}}\}A^T = \sigma_0^2 AN^{-1}A^T,} \qquad\qquad (3.15\text{a})$$

and therefore the cofactor matrix for the adjusted observations is written as

$$Q_{\hat{\boldsymbol{\mu}}_y} := AN^{-1}A^T. \qquad\qquad (3.15\text{b})$$

As an aside, the cofactor matrix $Q_{\hat{\boldsymbol{\mu}}_y}$ has been called "hat matrix" by authors such as Draper and Smith (1998, pp. 205, 207) in the context of linear regression when the weight matrix is the identity matrix, since in that case $\hat{\boldsymbol{\mu}}_y = A(A^TA)^{-1}A\boldsymbol{y} =: H\boldsymbol{y}$, and thus H "puts a hat on $\boldsymbol{y}$" (they used $\hat{\boldsymbol{Y}}$ to denote adjusted observations). In the context of the GMM (3.1), where $P \neq I$, the hat matrix is obviously $A(A^TPA)^{-1}AP$ (according to (3.11)), which is non-symmetric. The esteemed American mathematician and statistician John W. Tukey is said to have coined the term hat matrix (Hoaglin and Welsch, 1978). However, in many disciplines, such as geodetic science, the term does not seem to be in common use.

Analogous to Figure 2.1, the projection of the observation vector $\boldsymbol{y}$ onto the range space of the design matrix A by the projection matrix $Q_{\tilde{e}}P$ is depicted in Figure 3.1.

Summarizing the above equations, the respective distributions for the estimated parameter vector, the residual vector, and the vector of adjusted observations are succinctly expressed by

$$\hat{\boldsymbol{\xi}} \sim \big(\boldsymbol{\xi}, \sigma_0^2 N^{-1}\big), \qquad\qquad (3.16\text{a})$$
$$\tilde{\boldsymbol{e}} \sim \big(\boldsymbol{0}, \sigma_0^2\big[P^{-1} - AN^{-1}A^T\big] = D\{\boldsymbol{y}\} - D\{\hat{\boldsymbol{\mu}}_y\}\big), \qquad\qquad (3.16\text{b})$$
$$\hat{\boldsymbol{\mu}}_y \sim \big(A\boldsymbol{\xi}, \sigma_0^2 AN^{-1}A^T\big). \qquad\qquad (3.16\text{c})$$

Since the variance component σ_0^2 is an unknown quantity, the dispersions shown in (3.16) cannot be computed unless either σ_0^2 is estimated or a value is specified for it. In the case where the estimated variance component is used in lieu of the true,

$$Q_{\tilde{e}}P := \left[I_n - A(A^TPA)^{-1}A^TP \right]$$
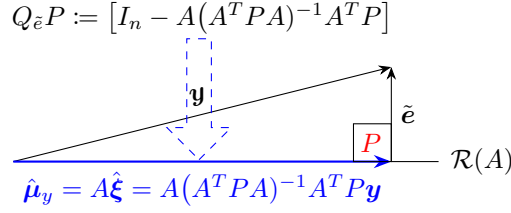
Figure 3.1: Depiction of P-orthogonality between residual vector $\tilde{e}$ and vector of adjusted observations $A\hat{\xi}$. The appearance of the weight matrix P in the box represents its roll in the orthogonality relationship. The effect of the projection matrix $Q_{\tilde{e}}P$ is depicted by the dashed arrow as projecting the observation vector $\boldsymbol{y}$ onto the (m-dimensional) range space of A. The vectors sum together as $\boldsymbol{y} = A\hat{\xi} + \tilde{e}$, just as they should.

but unknown, variance component, we speak of an *estimated dispersion matrix* for the estimated parameter vector, which is provided by

$$\hat{D}\{\hat{\boldsymbol{\xi}}\} = \hat{\sigma}_0^2 N^{-1}, \tag{3.17}$$

with obvious extension to other quantities, such as $\hat{D}\{\tilde{e}\}$ and $\hat{D}\{\hat{\boldsymbol{\mu}}_y\}$. See Section 3.3 for the derivation of the variance component estimate $\hat{\sigma}_0^2$, the formula for which is given in (3.30).

3.2.1 Example — Fitting a Parabola

Suppose n observations were taken of data that, when plotted in 2D, appear to approximate a parabola (Figure 3.2). The y-coordinates represent measured data with random errors having zero mean and iid dispersion. The x-coordinates are assumed to be known without error. This is a classical *regression problem*.

The observations equations of the Gauss-Markov Model (GMM) are set up as follows: The ith observation equation, $i = 1, \ldots, n$,

$$y_i = ax_i^2 + bx_i + c + e_i, \tag{3.18}$$

can be extended to a system of equations in matrix from as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\xi} + \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}}_{\boldsymbol{e}}. \tag{3.19}$$
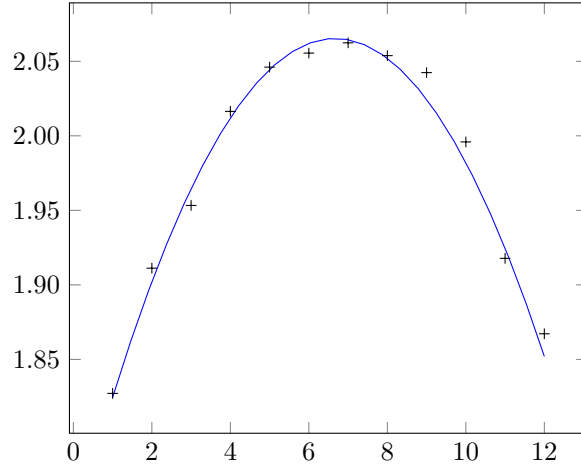
Figure 3.2: A fitted parabolic curve based on measured y-coordinates and given x-coordinates

where $\boldsymbol{\xi} := [a,\, b,\, c]^T$ is the vector of unknown parameters, which, together with the stochastic model $\boldsymbol{e} \sim (\boldsymbol{0}, \mathrm{iid})$, constitutes a Gauss-Markov Model. Note that in other examples of a GMM, the the random observation errors could have a heteroscedastic distribution, or their dispersion could be represented by a full cofactor matrix Q.

3.2.2   Correlation of Adjusted Observations and Predicted Residuals

Equation (3.14b) implies that the covariance between the vector of *adjusted observations* $\hat{\boldsymbol{\mu}}_y = A\hat{\boldsymbol{\xi}}$ and the vector of residuals $\tilde{\boldsymbol{e}}$ is zero. Since, according to (3.7) and (3.9), both vectors are a function of the random vector $\boldsymbol{y}$, this can also be shown by applying the law of covariance propagation as follows:

$$
\begin{aligned}
C\{A\hat{\boldsymbol{\xi}}, \tilde{\boldsymbol{e}}\} = AN^{-1}A^T P \cdot D\{\boldsymbol{y}\} \cdot \left(I_n - AN^{-1}A^T P\right)^T = \\
= \sigma_0^2 \left[AN^{-1}A^T - AN^{-1}\left(A^T P A\right)N^{-1}A^T\right] = \\
= \sigma_0^2 \left[AN^{-1}A^T - AN^{-1}A^T\right] = 0. \quad (3.20)
\end{aligned}
$$

Also, we have the following covariance between the adjusted and original observations:

$$
\begin{aligned}
C\{A\hat{\boldsymbol{\xi}}, \boldsymbol{y}\} = AN^{-1}A^T P D\{\boldsymbol{y}\} = \sigma_0^2 AN^{-1}A^T PP^{-1} = \\
= \sigma_0^2 AN^{-1}A^T = D\{A\hat{\boldsymbol{\xi}}\}. \quad (3.21)
\end{aligned}
$$

Zero correlation does not necessarily imply statistical independence, though the converse does hold. Analogous to (9.9a), the adjusted observations and predicted residuals are not statistically independent unless the expectation of their product is

equal to the product of their expectations. The following shows that this property is not satisfied: Since the trace of a scalar product is the scalar product itself, we start with

$$E\{(A\hat{\boldsymbol{\xi}})^T \tilde{\boldsymbol{e}}\} = E\{\text{tr}\,\hat{\boldsymbol{\xi}}^T A^T (I_n - AN^{-1}A^T P)\boldsymbol{y}\}.$$

But the trace is invariant with respect to a cyclic transformation (see (A.5)). Thus,

$$E\{(A\hat{\boldsymbol{\xi}})^T \tilde{\boldsymbol{e}}\} = E\{\text{tr}(A^T - A^T AN^{-1}A^T P)\boldsymbol{y}\hat{\boldsymbol{\xi}}^T\} =$$
$$= \text{tr}(A^T - A^T AN^{-1}A^T P)E\{\boldsymbol{y}\hat{\boldsymbol{\xi}}^T\} \neq 0 = E\{(A\hat{\boldsymbol{\xi}})^T\}E\{\tilde{\boldsymbol{e}}\}, \ \text{ since } E\{\tilde{\boldsymbol{e}}\} = \boldsymbol{0}.$$

3.2.3 *P*-Weighted Norm of the Residual Vector

The P-weighted norm of the residual vector $\tilde{\boldsymbol{e}}$ is an important quantity that can be used to check the overall ("global") fit of the adjustment. The norm is defined as

$$\boxed{\Omega := \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}},} \tag{3.22}$$

and it is guaranteed to be a minimum, since $\tilde{\boldsymbol{e}}$ was obtained by minimizing $\boldsymbol{e}^T P \boldsymbol{e}$ (cf. (3.5)). In the special case where $P = I_n$, the quadratic form Ω is often called the *sum of squared residuals*, or SSR, in the statistical literature. We use the term SSR in the following chapters even when P is not the identity matrix. Substituting (3.9) into (3.22) leads to some commonly used alternative forms for Ω.

$$\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} = (\boldsymbol{y} - A\hat{\boldsymbol{\xi}})^T P(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}) = \tag{3.23a}$$
$$= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{y}^T P A\hat{\boldsymbol{\xi}} - \hat{\boldsymbol{\xi}}^T A^T P \boldsymbol{y} + \hat{\boldsymbol{\xi}}^T A^T P A\hat{\boldsymbol{\xi}} =$$
$$= \boldsymbol{y}^T P \boldsymbol{y} - 2\boldsymbol{c}^T \hat{\boldsymbol{\xi}} + \boldsymbol{c}^T \hat{\boldsymbol{\xi}} =$$
$$= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T \hat{\boldsymbol{\xi}} = \tag{3.23b}$$
$$= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T N^{-1} \boldsymbol{c} = \tag{3.23c}$$
$$= \boldsymbol{y}^T P \boldsymbol{y} - (N\hat{\boldsymbol{\xi}})^T N^{-1} N\hat{\boldsymbol{\xi}} =$$
$$= \boldsymbol{y}^T P \boldsymbol{y} - \hat{\boldsymbol{\xi}}^T N\hat{\boldsymbol{\xi}} = \tag{3.23d}$$
$$= \boldsymbol{y}^T (P - PAN^{-1}A^T P)\boldsymbol{y} \tag{3.23e}$$

Note that the target function (3.5) could have been written explicitly as a function of the random error vector $\boldsymbol{e}$ with the introduction of a vector of Lagrange multipliers $\boldsymbol{\lambda}$ as follows:

$$\Phi(\boldsymbol{e}, \boldsymbol{\xi}, \boldsymbol{\lambda}) = \boldsymbol{e}^T P \boldsymbol{e} + 2\boldsymbol{\lambda}^T (\boldsymbol{y} - A\boldsymbol{\xi} - \boldsymbol{e}) = \text{stationary}. \tag{3.24}$$

The Euler-Lagrange necessary conditions (or first-order conditions) would then follow as

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}} = P\tilde{\boldsymbol{e}} - \hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \tag{3.25a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\xi}} = A^T \hat{\boldsymbol{\lambda}} \doteq \mathbf{0}, \tag{3.25b}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}} - \tilde{\boldsymbol{e}} \doteq \mathbf{0}. \tag{3.25c}$$

Equation (3.25a) leads immediately to $\hat{\boldsymbol{\lambda}} = P\tilde{\boldsymbol{e}}$ as an estimate of the unknown Lagrange multipliers and thus to yet another expression for the P-weighted norm, namely

$$\Omega = \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} = \tilde{\boldsymbol{e}}^T \hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\lambda}}^T P^{-1} \hat{\boldsymbol{\lambda}}. \tag{3.26}$$

Moreover, substituting (3.25a) into (3.25b) yields, in analogy to (2.13), the orthogonality condition $A^T P\tilde{\boldsymbol{e}} = \mathbf{0}$, which implies that the vector of adjusted observations is P-orthogonal to the vector of residuals, i.e., $(A\hat{\boldsymbol{\xi}})^T P\tilde{\boldsymbol{e}} = \mathbf{0}$.

Incidentally, when $P = I$ the orthogonality condition becomes $A^T\tilde{\boldsymbol{e}} = \mathbf{0}$, implying that the residual vector $\tilde{\boldsymbol{e}}$ is in the left nullspace of matrix A (nullspace of A^T) in that case. This explains why the residuals sum to zero in ordinary least-squares[1] problems where one column of A contains only ones, or any constant value (e.g., in line-fitting problems and the in parabola-fitting example above).

3.3   Estimated Variance Component Within the Gauss-Markov Model

As stated in Section 2.4, the variance component σ_0^2 is an unknown quantity in the Gauss-Markov Model (GMM). We now present the derivation of the estimated variance component $\hat{\sigma}_0^2$. As defined in (3.1), the dispersion matrix for the random error vector $\boldsymbol{e}$ is $D\{\boldsymbol{e}\} = \sigma_0^2 Q$. Also, by definition of dispersion we have $D\{\boldsymbol{e}\} = E\{(\boldsymbol{e} - E\{\boldsymbol{e}\})(\boldsymbol{e} - E\{\boldsymbol{e}\})^T\}$. But, for the error vector $E\{\boldsymbol{e}\} = \mathbf{0}$; therefore

$$D\{\boldsymbol{e}\} = E\{\boldsymbol{e}\boldsymbol{e}^T\} = \sigma_0^2 Q = \sigma_0^2 P^{-1}, \ \text{since} \ E\{\boldsymbol{e}\} = \mathbf{0}. \tag{3.27}$$

The following steps lead to an expression for the variance component σ_0^2 in terms of the quadratic product $\boldsymbol{e}^T P\boldsymbol{e}$.

$E\{\boldsymbol{e}\boldsymbol{e}^T\} = \sigma_0^2 Q$ (by definition)

$PE\{\boldsymbol{e}\boldsymbol{e}^T\} = \sigma_0^2 I_n$ (multiply both sides by P)

$\operatorname{tr} PE\{\boldsymbol{e}\boldsymbol{e}^T\} = \sigma_0^2 \operatorname{tr} I_n = n\sigma_0^2$ (apply the trace operator)

$\operatorname{tr} E\{P\boldsymbol{e}\boldsymbol{e}^T\} = n\sigma_0^2$ (move the constant matrix P into the expectation)

$E\{\operatorname{tr} P\boldsymbol{e}\boldsymbol{e}^T\} = n\sigma_0^2$

         (interchange the trace and expectation operators—both linear)

$E\{\operatorname{tr} \boldsymbol{e}^T P\boldsymbol{e}\} = n\sigma_0^2$

       (the trace is invariant with respect to a cyclic transformation)

---

[1] Some authors refer to the case of $P = I$ as *ordinary least-squares*, e.g., Harville (2000, p. 267).

$$E\{\boldsymbol{e}^T P \boldsymbol{e}\} = n\sigma_0^2 \quad \text{(a quadratic product is a scalar; trace of scalar is scalar itself)}$$

$$\sigma_0^2 = E\{\frac{\boldsymbol{e}^T P \boldsymbol{e}}{n}\} \qquad \text{(dividing through by } n \text{ and placing } n \text{ inside } E\{\cdot\})$$

$$\bar{\sigma}_0^2 := \frac{\boldsymbol{e}^T P \boldsymbol{e}}{n} \qquad \text{(define a symbol for the term inside } E\{\cdot\})$$

$$E\{\bar{\sigma}_0^2\} = \sigma_0^2 \qquad \text{(by substitution)}$$

Thus we can say that $(\boldsymbol{e}^T P \boldsymbol{e})/n$ provides an unbiased representation of σ_0^2. However, we do not actually know the true random error vector $\boldsymbol{e}$, but we do know its predicted value $\tilde{\boldsymbol{e}}$.

We now work with the residual vector $\tilde{\boldsymbol{e}}$ to find an unbiased estimate of σ_0^2. Combining steps similar to those explained above, we can write

$$E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\} = \text{tr}\, E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\} = \text{tr}\, E\{\tilde{\boldsymbol{e}}\tilde{\boldsymbol{e}}^T\} P = \text{tr}\, D\{\tilde{\boldsymbol{e}}\} P. \qquad (3.28)$$

According to (3.14a), the dispersion of the residual vector is $D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2\big(P^{-1} - AN^{-1}A^T\big)$. Substituting this result into (3.28) gives

$$\begin{aligned}
E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\} &= \text{tr}\, \sigma_0^2\big(P^{-1} - AN^{-1}A^T\big)P = \\
&= \sigma_0^2\big(\text{tr}\, I_n - \text{tr}\, AN^{-1}A^T P\big) = \\
&= \sigma_0^2\big(\text{tr}\, I_n - \text{tr}\, N^{-1}A^T PA\big) = \qquad \text{(using (A.5))} \\
&= \sigma_0^2(n - \text{rk}\, N) = \sigma_0^2(n - \text{rk}\, A). \qquad \text{(using (1.7c))}
\end{aligned}$$

Finally, we arrive at

$$E\{\frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{n - \text{rk}\, A}\} = \sigma_0^2. \qquad (3.29)$$

Now, we simply label the argument of the expectation operator on the left side of (3.29) as $\hat{\sigma}_0^2$, which allows us to write the expression for the estimated variance component as

$$\boxed{\hat{\sigma}_0^2 = \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{n - \text{rk}\, A}.} \qquad (3.30)$$

Obviously, $\hat{\sigma}_0^2$ is a uniformly unbiased estimate of σ_0^2, since $E\{\hat{\sigma}_0^2\} = \sigma_0^2$. In the case of the Model of Direct Observations, we replace A with τ, which has rank of 1, and thus we have $\hat{\sigma}_0^2 := \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}/(n-1)$, which verifies (2.38). Alternative expressions for $\hat{\sigma}_0^2$ can be reached by use of (3.23) and (3.26).

The above derivations imply the following relationship between $E\{\boldsymbol{e}^T P \boldsymbol{e}\}$ and $E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\}$:

$$\frac{E\{\boldsymbol{e}^T P \boldsymbol{e}\}}{n} = \frac{E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\}}{n - \text{rk}\, A} = \sigma_0^2 \Rightarrow \qquad (3.31a)$$

$$E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\} < E\{\boldsymbol{e}^T P \boldsymbol{e}\} \qquad (3.31b)$$

According to Grafarend and Schaffrin (1993, pg. 103), and Schaffrin (1997b), the dispersion, and estimated dispersion, respectively, of $\hat{\sigma}_0^2$ are given by

$$D\{\hat{\sigma}_0^2\} = (n - m)^{-1} \cdot 2\big(\sigma_0^2\big)^2 \tag{3.32}$$

and

$$\hat{D}\{\hat{\sigma}_0^2\} = (n - m)^{-1} \cdot 2\big(\hat{\sigma}_0^2\big)^2, \tag{3.33}$$

where it is assumed that $m = \operatorname{rk} A$; cf. Searle and Khuri (2017, Eq. (10.35)).

After a least-squares adjustment, the estimated variance component can be tested for statistical agreement with a specified value, say 1, by means of hypothesis testing as described in Section 9.4.2.

3.4   Linearized Observation Equations and a Corresponding Algorithm for an Iterative Least-Squares Solution

When the the observations $\boldsymbol{y}$ are expressed as *nonlinear* functions of the unknown parameters $\boldsymbol{\xi}$, the observation equations and dispersion matrix can be represented as

$$E\{\boldsymbol{y}\} = \boldsymbol{a}(\boldsymbol{\xi}), \tag{3.34a}$$

$$D\{\boldsymbol{y}\} = \sigma_0^2 P^{-1} = D\{\boldsymbol{e}\}, \quad \boldsymbol{e} := \boldsymbol{y} - E\{\boldsymbol{y}\}, \tag{3.34b}$$

implying that

$$\boldsymbol{y} = \boldsymbol{a}(\boldsymbol{\xi}) + \boldsymbol{e}, \tag{3.34c}$$

where $\boldsymbol{a}(\boldsymbol{\xi})$ is a vector of functions that maps $\mathbb{R}^m$ to $\mathbb{R}^n$ and σ_0^2 and $\boldsymbol{e}$ are defined as usual. Given a vector of *approximate* parameter values $\boldsymbol{\xi}_{(0)}$, a Taylor-series expansion permits (3.34a) to be rewritten as

$$E\{\boldsymbol{y}\} = \boldsymbol{a}(\boldsymbol{\xi}_{(0)}) + \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\xi}^T}\bigg|_{\boldsymbol{\xi}=\boldsymbol{\xi}_{(0)}} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_{(0)}) + \cdots \tag{3.35a}$$

leading to

$$\boxed{E\{\boldsymbol{y} - \boldsymbol{a}(\boldsymbol{\xi}_{(0)})\} = A \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_{(0)})} + \text{higher order terms}, \tag{3.35b}$$

where A is an $n \times m$ coefficient matrix containing the partial derivatives evaluated at $\boldsymbol{\xi}_{(0)}$; such a matrix is also called a Jacobian matrix.

By truncating the Taylor-series expansion (i.e. dropping the higher-order terms) and working with *observation increments* $\boldsymbol{y} - \boldsymbol{a}(\boldsymbol{\xi}_{(0)})$ and *parameter increments* $\boldsymbol{\xi} - \boldsymbol{\xi}_{(0)}$, we may form the system of linearized least-squares *normal equations*

$$\boxed{(A^T P A)(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}_{(0)}) = A^T P(\boldsymbol{y} - \boldsymbol{a}(\boldsymbol{\xi}_{(0)})),} \tag{3.36}$$

leading to

$$\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}_{(0)} + (A^T P A)^{-1} A^T P(\boldsymbol{y} - \boldsymbol{a}(\boldsymbol{\xi}_{(0)})) \tag{3.37a}$$

and

$$D\{\hat{\boldsymbol{\xi}}\} = D\{\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}_{(0)}\} = \sigma_0^2 (A^T P A)^{-1} \tag{3.37b}$$

for the estimate of $\boldsymbol{\xi}$ and its dispersion matrix, respectively.

The chosen approximate values for $\boldsymbol{\xi}_{(0)}$ may be less precise than we prefer, which, in turn, might affect the accuracy and precision of the estimates $\hat{\boldsymbol{\xi}}$. In practice, $\boldsymbol{\xi}_{(0)}$ may be taken from a solution based on only a minimum subset of the observation equations (i.e., only m of them to solve for m unknowns). Such approximate values could be improved by replacing them with values obtained from an initial least-squares estimate $\hat{\boldsymbol{\xi}}$. Then, the system of normal equations could be updated and solved again, leading to more precise values for $\hat{\boldsymbol{\xi}}$. This process could be repeated until the difference between $\hat{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}_{(0)}$ becomes arbitrarily small. This technique is called *iterative least-squares solution*.

In the jth iteration step ($j > 0$) of an algorithm for an iterative least-squares solution, the vector of approximate values $\boldsymbol{\xi}_{(j)}$ is specified by

$$\boldsymbol{\xi}_{(j)} := \hat{\boldsymbol{\xi}}_{j-1} - \underset{\sim}{\boldsymbol{0}}, \tag{3.38a}$$

where the subtraction of a *random zero vector* $\underset{\sim}{\boldsymbol{0}}$ is a formality that reflects that the approximate vector $\boldsymbol{\xi}_{(j)}$ is non-random, as it must be. Thus, we say that the subtraction of $\underset{\sim}{\boldsymbol{0}}$ strips $\hat{\boldsymbol{\xi}}_{j-1}$ of its randomness (note that the subtraction of a vector of random zeros does not change the numerical values of $\hat{\boldsymbol{\xi}}_{j-1}$). The iterations are repeated until

$$\left\| \hat{\boldsymbol{\xi}}_j - \hat{\boldsymbol{\xi}}_{(j-1)} \right\| < \epsilon \tag{3.38b}$$

for some chosen, small ϵ. Such an iterative algorithm is called a Gauss-Newton algorithm.

Summarizing notation, at iteration step j, the symbol $\boldsymbol{\xi}_{(j)}$ has been used to indicate approximate values for the Taylor-series expansion point, while $\boldsymbol{\xi}$ is used for the true (unknown) parameter vector as usual, and $\hat{\boldsymbol{\xi}}_j$ denotes the vector of estimated parameters.

Iterative algorithm to solve a linearized system of normal equations:

1. Initialization: Specify initial values for $\boldsymbol{\xi}_{(0)}$, e.g., based on a minimum number of observation equations. Compute $\boldsymbol{a}(\boldsymbol{\xi}_{(0)})$ to form matrix $A_{(0)}$. Then set the iteration index to $j = 1$.

2. Compute the jth solution

$$\hat{\boldsymbol{\xi}}_j = \boldsymbol{\xi}_{(j-1)} + \left[A_{(j-1)}^T P A_{(j-1)}\right]^{-1} A_{(j-1)}^T P\left[\boldsymbol{y} - \boldsymbol{a}(\boldsymbol{\xi}_{(j-1)})\right]. \tag{3.39a}$$

Check for *convergence*: If the inequality

$$\left\| \hat{\boldsymbol{\xi}}_j - \boldsymbol{\xi}_{(j-1)} \right\| < \epsilon \qquad (3.39\text{b})$$

holds for some chosen ϵ, then the solution has converged; thus go to step 4.

3. Increment the iteration counter j by one. Update the expansion point $\boldsymbol{\xi}_{(j-1)}$ according to (3.38a). Update the partial derivatives in the Jacobian matrix $A_{(j-1)}$. Repeat step 2.

4. Upon convergence, and dropping the iteration index j, the dispersion matrix for the estimated parameters becomes

$$D\{\hat{\boldsymbol{\xi}}\} = (A^T P A)^{-1}. \qquad (3.39\text{c})$$

The residual vector is provided by

$$\tilde{\boldsymbol{e}} = \boldsymbol{y} - \boldsymbol{a}(\hat{\boldsymbol{\xi}}), \qquad (3.39\text{d})$$

the vector of adjusted observations by

$$\boldsymbol{y} - \tilde{\boldsymbol{e}} = \boldsymbol{a}(\hat{\boldsymbol{\xi}}), \qquad (3.39\text{e})$$

and the estimated variance component by

$$\hat{\sigma}_0^2 = (\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}})/r. \qquad (3.39\text{f})$$

Checks similar to those discussed in Section 2.5.1 should also be made.

3.5   Introduction of Datum Information to Treat the Rank-Deficient Gauss-Markov Model

A rank-deficient Gauss-Markov Model (GMM) is one in which the rank of the coefficient matrix A is less than its number of columns. This means that at least one column of A is either a scalar multiple of a different column or that it can be expressed as a linear combination of other columns. Such rank deficiency is expressed mathematically as $\operatorname{rk} A < m$, where m is the number of columns of A. It implies also that $\operatorname{rk} N = \operatorname{rk} A^T P A < m$, which means that the unknown parameters cannot be estimated by (3.7). Put another way, a rank-deficient model is one in which there are more parameters than can be estimated from the data. In fact, the rank of the coefficient matrix reveals the number of *estimable parameters* of the model.

Rank deficiency often arises in the context of network adjustments where station coordinates must be estimated but the observations do not contain sufficient information to define the underlying coordinate system, also called *datum* in this context. Thus we speak of a network *datum deficiency*. As noted in Chapter 5, a 2-D network where only angles and distance have been measured provides an example of a datum deficiency of three, owing to the unknown origin and orientation parameters of the network. However, if *certain values* (or known values) are provided for enough parameters, this "datum information" can be used to eliminate the rank deficiency of the model. The method is described in the following.

Consider the following (linearized) GMM with rank-deficient matrix A:

$$\boldsymbol{y} = A\boldsymbol{\xi} + \boldsymbol{e}, \ \ \boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 P^{-1}\right), \ \ \mathrm{rk}\, A =: q < m. \tag{3.40a}$$

We can partition the matrix A as

$$\underset{n \times m}{A} = \left[\underset{n \times q}{A_1} \ \middle| \ \underset{n \times (m-q)}{A_2}\right], \ \ \text{with } \mathrm{rk}\, A_1 = q := \mathrm{rk}\, A, \tag{3.40b}$$

so that matrix A_1 has full column rank. A compatible partitioning of the parameter vector $\boldsymbol{\xi}$, i.e.,

$$\boldsymbol{\xi} = \begin{bmatrix} \underset{q \times 1}{\boldsymbol{\xi}_1} \\ \\ \underset{(m-q) \times 1}{\boldsymbol{\xi}_2} \end{bmatrix}, \tag{3.40c}$$

leads to the following system of partitioned normal equations:

$$\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} P \begin{bmatrix} A_1, & A_2 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} P\boldsymbol{y} = \left[\begin{array}{c|c} A_1^T P A_1 & A_1^T P A_2 \\ \hline A_2^T P A_1 & A_2^T P A_2 \end{array}\right] \begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} A_1^T P\boldsymbol{y} \\ A_2^T P\boldsymbol{y} \end{bmatrix} =$$

$$= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{bmatrix}. \tag{3.41}$$

The sub-scripted terms in (3.41) may be defined more succinctly as

$$\left[N_{ij}, \ \boldsymbol{c}_i\right] := A_i^T P\left[A_j, \ \boldsymbol{y}\right], \ \ \text{for } i, j \in \{1, 2\}. \tag{3.42}$$

Defining a datum for $m-q$ parameters means that values for them must be specified. Mathematically, a datum is defined by $\hat{\boldsymbol{\xi}}_2 \to \boldsymbol{\xi}_2^0$, where $\boldsymbol{\xi}_2^0$ is known. The rank of A_1 given in (3.40b) implies that the inverse of the $q \times q$ matrix N_{11} exists. Therefore, from the top row of (3.41), and with a *given datum* $\boldsymbol{\xi}_2^0$ substituted for $\hat{\boldsymbol{\xi}}_2$, we can write

$$N_{11}\hat{\boldsymbol{\xi}}_1 = \boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0 \Rightarrow \tag{3.43a}$$

$$\boxed{\hat{\boldsymbol{\xi}}_1 = N_{11}^{-1}\left(\boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0\right).} \tag{3.43b}$$

Equation (3.43b) shows that datum values can be specified or modified after observations have been made and matrix N_{11} has been inverted. Moreover, since the only random component in (3.43b) is $\boldsymbol{c}_1$, we have

$$\boxed{D\{\hat{\boldsymbol{\xi}}_1\} = \sigma_0^2 N_{11}^{-1}} \tag{3.44}$$

for the dispersion of the vector of estimated parameters $\hat{\boldsymbol{\xi}}_1$.

The predicted random error (residual) vector and its dispersion are then defined as follows:

$$\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}} = \boldsymbol{y} - \left[A_1 \mid A_2 \right] \begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \boldsymbol{\xi}_2^0 \end{bmatrix} = \boldsymbol{y} - A_1\hat{\boldsymbol{\xi}}_1 - A_2\boldsymbol{\xi}_2^0, \tag{3.45a}$$

$$D\{\tilde{\boldsymbol{e}}\} = D\{\boldsymbol{y}\} - D\{A_1\hat{\boldsymbol{\xi}}_1\} = \sigma_0^2 \left(P^{-1} - A_1 N_{11}^{-1} A_1^T \right). \tag{3.45b}$$

Note that $C\{\boldsymbol{y}, A_1\hat{\boldsymbol{\xi}}_1\} = 0$, which is implied by (3.45b). After computing the residuals, it is straightforward to compute the vector of adjusted observations and it dispersion matrix, respectively, by

$$\widehat{E\{\boldsymbol{y}\}} =: \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}} = A_1\hat{\boldsymbol{\xi}}_1 + A_2\boldsymbol{\xi}_2^0, \tag{3.46a}$$

$$D\{\hat{\boldsymbol{\mu}}_y\} = D\{A_1\hat{\boldsymbol{\xi}}_1\} = \sigma_0^2 \cdot A_1 N_{11}^{-1} A_1^T. \tag{3.46b}$$

Here, $\hat{\boldsymbol{\mu}}_y$ is also interpreted as an estimate of the true, and thus unknown, vector of observables $\boldsymbol{\mu}_y$, where $E\{\boldsymbol{y}\} = \boldsymbol{\mu}_y$.

The sum of squared residuals (SSR) is given by

$$\Omega = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}, \tag{3.47}$$

while the redundancy of the model is provided by

$$r = n - \operatorname{rk} A = n - q. \tag{3.48}$$

Substituting (3.45a) into (3.47), and considering (3.43a), leads to

$$\hat{\sigma}_0^2 = \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{r} = \frac{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}_1^T \hat{\boldsymbol{\xi}}_1 - \boldsymbol{c}_2^T \boldsymbol{\xi}_2^0}{n - q} \tag{3.49}$$

as an estimate for the unknown variance component σ_0^2. Here, the relation $\hat{\boldsymbol{\xi}}_1^T N_{11} \hat{\boldsymbol{\xi}}_1 + \hat{\boldsymbol{\xi}}_1^T N_{12} \hat{\boldsymbol{\xi}}_2 = \hat{\boldsymbol{\xi}}_1^T \boldsymbol{c}_1$ has been used. However, since $\operatorname{rk} A_1 = \operatorname{rk} A = q$, the $n \times (m-q)$ submatrix A_2 must be in the column space of the $n \times q$ matrix A_1 so that

$$A_2 = A_1 L \tag{3.50a}$$

for some $q \times (m-q)$ matrix L. Therefore,

$$N_{12} = A_1^T P A_2 = A_1^T P A_1 L = N_{11} L \Rightarrow \tag{3.50b}$$

$$N_{11}^{-1} N_{12} = L. \tag{3.50c}$$

With this result, and using (3.43b), we have

$$\boldsymbol{c}_1^T \hat{\boldsymbol{\xi}}_1 + \boldsymbol{c}_2^T \boldsymbol{\xi}_2^0 = \boldsymbol{y}^T P A_1 \left(N_{11}^{-1} \boldsymbol{c}_1 - N_{11}^{-1} N_{12} \boldsymbol{\xi}_2^0 \right) + \boldsymbol{y}^T P A_2 \boldsymbol{\xi}_2^0 =$$

$$= \boldsymbol{y}^T P A_1 \left(N_{11}^{-1} \boldsymbol{c}_1 - L\boldsymbol{\xi}_2^0 \right) + \boldsymbol{y}^T P A_2 \boldsymbol{\xi}_2^0 =$$
$$= \boldsymbol{y}^T P A_1 N_{11}^{-1} \boldsymbol{c}_1 - \boldsymbol{y}^T P \left(A_1 L \right) \boldsymbol{\xi}_2^0 + \boldsymbol{y}^T P A_2 \boldsymbol{\xi}_2^0 =$$
$$= \boldsymbol{y}^T P A_1 N_{11}^{-1} \boldsymbol{c}_1 = \boldsymbol{c}_1^T N_{11}^{-1} \boldsymbol{c}_1, \tag{3.51}$$

which, upon substitution into (3.49), leads to

$$\boxed{\hat{\sigma}_0^2 = \frac{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}_1^T N_{11}^{-1} \boldsymbol{c}_1}{n - q}} \tag{3.52}$$

as an alternative form for the estimated variance component.

It is instructive to compare the dispersion of $\hat{\boldsymbol{\xi}}_1$ shown in (3.44) with the corresponding dispersion in the case that matrix A has full row rank, i.e., $\operatorname{rk} A = m$. In the full-rank case, we could invert the coefficient matrix of (3.41) and find the upper $q \times q$ block of the inverse, scaled by σ_0^2, to be the dispersion of $\hat{\boldsymbol{\xi}}_1$. Referring to (A.15) for the inverse of the partitioned matrix N, we find

$$\underbrace{D\{\hat{\boldsymbol{\xi}}_1\}}_{\text{no datum}} = \sigma_0^2 \left[N_{11}^{-1} + N_{11}^{-1} N_{12} \left(N_{22} - N_{21} N_{11}^{-1} N_{12} \right)^{-1} N_{21} N_{11}^{-1} \right] =$$

$$= \sigma_0^2 \left(N_{11} - N_{12} N_{22}^{-1} N_{21} \right)^{-1} > \sigma_0^2 N_{11}^{-1} = \underbrace{D\{\hat{\boldsymbol{\xi}}_1\}}_{\text{datum supplied}} . \tag{3.53}$$

The smaller dispersion in the last line of (3.53) shows that if a datum is introduced (increase in information), the unknown parameters $\boldsymbol{\xi}$ are estimated with smaller variance.

Minimally constrained adjustment The type of least-squares adjustment described in this section belongs to a class of *minimally constrained adjustment*, a subject treated in much greater detail in Part II. The reason that the adjustment is of type minimally constrained is because the datum information only provides information on $m - q$ of the parameters, which is just enough to overcome the rank deficiency of the model. The result is a *unique solution* for the residual vector $\tilde{\boldsymbol{e}}$, the adjusted observations $A\hat{\boldsymbol{\xi}}$, and the estimated variance component $\hat{\sigma}_0^2$. This means that the specification for $\boldsymbol{\xi}_2^0$ will not affect the computed values of these quantities. Put another way, we can say that they are invariant to the choice of the datum. On the other hand, the vector of estimated parameters $\hat{\boldsymbol{\xi}}$ will not be unique; it does depend on the specification of $\boldsymbol{\xi}_2^0$.

3.6 Practice Problems

1. Starting with the Lagrange target function (3.24), derive the least-squares solution (LESS) within the Gauss-Markov Model for the unknown parameter vector $\boldsymbol{\xi}$ and the unknown vector of Lagrange multipliers $\boldsymbol{\lambda}$.

2. Based on your answer in the preceding problem, show that the identity $\Omega = \hat{\boldsymbol{\lambda}}^T P^{-1} \hat{\boldsymbol{\lambda}}$ in (3.26) holds.

3. In order to determine the height of point F, leveling measurements have been taken in forward and reverse directions from three different points A, B, and C, each with known height. The relevant data are given in Table 3.1.

Table 3.1: Leveling data for Problem 3.

Point	Height [m]	Forward obs. to F [m]	Length of path [km]	Reverse obs. from F [m]
A	100.055	10.064	2.5	-10.074
B	102.663	7.425	4	-7.462
C	95.310	14.811	6	-14.781

Assume that the standard deviations of the observations are $\sigma = 3\,\mathrm{mm}$ per every one km of leveling and that all measurements are uncorrelated. Setup the Gauss-Markov Model and compute the LESS of:

(a) The height at point F and its estimated dispersion.

(b) The vector of residuals and its estimated dispersion matrix.

(c) The estimated variance component.

(d) Compute the trace of the product $\sigma_0^{-2}\cdot D\{\tilde{e}\}\cdot P$ and confirm that it equals the redundancy of the model.

4. Elevations were observed with a digital level at nodes of a 2D-grid. The horizontal coordinates of the nodes $(\boldsymbol{X},\boldsymbol{Y})$ are assumed be be known with certainty, while the random errors of the observed elevations have a homogeneous distribution with zero mean and $\sigma^2 = (10\,\mathrm{mm})^2$ variance. The data are listed in Table 3.2.

(a) Use the LESS within the GMM to estimate the parameters of a fitted plane assuming the observation equations can be modeled by

$$E\{y_i\} = aX_i + bY_i + c, \quad i = 1, \ldots, n,$$

with unknown parameters $\boldsymbol{\xi} = [a,\, b,\, c]^T$.

(b) Use the LESS within the GMM to estimate parameters for a quadratic surface assuming the observation equations can be modeled by

$$E\{y_i\} = aX_i^2 + bY_i^2 + cX_iY_i + dX_i + eY_i + f, \quad i = 1, \ldots, n,$$

with unknown parameters $\boldsymbol{\xi} = [a,\, b,\, c,\, d,\, e,\, f]^T$.

(c) Which of the two above observational models, the planar one or the quadratic one, fit the data best? Give the reason for your answer.

Table 3.2: Elevation data y_i observed at known grid locations

i	X_i	Y_i	y_i	i	X_i	Y_i	y_i
1	−20	−20	9.869	14	0	10	10.019
2	−20	−10	9.920	15	0	20	10.037
3	−20	0	9.907	16	10	−20	9.946
4	−20	10	9.957	17	10	−10	9.988
5	−20	20	9.959	18	10	0	10.035
6	−10	−20	9.889	19	10	10	10.055
7	−10	−10	9.937	20	10	20	10.066
8	−10	0	9.973	21	20	−20	9.963
9	−10	10	10.025	22	20	−10	9.986
10	−10	20	10.026	23	20	0	10.037
11	0	−20	9.917	24	20	10	10.068
12	0	−10	10.000	25	20	20	10.069
13	0	0	10.007				
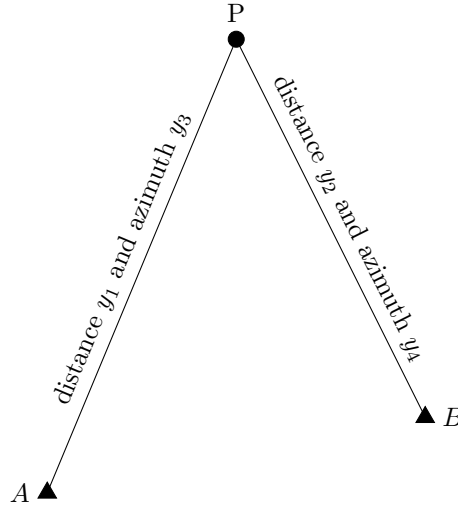
Figure 3.3: Two distances and two azimuths measured from known points A and B to determine coordinates of point P

5. To determine the coordinates of an unknown point $P(x, y)$, some measurements were carried out from two given points $A(50, 30)$ and $B(100, 40)$ in meters.

   Two distances were observed, the first from A to P and the second from B to P. The observed distances are $y_1 = 66.137$ m and $y_2 = 58.610$ m, and they are considered to be uncorrelated with variance $\sigma^2 = (1\,\text{cm})^2$.

In addition, two azimuths were observed independently. The observed azimuth from A to P is $y_3 = 20°20'55''$ and the observed azimuth from B to P is $y_4 = 332°33'41''$. The standard deviation of both azimuths is $\sigma_\alpha = 5''$.

Compute the following:

(a) The estimated coordinates of point P.

(b) The estimated variances of the coordinates and their correlation coefficient.

(c) The residual vector $\tilde{e}$.

(d) The estimated variance component $\hat{\sigma}_0^2$.

6. To determine the coordinates of a new point P, distances were measured to four given points having known coordinates. One angle was also measured. The coordinates of the given points are listed in Table 3.3, and the observations, along with their standard deviations, are listed in Table 3.4.
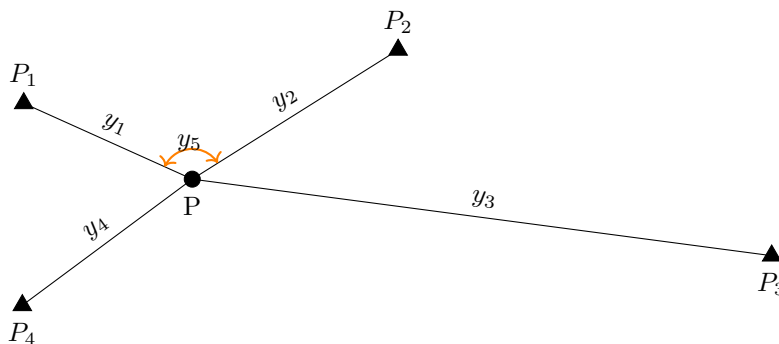
Figure 3.4: Four distances and one angle measured to determine point P

Table 3.3: Coordinates of known points in meters

Point	x_i [m]	y_i [m]
P_1	842.281	925.523
P_2	1337.544	996.249
P_3	1831.727	723.962
P_4	840.408	658.345

Table 3.4: Observations: distances and their standard deviation are in units of meters.

Observable	y_i	σ_i
P_1P	244.457	0.006
P_2P	321.622	0.010
P_3P	773.129	0.024
P_4P	280.019	0.080
$\angle P_1PP_2$	$123°38'20''$	$5''$

(a) Setup the observation equations and form the normal equations.

(b) Compute the LESS for the coordinates of points P and compute their variances and covariances.

(c) Compute the residual vector $\tilde{e}$, the adjusted observations, and the dispersion matrices of both.

(d) Compute the estimated variance component $\hat{\sigma}_0^2$.

7. Pearson (1901) presented the data in Table 3.5 for a line-fitting problem. Considering the x-coordinates to be known with certainty and the y-coordinates to be observed with random errors having zero mean and iid dispersion, complete the following:

(a) Setup a Gauss-Markov Model to estimate the slope and y-intercept of a line and compute those estimates.

(b) Compute the residual vector $\tilde{e}$ and the estimated variance component $\hat{\sigma}_0^2$.

(c) Plot the data along with the fitted line.

Table 3.5: Pearson (1901) data for a fitted line

Point i	x_i	y_i
1	0.0	5.9
2	0.9	5.4
3	1.8	4.4
4	2.6	4.6
5	3.3	3.5
6	4.4	3.7
7	5.2	2.8
8	6.1	2.8
9	6.5	2.4
10	7.4	1.5

8. The affine 2-D transformation is based on six unknown parameters:

   - ξ_1, ξ_2 for the *translation* of the origin of the coordinate frame,
   - β, $\beta + \epsilon$ for the *rotation* angles of the respective axes.
   - ω_1, ω_2 for the *scale* factors of the respective axes.

   For a point having coordinates (x_i, y_i) in the source coordinate frame and (X_i, Y_i) in the target coordinate frame, the transformation is described by

   $$\begin{bmatrix} X_i \\ Y_i \end{bmatrix} = \begin{bmatrix} \omega_1 \cdot \cos\beta & -\omega_2 \cdot \sin(\beta + \epsilon) \\ \omega_1 \cdot \sin\beta & \omega_2 \cdot \cos(\beta + \epsilon) \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} e_{X_i} \\ e_{Y_i} \end{bmatrix}. \tag{3.54a}$$

   Here

   - (x_i, y_i) are given coordinates in the *source system*;
   - (X_i, Y_i) are observed coordinates in the *target system*;
   - i denotes the point number, $i \in \{1, 2, \ldots, n/2\}$.

   Making the substitutions

   $$\xi_3 := \omega_1 \cos\beta, \ \xi_4 := \omega_2 \sin(\beta + \epsilon), \ \xi_5 := \omega_1 \sin\beta, \ \xi_6 := \omega_2 \cos(\beta + \epsilon) \tag{3.54b}$$

   results in the linear system of observation equations

   $$\begin{aligned} X_i &= x_i \cdot \xi_3 - y_i \cdot \xi_4 + \xi_1 + e_{X_i}, \\ Y_i &= x_i \cdot \xi_5 + y_i \cdot \xi_6 + \xi_2 + e_{Y_i}, \end{aligned} \quad \begin{bmatrix} e_{X_i} \\ e_{Y_i} \end{bmatrix} \sim (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} (Q_{XX})_{ii} & (Q_{XY})_{ii} \\ (Q_{XY}^T)_{ii} & (Q_{YY})_{ii} \end{bmatrix}), \tag{3.54c}$$

where Q_{XX}, Q_{YY}, and Q_{XY} are given cofactor matrices.

Using the data from Table 3.6, which is copied from Wolf (1983, p. 586), and assuming the random observation errors are iid, complete the following:

(a) Compute the least-squares estimates of $\hat{\boldsymbol{\xi}}$ and then the derived quantities $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\omega}_1$, and $\hat{\omega}_2$.

(b) Sketch a diagram showing the axes of both coordinate systems. Annotate the diagram with labels for the rotation angles between the axes and the translations between the two origins.

(c) Use the estimated parameters to compute coordinates in the xy system for points 1–3 shown in Table 3.6.

Table 3.6: Calibrated (known) and comparator (measured) coordinates from Wolf (1983, p. 586)

	Comparator coordinates		Calibrated coordinates	
Point	X [mm]	Y [mm]	x [mm]	y [mm]
Fiducial A	55.149	159.893	-113.000	0.000
Fiducial B	167.716	273.302	0.000	113.000
Fiducial C	281.150	160.706	113.000	0.000
Fiducial D	168.580	47.299	0.000	-113.000
1	228.498	105.029		
2	270.307	199.949		
3	259.080	231.064		

9. The spirit leveling data in Table 3.7 come from Rainsford (1968), where orthometric corrections have already been applied to the recorded observations. The weight of each observation was taken as the distance in miles divided by 100. All random observation errors are uncorrelated. The unknown parameters are the heights of points A, B, C, D, E, and F (Figure 3.5). Since the observations pertain to height differences, the model has a rank deficiency (datum deficiency) of one. Therefore, datum information is introduced as in Section 3.5 by specifying the height of point D as 1928.277 ft.

Complete the following:

(a) Set up a partitioned Gauss-Markov Model and the corresponding partitioned least-squares normal equations according to Section 3.5.

(b) Compute the LESS for the estimated heights of points A, B, C, E, and F.

(c) Compute the residual vector and the estimated variance component.

(d) Compute the adjusted observations and then sum them for each of the four closed loops in the network that pass through either point B or C.
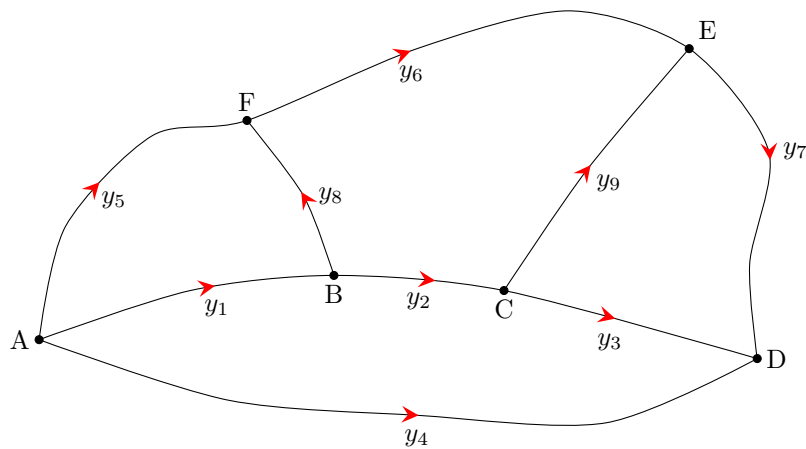
Figure 3.5: Leveling network after Rainsford (1968)

Table 3.7: Leveling data from Rainsford (1968)

From	To	No.	Observed height diff. [ft]	Length [miles]
A	B	1	$+124.632$	68
B	C	2	$+217.168$	40
C	D	3	-92.791	56
A	D	4	$+248.754$	171
A	F	5	-11.418	76
F	E	6	-161.107	105
E	D	7	$+421.234$	80
B	F	8	-135.876	42
C	E	9	-513.895	66

Also sum them for the closed perimeter loop that contains all points except B and C.

(e) Repeat all your computations using a height of 1679.432 ft for point A as datum information. Which results are different and which are the same between the two adjustments? Can you explain the differences and similarities?

3.7 Summary Formulas for the Introduction of Datum Information for the Least-Squares Solution Within the Rank Deficient Gauss-Markov Model

The rank deficient Gauss-Markov Model is given by

$$
\underset{n\times 1}{\boldsymbol{y}} = \left[\begin{array}{c|c} \underset{n\times q}{A_1} & \underset{n\times(m-q)}{A_2} \end{array}\right] \left[\begin{array}{c} \underset{q\times 1}{\boldsymbol{\xi}_1} \\ \underset{(m-q)\times 1}{\boldsymbol{\xi}_2} \end{array}\right] + \boldsymbol{e}, \ \ \boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 P^{-1}\right),
$$
$$
\operatorname{rk} A =: q < m \ \text{ and } \ \operatorname{rk} A_1 = q.
$$

Table 3.8: Summary formulas for the introduction of datum information $(\hat{\boldsymbol{\xi}}_2 \to \boldsymbol{\xi}_2^0)$ for the LESS within the rank deficient Gauss-Markov Model

Quantity	Formula	Eq.
Model redundancy	$r = n - \operatorname{rk} A = n - q$	(3.48)
Vector of estimated parameters, with given $\boldsymbol{\xi}_2^0$	$\hat{\boldsymbol{\xi}}_1 = N_{11}^{-1}\big(\boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0\big)$	(3.43b)
Dispersion matrix for estimated parameters	$D\{\hat{\boldsymbol{\xi}}_1\} = \sigma_0^2 \cdot N_{11}^{-1}$	(3.44)
Vector of predicted residuals, with given $\boldsymbol{\xi}_2^0$	$\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}} = \boldsymbol{y} - A_1\hat{\boldsymbol{\xi}}_1 - A_2\boldsymbol{\xi}_2^0$	(3.45b)
Dispersion matrix for residuals	$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot \big(P^{-1} - A_1 N_{11}^{-1} A_1^T\big)$	(3.45b)
Sum of squared residuals (SSR)	$\Omega = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$	(3.47)
Estimated variance component, with given $\boldsymbol{\xi}_2^0$	$\hat{\sigma}_0^2 = (\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}})/r =$ $(\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}_1^T \hat{\boldsymbol{\xi}}_1 - \boldsymbol{c}_2^T \boldsymbol{\xi}_2^0)/(n-q)$	(3.49)

Continued on next page

Quantity	Formula	Eq.
Vector of adjusted observations	$\widehat{E\{\boldsymbol{y}\}} =: \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}} = A_1\hat{\boldsymbol{\xi}}_1 + A_2\boldsymbol{\xi}_2^0$	(3.46a)
Dispersion matrix for adjusted observations	$D\{\hat{\boldsymbol{\mu}}_y\} = \sigma_0^2 {\cdot} A_1 N_{11}^{-1} A_1^T$	(3.46b)

Continued from previous page

3.8   Summary Formulas for the Least-Squares Solution Within the Gauss-Markov Model With Full Rank

The Gauss-Markov Model with full column rank coefficient matrix A is given by

$$\underset{n \times 1}{\boldsymbol{y}} = \underset{n \times m}{A}\,\boldsymbol{\xi} + \boldsymbol{e}, \;\; \boldsymbol{e} \sim \big(\boldsymbol{0}, \sigma_0^2 P^{-1}\big),$$
$$\operatorname{rk} A = m.$$

Table 3.9: Summary formulas for the LESS within the Gauss-Markov Model with full rank

Quantity	Formula	Eq.
Model redundancy	$r = n - \operatorname{rk} A = n - m$	(3.3)
Vector of estimated parameters	$\hat{\boldsymbol{\xi}} = N^{-1}\boldsymbol{c}, \; [N, \boldsymbol{c}] := A^T P[A, \boldsymbol{y}]$	(3.7)
Dispersion matrix for estimated parameters	$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 {\cdot} N^{-1}$	(3.13)
Vector of predicted residuals	$\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}} = \big(I_n - AN^{-1}A^T P\big)\boldsymbol{y}$	(3.9)
Dispersion matrix for residuals	$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 {\cdot}\big(P^{-1} - AN^{-1}A^T\big)$	(3.14a)
Sum of squared residuals (SSR)	$\Omega = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$	(3.22)

Quantity	Formula	Eq.
Estimated variance component	$\hat{\sigma}_0^2 = (\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}})/(n - \text{rk}\, A)$	(3.30)
Vector of adjusted observations	$\widehat{E\{\boldsymbol{y}\}} =: \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}$	(3.11)
Dispersion matrix for adjusted observations	$D\{\hat{\boldsymbol{\mu}}_y\} = \sigma_0^2 {\cdot} A N^{-1} A^T$	(3.15a)

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Chapter 4

The Model of Condition Equations

4.1 Model Definition

In the least-squares adjustment within the *model of condition equations*, the unknown parameters $\boldsymbol{\xi}$ are not estimated directly, rather the random error vector $\boldsymbol{e}$ is predicted. This approach might be taken if the parameters are of no particular interest, or it might be done to make the problem easy to formulate. An example of the latter is the adjustment of leveling networks, where the parameters (heights of the stations) are of primary interest, but because closed "level loops" within the network sum to zero (a necessary condition), it is convenient to difference the observations along these loops before performing the adjustment (see level-loop example in Section 4.4). Another motivation for using the model of condition equations is that the size of the matrix to invert in the least-squares solution (LESS) may be smaller than that in the corresponding LESS within the Gauss-Markov Model (GMM).

Let the $r \times n$ matrix B represent a difference operator such that when it is applied to the $n \times 1$ observation equations $\boldsymbol{y} = A\boldsymbol{\xi} + \boldsymbol{e}$, the parameters are eliminated. More specifically, we require that $BA = 0$, which implies that $B\boldsymbol{y} = B(A\boldsymbol{\xi} + \boldsymbol{e}) = B\boldsymbol{e}$. Therefore, by applying the difference operator B, the GMM is transformed to the following model of condition equations:

$$\boldsymbol{w} := \underset{r \times n}{B}\, \boldsymbol{y} = B\boldsymbol{e}, \quad \underset{n \times 1}{\boldsymbol{e}} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}), \tag{4.1a}$$

$$r := n - q = \operatorname{rk} B, \tag{4.1b}$$

where the variable r denotes the redundancy of the model, and q is the rank of the $n \times m$ matrix A from the GMM (3.1). Equation (4.1b) requires that matrix B has full row rank. Moreover, it shows that the redundancy of the model is not changed from that of the GMM by eliminating parameters.

4.2　The Least-Squares Solution Within the Model of Condition Equations

The least-squares criterion for minimizing the (P-weighted, squared) norm of the vector of observational random errors is written as

$$\min \boldsymbol{e}^T P \boldsymbol{e} \ \text{ subject to } \boldsymbol{w} = B\boldsymbol{e}, \tag{4.2}$$

for which the Lagrange target function

$$\Phi(\boldsymbol{e}, \boldsymbol{\lambda}) := \boldsymbol{e}^T P \boldsymbol{e} + 2\boldsymbol{\lambda}^T (\boldsymbol{w} - B\boldsymbol{e}) \tag{4.3}$$

can be written, which must be made stationary with respect to the unknown terms $\boldsymbol{e}$ and $\boldsymbol{\lambda}$. Here, $\boldsymbol{\lambda}$ is an $r \times 1$ vector of Lagrange multipliers. Taking the first partial derivatives of (4.3) leads to the Euler-Lagrange necessary conditions

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}} = P\tilde{\boldsymbol{e}} - B^T \hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \tag{4.4a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = \boldsymbol{w} - B\tilde{\boldsymbol{e}} \doteq \boldsymbol{0}. \tag{4.4b}$$

The sufficient condition, required to ensure a minimum is reached, is satisfied by $\partial^2 \Phi / \partial \boldsymbol{e} \partial \boldsymbol{e}^T = 2P$, which is positive definite since the weight matrix P is invertible according to (4.1a). The simultaneous solution of (4.4a) and (4.4b) leads to the *Best LInear Prediction* (BLIP) of $\boldsymbol{e}$ as derived in the following: Equation (4.4a) leads to

$$\tilde{\boldsymbol{e}} = P^{-1}B^T \hat{\boldsymbol{\lambda}}. \tag{4.5a}$$

Then, (4.4b) and (4.5a) allows

$$\boldsymbol{w} = B\tilde{\boldsymbol{e}} = \big(BP^{-1}B^T\big)\hat{\boldsymbol{\lambda}} \Rightarrow \tag{4.5b}$$

$$\hat{\boldsymbol{\lambda}} = \big(BP^{-1}B^T\big)^{-1}\boldsymbol{w} \Rightarrow \tag{4.5c}$$

$$\tilde{\boldsymbol{e}} = P^{-1}B^T \big(BP^{-1}B^T\big)^{-1}\boldsymbol{w}, \tag{4.5d}$$

finally leading to the predicted random error vector

$$\boxed{\tilde{\boldsymbol{e}} = P^{-1}B^T \big(BP^{-1}B^T\big)^{-1}B\boldsymbol{y}.} \tag{4.5e}$$

Note that the matrix product $BP^{-1}B^T$ results in a symmetric, positive-definite matrix of size $r \times r$, since B has full row rank. The predicted random error vector $\tilde{\boldsymbol{e}}$ is also called the *residual* vector. The expectation of the given observation vector is expressed as $E\{\boldsymbol{y}\} = \boldsymbol{\mu}_y$, where $\boldsymbol{\mu}_y$ is the true, but unknown, vector of observables. Thus we write the vector of adjusted observations as

$$\boxed{\widehat{E\{\boldsymbol{y}\}} = \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}.} \tag{4.6}$$

Nota bene: Implicit in the term $B\boldsymbol{y}$ is the subtraction of a constant term $\boldsymbol{\kappa}$ if necessary, viz. $(B\boldsymbol{y} - \boldsymbol{\kappa}) - B\boldsymbol{e} = \boldsymbol{0}$, implying that $B\boldsymbol{y} \to B\boldsymbol{y} - \boldsymbol{\kappa}$. An example is the condition that the n interior angles of a simple polygon in a plane must sum to $\kappa = (n-2)180°$. Then the condition equation would read

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 - e_1 \\ y_2 - e_2 \\ \vdots \\ y_n - e_n \end{bmatrix} - (n-2)\pi = 0.$$

Thus, for numerical computations, we may need to modify (4.5e) to read

$$\tilde{\boldsymbol{e}} = P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}(B\boldsymbol{y} - \boldsymbol{\kappa}), \tag{4.7}$$

which has no effect on the dispersion formulas that follow.

The square of the P-weighted residual norm Ω, also called the sum of squared residuals (SSR), is computed by

$$\Omega = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} = \tilde{\boldsymbol{e}}^T B^T \hat{\boldsymbol{\lambda}} = \boldsymbol{w}^T \hat{\boldsymbol{\lambda}} = \boldsymbol{w}^T (BP^{-1}B^T)^{-1}\boldsymbol{w} = \tag{4.8a}$$

$$= \boldsymbol{y}^T B^T (BP^{-1}B^T)^{-1}B\boldsymbol{y}, \tag{4.8b}$$

leading to the estimated variance component

$$\hat{\sigma}_0^2 = \frac{\Omega}{r} = \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{r}, \tag{4.9}$$

with $r = \operatorname{rk} B$. In words, it is described as the squared P-weighted residual norm divided by the degrees of freedom (redundancy) of the model.

Applying the law of error propagation, the dispersion of the residual vector is computed by

$$D\{\tilde{\boldsymbol{e}}\} = P^{-1}B^T(BP^{-1}B^T)^{-1}B \cdot D\{\boldsymbol{y}\} \cdot B^T(BP^{-1}B^T)^{-1}BP^{-1} =$$

$$= P^{-1}B^T(BP^{-1}B^T)^{-1}B(\sigma_0^2 P^{-1})B^T(BP^{-1}B^T)^{-1}BP^{-1} \Rightarrow$$

$$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot P^{-1}B^T(BP^{-1}B^T)^{-1}BP^{-1}. \tag{4.10}$$

As we did earlier within the GMM (Section 3.2.2), we compute the covariance between the residual vector $\tilde{\boldsymbol{e}}$ and the vector adjusted observations $\hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}$ as follows:

$$C\{\hat{\boldsymbol{\mu}}_y, \tilde{\boldsymbol{e}}\} = C\{\left[I - P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right]\boldsymbol{y}, \, P^{-1}B^T\left(BP^{-1}B^T\right)B\boldsymbol{y}\} =$$

$$= \left[I - P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right] \cdot D\{\boldsymbol{y}\} \cdot \left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right]^T =$$

$$= \left[I - P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right] \cdot \sigma_0^2 P^{-1} \cdot B^T\left(BP^{-1}B^T\right)^{-1}BP^{-1} =$$

$$= \sigma_0^2 \big[P^{-1}B^T \big(BP^{-1}B^T\big)^{-1}BP^{-1} - P^{-1}B^T\big(BP^{-1}B^T\big)^{-1}BP^{-1}B^T \cdot$$
$$\cdot \big(BP^{-1}B^T\big)^{-1}BP^{-1} \big] = 0 \tag{4.11}$$

Thus, it has been shown that the residuals and adjusted observations are uncorrelated, and therefore the dispersion of the adjusted observations can be written as

$$\boxed{D\{\hat{\boldsymbol{\mu}}_y\} = D\{\boldsymbol{y}\} - D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \big[P^{-1} - P^{-1}B^T(BP^{-1}B^T)^{-1}BP^{-1}\big].} \tag{4.12}$$

Note that B is not a unique matrix, but regardless of how B is chosen the results of the adjustment will be the same, provided the following necessary conditions for B are satisfied:

(i) *Dimensionality:* $\operatorname{rk} B = n - \operatorname{rk} A = n - q = r$, which means that $\operatorname{rk} B + \operatorname{rk} A = (n - q) + q = n$.

(ii) *Orthogonality:* $BA = 0$.

4.3 Equivalence Between LESS Within the Gauss-Markov Model and the Model of Condition Equations

To show the equivalence between the least-squares adjustments within the GMM and the model of condition equations, it must be shown that the predicted random error vectors (residuals) from both adjustments are equivalent. The residual vector $\tilde{\boldsymbol{e}}$ from each adjustment can be expressed as a projection matrix times the true random error vector $\boldsymbol{e}$ (or equivalently, times the observation vector $\boldsymbol{y}$) as shown below.

The residual vector within the GMM can be written as

$$\tilde{\boldsymbol{e}} = \big[I_n - AN^{-1}A^T P\big]\boldsymbol{e}. \tag{4.13}$$

And the residual vector within the model of condition equations can be written as

$$\tilde{\boldsymbol{e}} = \big[P^{-1}B^T\big(BP^{-1}B^T\big)^{-1}B\big]\boldsymbol{e}. \tag{4.14}$$

Note that the right sides of (4.13) and (4.14) cannot actually be computed since $\boldsymbol{e}$ is unknown, but the equations do hold since, for the GMM,

$$\tilde{\boldsymbol{e}} = \big[I_n - AN^{-1}A^T P\big]\boldsymbol{y} =$$
$$= \big[I_n - AN^{-1}A^T P\big]\big(A\boldsymbol{\xi} + \boldsymbol{e}\big) =$$
$$= \big[A\boldsymbol{\xi} - AN^{-1}(A^T PA)\boldsymbol{\xi}\big] + \big[I_n - AN^{-1}A^T P\big]\boldsymbol{e} \Rightarrow$$
$$\tilde{\boldsymbol{e}} = \big[I_n - AN^{-1}A^T P\big]\boldsymbol{e}, \tag{4.15}$$

and, for the model of condition equations,

$$\tilde{\boldsymbol{e}} = P^{-1}B^T\big(BP^{-1}B^T\big)^{-1}B\boldsymbol{y} =$$

$$= P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B(A\boldsymbol{\xi}+\boldsymbol{e}) \Rightarrow$$

$$\tilde{\boldsymbol{e}} = \left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right]\boldsymbol{e}, \tag{4.16}$$

using the fact that $BA = 0$.

To show that (4.13) and (4.14) are equivalent, it must be shown that the range spaces and the nullspaces are equivalent for their respective projection matrices

$$\bar{P}_1 := [I_n - AN^{-1}A^T P] \ \text{ and } \ \bar{P}_2 := [P^{-1}B^T(BP^{-1}B^T)^{-1}B].$$

(i) Equivalent range spaces: Show that

$$\mathcal{R}\left[I_n - AN^{-1}A^T P\right] = \mathcal{R}\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right].$$

Proof: Since $A^T P P^{-1}B^T = A^T B^T = 0$, then

$$\left[I_n - AN^{-1}A^T P\right]\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right]\boldsymbol{z} =$$

$$= \left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right]\boldsymbol{z} - \boldsymbol{0} \ \text{ for all } \boldsymbol{z} \in \mathbb{R}^n,$$

which, according to (1.3), implies that

$$\mathcal{R}\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right] \subset \mathcal{R}\left[I_n - AN^{-1}A^T P\right].$$

Also:

$$\dim \mathcal{R}\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right] =$$

$$= \mathrm{rk}\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right] = \qquad \text{using (A.45a)}$$

$$= \mathrm{tr}\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right] = \qquad \text{using (1.7c)}$$

$$= \mathrm{tr}\left[BP^{-1}B^T\left(BP^{-1}B^T\right)^{-1}\right] = \qquad \text{using (A.5)}$$

$$= \mathrm{tr}\, I_r = r.$$

Furthermore:

$$\dim \mathcal{R}\left[I_n - AN^{-1}A^T P\right] =$$

$$= \mathrm{rk}\left(I_n - AN^{-1}A^T P\right) = \qquad \text{using (A.45a)}$$

$$= \mathrm{tr}\left(I_n - AN^{-1}A^T P\right) = \qquad \text{using (1.7c)}$$

$$= \mathrm{tr}\, I_n - \mathrm{tr}\left(N^{-1}A^T PA\right) = \qquad \text{using (A.5)}$$

$$= n - \mathrm{rk}\, N = n - \mathrm{rk}\, A =$$

$$= n - q = r,$$

which implies that

$$\boxed{\mathcal{R}\left[I_n - AN^{-1}A^T P\right] = \mathcal{R}\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right],} \tag{4.17}$$

since one range space contains the other and both have the same dimension. Thus we have shown that the range spaces (column spaces) of $\bar{P}_1$ and $\bar{P}_2$ are equivalent. Now we turn to the nullspaces.

(ii) Equivalent Nullspaces: Show that

$$\mathcal{N}\big[I_n - AN^{-1}A^T P\big] = \mathcal{N}\big[P^{-1}B^T \big(BP^{-1}B^T\big)^{-1}B\big].$$

Proof:

First show that $\mathcal{N}\big[I_n - AN^{-1}A^T P\big] = \mathcal{R}(A)$.

We begin with

$$\big[I_n - AN^{-1}A^T P\big]A\boldsymbol{\alpha} = \mathbf{0} \;\text{ for all } \boldsymbol{\alpha},$$

which implies that

$$\mathcal{R}(A) \subset \mathcal{N}\big[I_n - AN^{-1}A^T P\big], \;\text{ since } A\boldsymbol{\alpha} \in \mathcal{R}(A);$$

also

$$\dim \mathcal{R}(A) = \operatorname{rk} A = q.$$

Equations (A.45a) and (A.45b) reveal that the sum of the dimensions of the range space and nullspace of a matrix is equal to its number of columns. Using this property, and results from (i), we find that

$$\dim \mathcal{N}\big[I_n - AN^{-1}A^T P\big] =$$
$$= n - \dim \mathcal{R}\big[I_n - AN^{-1}A^T P\big] = n - r = q.$$

Therefore,

$$\mathcal{N}\big[I_n - AN^{-1}A^T P\big] = \mathcal{R}(A).$$

Also, we have

$$\big[P^{-1}B^T \big(BP^{-1}B^T\big)^{-1}B\big]A = 0,$$

since $BA = 0$. The preceding development implies that

$$\mathcal{R}(A) = \mathcal{N}\big[I_n - AN^{-1}A^T P\big] \subset \mathcal{N}\big[P^{-1}B^T \big(BP^{-1}B^T\big)^{-1}B\big],$$

or

$$\mathcal{N}(\bar{P}_1) \subset \mathcal{N}(\bar{P}_2).$$

We showed in part (i) that the dimensions of the range spaces of the respective projection matrices are equivalent. And, since

$$\dim \mathcal{N}(\bar{P}_1) = n - \dim \mathcal{R}(\bar{P}_1) = n - \dim \mathcal{R}(\bar{P}_2),$$

it follows that

$$\dim \mathcal{N}(\bar{P}_1) = \dim \mathcal{N}(\bar{P}_2).$$

As already stated in part (i), if one vector space is a subset of another and both spaces have the same dimension, then the subspaces are equivalent. Therefore, we can say that

$$\mathcal{N}(\bar{P}_1) = \mathcal{N}(\bar{P}_2),$$

or

$$\mathcal{N}\left[I_n - AN^{-1}A^T P\right] = \mathcal{N}\left[P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}B\right]. \qquad (4.18)$$

We have shown that both the range spaces and nullspaces of the projection matrices $\bar{P}_1$ and $\bar{P}_2$ are equivalent, proving that *the residual vectors from the two adjustments are the same and thus that the two adjustments are indeed equivalent.*

4.4 Examples — Linear and Nonlinear

4.4.1 Linear Example — a Small Leveling Network

The following example is borrowed from Mikhail and Gracie (1981, Problem 4-8). It involves a leveling network comprised of two closed loops as shown in Figure 4.1. The data are listed in Table 4.1.
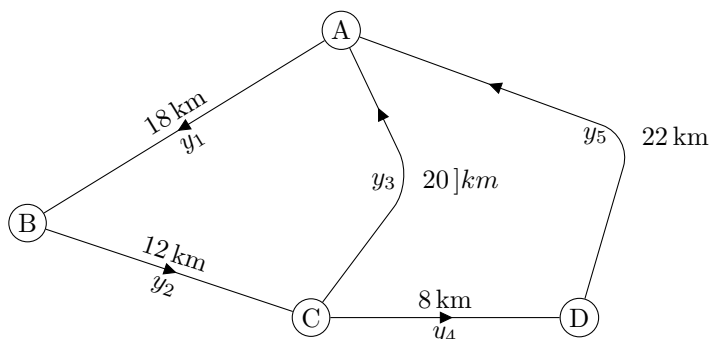
Figure 4.1: Example leveling network

Table 4.1: Leveling network data

Line	Element of $\boldsymbol{y}$	Observed elevation difference (m)	Length (km)
A to B	y_1	-12.386	18
B to C	y_2	-11.740	12
C to A	y_3	24.101	20
C to D	y_4	-8.150	8
D to A	y_5	32.296	22

In a leveling network, one condition equation can be written for each closed loop. Connecting observations in a counter-clockwise order, two condition equations may

be written as

$$(y_1 - e_1) + (y_2 - e_2) + (y_3 - e_3) = 0,$$
$$-(y_3 - e_3) + (y_4 - e_4) + (y_5 - e_5) = 0,$$

or, in matrix form, as

$$B\boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -12.386 \\ -11.740 \\ 24.101 \\ -8.150 \\ 32.296 \end{bmatrix} = B\boldsymbol{e}.$$

The observations' weights are inversely proportional to the distances in km, so that

$$P^{-1} = 10^{-6} \cdot \text{diag}(18, 12, 20, 8, 22) \cdot \text{m}^2$$

appears to be a reasonable weight matrix. The residuals are then computed by

$$\tilde{\boldsymbol{e}} = P^{-1}B^T(BP^{-1}B^T)^{-1}B\boldsymbol{y} = \begin{bmatrix} -0.003 \\ -0.002 \\ -0.020 \\ 0.007 \\ 0.018 \end{bmatrix} \text{m}.$$

The redundancy of the model is given by $r = \text{rk}\,B = 2$. The adjusted observations are computed by

$$\hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}} = \begin{bmatrix} -12.383 \\ -11.738 \\ 24.121 \\ -8.157 \\ 32.278 \end{bmatrix} \text{m}.$$

The dispersion matrix for the residuals is

$$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot P^{-1}B^T(BP^{-1}B^T)^{-1}BP^{-1} =$$

$$= \sigma_0^2 \cdot \begin{bmatrix} 7.7 & 5.1 & 5.1 & 1.4 & 3.8 \\ 5.1 & 3.4 & 3.4 & 0.9 & 2.5 \\ 5.1 & 3.4 & 11.4 & -2.3 & -6.3 \\ 1.4 & 0.9 & -2.3 & 1.5 & 4.2 \\ 3.8 & 2.5 & -6.3 & 4.2 & 11.5 \end{bmatrix} \text{mm}^2.$$

The weighted sum of squared residuals is $\Omega := \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} = (6.454972)^2$, leading to the estimated variance component $\hat{\sigma}_0^2 = \Omega/r = (4.564355)^2$. The estimated variance

component may be used to compute the *estimated dispersion matrix* for the residuals as

$$\hat{D}\{\tilde{e}\} = \hat{\sigma}_0^2 \cdot P^{-1}B^T(BP^{-1}B^T)^{-1}BP^{-1} =$$

$$= \begin{bmatrix} 1.61 & 1.07 & 1.07 & 0.29 & 0.79 \\ 1.07 & 0.71 & 0.71 & 0.19 & 0.52 \\ 1.07 & 0.71 & 2.38 & -0.48 & -1.31 \\ 0.29 & 0.19 & -0.48 & 0.32 & 0.87 \\ 0.79 & 0.52 & -1.31 & 0.87 & 2.40 \end{bmatrix} \text{cm}^2.$$

Now, if the same problem were to be modeled within the Gauss-Markov Model with the unknown parameters being the heights of the points denoted by $\xi = [H_A, H_B, H_C, H_D]^T$, then the coefficient matrix would be written as

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

Obviously, the conditions $r = n - \text{rk}\, A = 5 - 3 = 2 = \text{rk}\, B$ and $BA = 0$ are satisfied. Indeed, one can easily verify that the LESS within the GMM will produce the same residual vector and same estimated variance component as shown above.

4.4.2 Nonlinear Example — Observations of a Triangle

Table 4.2 lists distance observations for all sides of a triangle and two of its angles as depicted in Figure 4.2. The standard deviations of the observations are shown in the last column of the table. The observations are to be adjusted by computing the residual vector within the model of condition equations.

Table 4.2: Observations of sides and angles of a triangle

Obs. no.	Observation	Std. dev.
y_1	120.01 m	1 cm
y_2	105.02 m	1 cm
y_3	49.98 m	1 cm
y_4	94°47′10″	20″
y_5	60°41′20″	20″

The following two nonlinear condition equations can be written as a function of the unknown 5×1 random error vector e, the first based on the law of sines and
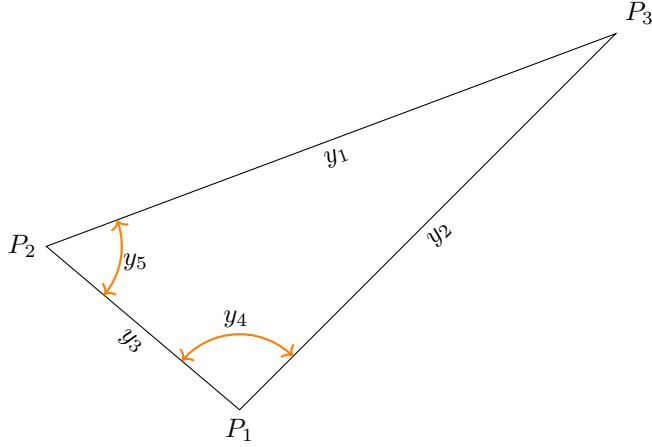
Figure 4.2: Observations of sides and angles of a triangle

the second on the law of cosines for a triangle in a plane:

$$f_1(\boldsymbol{e}) = (y_2 - e_2) \cdot \sin(y_4 - e_4) - (y_1 - e_1) \cdot \sin(y_5 - e_5) = 0 \qquad (4.19\text{a})$$

$$\begin{aligned} f_2(\boldsymbol{e}) = {}& (y_1 - e_1)^2 + (y_2 - e_2)^2 - (y_3 - e_3)^2 - \\ & - 2 \cdot (y_1 - e_1)(y_2 - e_2) \cdot \cos(\pi - y_4 + e_4 - y_5 + e_5) = 0. \end{aligned} \qquad (4.19\text{b})$$

The following total derivatives are written for the sake of forming partial derivatives that are needed for linearization:

$$\begin{aligned} \mathrm{d}f_1 = {}& -\sin(y_4 - e_4)\mathrm{d}e_2 - (y_2 - e_2)\cos(y_4 - e_4)\mathrm{d}e_4 + \sin(y_5 - e_5)\mathrm{d}e_1 + \\ & + (y_1 - e_1)\cos(y_5 - e_5)\mathrm{d}e_5, \end{aligned} \qquad (4.20\text{a})$$

$$\begin{aligned} \mathrm{d}f_2 = {}& \big[-2(y_1 - e_1) + 2(y_2 - e_2)\cos(\pi - y_4 + e_4 - y_5 + e_5)\big]\mathrm{d}e_1 + \\ & + \big[-2(y_2 - e_2) + 2(y_1 - e_1)\cos(\pi - y_4 + e_4 - y_5 + e_5)\big]\mathrm{d}e_2 + \\ & + 2(y_3 - e_3)\mathrm{d}e_3 + \big[2(y_1 - e_1)(y_2 - e_2)\sin(\pi - y_4 + e_4 - y_5 + e_5)\big](\mathrm{d}e_4 + \mathrm{d}e_5). \end{aligned}$$
$$(4.20\text{b})$$

From these equations we get the partial derivatives $\partial f_1 / \partial e_2 = -\sin(y_4 - e_4)$, etc., leading to the Jacobian matrix

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial e_1} & \frac{\partial f_1}{\partial e_2} & \frac{\partial f_1}{\partial e_3} & \frac{\partial f_1}{\partial e_4} & \frac{\partial f_1}{\partial e_5} \\ \frac{\partial f_2}{\partial e_1} & \frac{\partial f_2}{\partial e_2} & \frac{\partial f_2}{\partial e_3} & \frac{\partial f_2}{\partial e_4} & \frac{\partial f_2}{\partial e_5} \end{bmatrix}, \qquad (4.20\text{c})$$

which must have rank 2 (full row rank).

The problem is linearized by the truncated Taylor series

$$\boldsymbol{f}(\boldsymbol{e}) \approx \boldsymbol{f}(\boldsymbol{e}_0) + \left. \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{e}^T} \right|_{\boldsymbol{e}=\boldsymbol{e}_0} \cdot (\boldsymbol{e} - \boldsymbol{e}_0) = \boldsymbol{0} \qquad (4.21\text{a})$$

about the expansion point e_0, being an approximate value for the unknown vector of random errors e. Using matrix B, evaluated at e_0, to represent the partial derivatives, and introducing $\Delta e := e - e_0$ as an unknown, incremental vector of residuals, leads to the formula

$$-f(e_0) = B \cdot \Delta e, \tag{4.21b}$$

which is in the form of

$$w = Be \tag{4.21c}$$

given in the model of condition equations. Therefore, we can setup an iterative algorithm to predict Δe as follows:

1. Set $e_0 = 0$ and choose a convergence criterion ϵ.

2. Then for $j = 1, 2, \ldots$, while $\widetilde{\Delta e}_j > \epsilon$, compute:

$$\widetilde{\Delta e}_j = P^{-1} B_j^T (B_j P^{-1} B_j^T)^{-1} w_j \tag{4.22a}$$

$$\tilde{e}_j = e_j + \widetilde{\Delta e}_j. \tag{4.22b}$$

Then update the expansion point, the Jacobian matrix, and the vector w for the next iteration as follows:

$$e_{j+1} = \tilde{e}_j - \underset{\sim}{0}, \quad B_{j+1} = B|_{e_{j+1}}, \quad \text{and} \quad w_{j+1} = -f(e_{j+1}). \tag{4.22c}$$

For the first iteration, the matrix B and vector w read

$$B = \begin{bmatrix} 0.08719744 & -0.09965131 & 0 & 8.762479 & 58.75108 \\ -48.92976 & 8.325453 & 99.96000 & 10463.14 & 10463.14 \end{bmatrix}$$

and

$$w = \begin{bmatrix} -0.00816522 \\ -0.86019942 \end{bmatrix}.$$

Upon convergence the predicted residual vector turns out to be

$$\tilde{e} = \begin{bmatrix} -0.0021\,\text{m} \\ 0.0035\,\text{m} \\ -0.0024\,\text{m} \\ -5.6'' \\ -9.2'' \end{bmatrix}. \tag{4.23}$$

Note that when choosing a numerical value for the convergence criterion ϵ, one must be mindful of the units involved in the residual vector. In this example, we have units of meters, for which a change of less than 0.1 mm might be satisfactory, but we also have units of radians for the angles, for which a change of less than 5×10^{-6} rad might be required. In such cases it may be prudent to check the elements of $\widetilde{\Delta e}_j$ individually, using separate convergence criterion for different observation types. Then, the algorithm would be considered to have converged when all the convergence criteria have been satisfied.

4.5   Generation of Equivalent Condition Equations When the Gauss-Markov Model is Rank Deficient

We may also wish to transform the rank-deficient model (3.40a) into a model of condition equations. To do so, consider the further splitting of the rank-deficient matrix A defined in (3.40b) as follows:

$$\underset{n\times m}{A} = \begin{bmatrix} A_1 & | & A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{4.24a}$$

$$\dim(A_{11}) = q \times q \ \text{ and } \ \dim(A_{22}) = (n-q) \times (m-q). \tag{4.24b}$$

Also, we have $\operatorname{rk} A_{11} = q := \operatorname{rk} A$. And, with the introduction of the $q \times (m-q)$ matrix L in (3.50a), satisfying $A_2 = A_1 L$, we may write

$$A_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = A_1 L = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} L \Rightarrow A = \begin{bmatrix} A_1 & | & A_1 L \end{bmatrix}. \tag{4.25}$$

Now, the matrix B within the model of condition equations could be chosen as

$$\underset{r\times n}{B} := \begin{bmatrix} A_{21} A_{11}^{-1} & | & -I_{n-q} \end{bmatrix}, \tag{4.26}$$

with r being the redundancy of the model as shown in (3.48) and (4.1b). This is a legitimate choice for B as long as the two conditions discussed in Section 4.2, are satisfied, viz the dimensionality condition and the orthogonality condition.

The first condition requires that the dimensions of the column spaces of A and B sum to the number of observations n. The second condition requires that the rows of matrix B are orthogonal to the columns of A, i.e., $BA = 0$. Taken together, these conditions mean that A and B^T are orthogonal complements in n-dimensional space, or, stated more succinctly,

$$\mathcal{R}(A) \overset{\perp}{\oplus} \mathcal{R}(B^T) = \mathbb{R}^n. \tag{4.27}$$

Both conditions i and ii are satisfied for (4.26) as shown below.

i. Dimensionality condition:

$$\operatorname{rk} B = r = n - q = n - \operatorname{rk} A \Rightarrow \operatorname{rk} A + \operatorname{rk} B = n. \tag{4.28a}$$

ii. Orthogonality condition:

$$BA = B \begin{bmatrix} A_1 & | & A_2 \end{bmatrix} = BA_1 \begin{bmatrix} I_q & | & L \end{bmatrix}, \tag{4.28b}$$

but

$$BA_1 = \begin{bmatrix} A_{21} A_{11}^{-1} & | & -I_{n-q} \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = A_{21} A_{11}^{-1} A_{11} - A_{21} = 0, \tag{4.28c}$$

and therefore

$$BA = 0. \tag{4.28d}$$

Note that as long as the rank of matrix A is known, we can always generate a splitting of A as shown in (4.24a); however, we may need to reorder the columns of A (tantamount to reordering the elements of the parameter vector) to ensure that A_{11} has full column rank.

4.6   Practice Problems

1. Practice deriving the formula for the residual vector $\tilde{e}$ as shown in Section 4.2 until you can do it without referring to the notes.

2. Compute the residual vector of Problem 9 of Section 3.6 using the LESS within the model of condition equations. Confirm that the rank of matrix B is $n - 5$ and that $BA = 0$, where A is the coefficient matrix from problem 9.

3. The observations listed in Table 4.3 are depicted in Figure 4.3. Assume that the listed angles were derived from differences of independently observed directions measured with a theodolite. For example, observation y_2 was derived from subtracting the observed direction from point P_2 to point P_3 from the direction from P_2 to P_4. The variance of each direction is $\sigma^2 = (10'')^2$.

   (a) Determine the variance of each of the six angles as well as the covariance between angles y_2 and y_3 and the covariance between angles y_4 and y_5. Based on these results, write down the covariance matrix Q.

   (b) Write down suitable condition equations and determine the redundancy of the model.

   (c) Using the LESS within the model of condition equations, compute the residual vector $\tilde{e}$ and its dispersion matrix $D\{\tilde{e}\}$.

   (d) Compute the estimated variance component $\hat{\sigma}_0^2$.

Table 4.3: Six measured angles between four points

Element of y	Observation
y_1	$37°52'35''$
y_2	$46°56'10''$
y_3	$57°18'50''$
y_4	$37°52'40''$
y_5	$53°44'50''$
y_6	$31°03'20''$
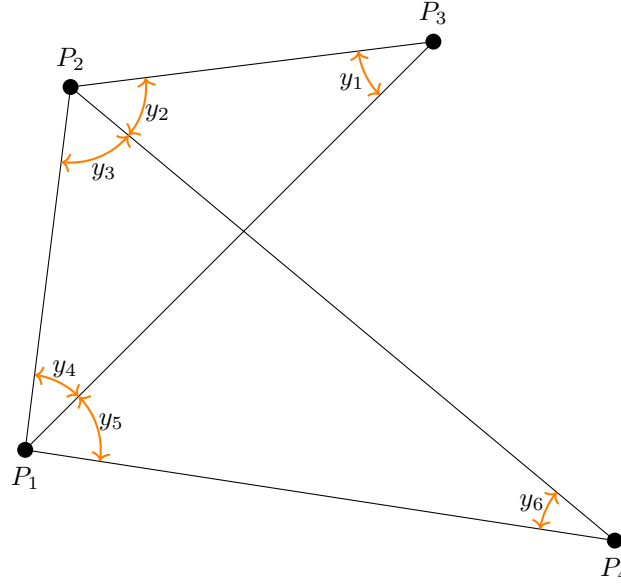
Figure 4.3: Six measured angles between four points

4. Four distances were measured between three points A, B, C as shown in Figure 4.4. The observed distances are $y_1 = 300.013\,\text{m}$, $y_2 = 300.046\,\text{m}$, $y_3 = 200.055\,\text{m}$, and $y_4 = 500.152\,\text{m}$. There are no correlations between the distances, and their standard deviations are defined by $\sigma = (5 + 10d)\,\text{mm}$, where d is the measured distance in km. Perform a least-squares adjustment within the model of condition equations to find the adjusted distance between points A and C and its estimated variance.
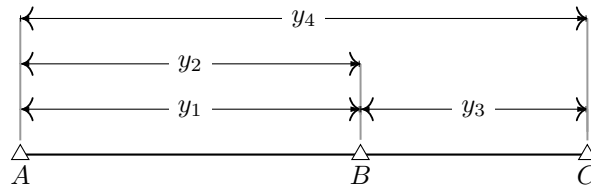
Figure 4.4: Four distances measured between three points A, B, C

5. Four angles are depicted in Figure 4.5. Angles y_1 and y_2 were derived from differencing among three observed directions. Angle y_3 was derived from an independent set of two directions. Likewise, angle y_4 was derived from yet another independent set of two directions. All directions are considered uncorrelated with standard deviation $\sigma = 10''$. The derived angles are $y_1 = 60°22'15''$, $y_2 = 75°39'20''$, $y_3 = 223°58'40''$, $y_4 = 136°01'30''$.

Note: The observed directions are uncorrelated, but some of the derived angles are not.

Use the LESS within the model of condition equations to compute the adjusted angles for y_1 and y_2. Also compute their variances.
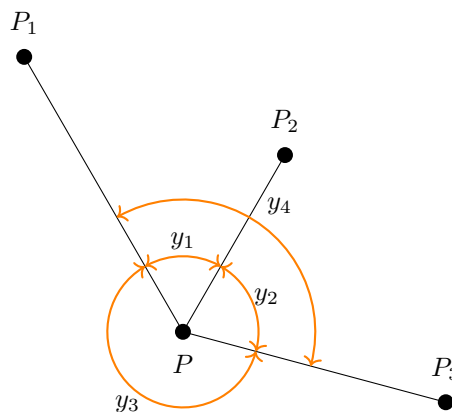
Figure 4.5: Four angles derived from three sets of directions

6. Using the data from problem 7 of Section 3.6, compute the residual vector $\tilde{e}$ by using the LESS within the model of condition equations. Confirm that the rank of matrix B is $n-2$ and that $BA = 0$, where A is the coefficient matrix from problem 7.

Hint: The slope between the first point and the jth point must equal the slope between the jth point and the $(j+1)$th point for $j = 2, 3, \ldots, n-1$.

4.7 Summary Formulas for the Least-Squares Solution Within the Model of Condition Equations

The model of condition equations is given by

$$\boxed{\begin{array}{c} \underset{r \times 1}{\boldsymbol{w}} \coloneqq \underset{r \times n}{B} \, \boldsymbol{y} = B\boldsymbol{e}, \quad \boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}), \\[2mm] r \coloneqq \operatorname{rk} B. \end{array}}$$

Table 4.4: Summary formulas for the LESS within the model of condition equations

Quantity	Formula	Eq. No.
Model redundancy	$r = \operatorname{rk} B$	(4.1b)
Vector of predicted residuals	$\tilde{\boldsymbol{e}} = P^{-1} B^T \left(B P^{-1} B^T \right)^{-1} B \boldsymbol{y}$	(4.5e)
Dispersion matrix for residuals	$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot P^{-1} B^T (B P^{-1} B^T)^{-1} B P^{-1}$	(4.10)
Sum of squared residuals (SSR)	$\Omega = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$	(4.8a)
Estimated variance component	$\hat{\sigma}_0^2 = \Omega / r$	(4.9)
Vector of adjusted observations	$\widehat{E\{\boldsymbol{y}\}} =: \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}$	(4.6)
Dispersion matrix for adjusted observations	$D\{\hat{\boldsymbol{\mu}}_y\} = \sigma_0^2 \cdot P^{-1} - D\{\tilde{\boldsymbol{e}}\}$	(4.12)

Chapter 5

The Gauss-Markov Model with Constraints

When *prior information* about the values of certain parameters, or about functional relationships between them, is known before the adjustment, those quantities can be maintained through the adjustment by application of *constraints*. For example, one may already know the height difference between two points in a leveling network that is to be adjusted, or it could be that the azimuth between two points in a 2D network to be adjusted must maintain a specified value. In both cases, the prior information can be preserved through constraints added to the Gauss-Markov Model (GMM). We say that such information is known *a priori*. The term a priori is a Latin phrase that literally means "from the earlier." In geodetic science, it refers to knowledge or information possessed before an experiment is conducted or an adjustment is computed.

One case where constraints might be useful is when the design matrix A does not have full column rank, implying that the inverse N^{-1} of the normal equation matrix does not exist, which means that the parameters of the model cannot be estimated using (3.7). This problem can occur, for example, when network observations must be adjusted in the estimation of point coordinates, but the observations themselves do not provide complete information about the network datum (i.e., its size, shape, orientation, and origin). For example, distance measurements provide information about the scale (size) of a network, and angle measurements provide information about its shape. But neither measurement type provides information about the origin or orientation of the network figure, which is necessary for estimating coordinates of network points. In such a case in 2-D, applying a constraint on two coordinates (i.e., on two parameters) and one azimuth (a function of four parameters) would provide the lacking information. In this case, the specified constraint values could be somewhat arbitrary, but we still may speak of them as being "known" (i.e., specified a priori) in the context of adjustments with constraints.

Of course, we have already seen in Section 3.5 how a minimum number of constraints on the unknown parameters can be imposed via datum information, thereby

overcoming a datum (rank) deficiency in the model and permitting a minimally constrained adjustment of the observations. The model explored in this chapter can be used not only to handle datum deficiencies in a way that leads to a minimally constrained adjustment, it can also be used to handle a variety of fixed constraints, possibly leading to an *over-constrained adjustment*. The latter case is one in which the imposition of constraints will impact the values of the residual vector.

5.1   Model Definition and Minimization Problem

The Gauss-Markov Model (GMM) with constraints imposed on the unknown parameters (all or some of them) is written as

$$\underset{n \times 1}{\boldsymbol{y}} = \underset{n \times m}{A} \, \boldsymbol{\xi} + \boldsymbol{e}, \ \ \boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}), \ \ \operatorname{rk} A =: q \le m, \tag{5.1a}$$

$$\underset{l \times 1}{\boldsymbol{\kappa}_0} = \underset{l \times m}{K} \, \boldsymbol{\xi}, \ \ \operatorname{rk} K =: l \ge m - q, \tag{5.1b}$$

where the rank condition

$$\operatorname{rk}\big[A^T, \, K^T\big] = m \tag{5.1c}$$

must be satisfied. The terms of the model are as defined on page 63, but now with the addition of a known $l \times m$ coefficient matrix K and an $l \times 1$ vector of specified constants $\boldsymbol{\kappa}_0$. Symbols for the normal equations were introduced in (3.4) and are repeated here for convenience:

$$\big[N, \, \boldsymbol{c}\big] := A^T P\big[A, \, \boldsymbol{y}\big]. \tag{5.2}$$

Note that, in contrast to the model in (3.1), the coefficient matrix A in (5.1a) is not required to have full column rank, in which case the matrix inverse N^{-1} would not exist. However, the specified rank conditions imply that $(N+K^T K)^{-1}$ exists, and, if N^{-1} exists, so does $(KN^{-1}K^T)^{-1}$. This is because the range space of $[A^T, \, K^T]$ spans $\mathbb{R}^m$ as implied by the rank condition stated in (5.1c). The redundancy of the model is computed by

$$\boxed{r := n - m + \operatorname{rk} K = n - m + l.} \tag{5.3}$$

Introducing an $l \times 1$ vector of Lagrange multipliers $\boldsymbol{\lambda}$, the Lagrange target function to minimize is

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\lambda}) := (\boldsymbol{y} - A\boldsymbol{\xi})^T P(\boldsymbol{y} - A\boldsymbol{\xi}) - 2\boldsymbol{\lambda}^T\big(\boldsymbol{\kappa}_0 - K\boldsymbol{\xi}\big) = \text{stationary} = \tag{5.4a}$$

$$= \boldsymbol{y}^T P \boldsymbol{y} - 2\boldsymbol{\xi}^T A^T P \boldsymbol{y} + \boldsymbol{\xi}^T A^T P A \boldsymbol{\xi} - 2\boldsymbol{\lambda}^T(\boldsymbol{\kappa}_0 - K\boldsymbol{\xi}). \tag{5.4b}$$

Its first partial derivatives are taken to form the following Euler-Lagrange necessary conditions:

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\xi}} = N\hat{\boldsymbol{\xi}} - \boldsymbol{c} + K^T \hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \tag{5.5a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = -\boldsymbol{\kappa}_0 + K\hat{\boldsymbol{\xi}} \doteq \boldsymbol{0}. \tag{5.5b}$$

In matrix form (5.5a) and (5.5b) are expressed as

$$\begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{\kappa}_0 \end{bmatrix}, \tag{5.6}$$

where the vector on the left side contains $m + l$ unknowns to be estimated.

The sufficient condition, required for minimization, is satisfied by

$$(1/2)\big(\partial^2 \Phi/\partial \boldsymbol{\xi}\partial \boldsymbol{\xi}^T\big) = N, \tag{5.7}$$

which is positive (semi)definite. We refer to the matrix on the left side of (5.6) as the least-squares *normal equation matrix*. It is invertible if, and only if, $\mathrm{rk}\big[A^T,\ K^T\big] = m$. This rank condition means that, for the normal equation matrix,

- among the first m columns, at least $m - l$ must be linearly independent, and

- the additional l columns are complementary, meaning that when combined with the first $m - l$ columns they span $\mathbb{R}^m$.

5.2 Estimation of Parameters and Lagrange Multipliers

In the following, we consider two cases: (1) N is invertible (nonsingular or regular), and (2) N is singular. The LEast-Squares Solution (LESS) is developed for both cases in the following:

Case 1: N is invertible, implying that matrix A has full column rank, i.e., $\mathrm{rk}\,A = m$. Equations (5.5a) and (5.5b) then imply

$$\hat{\boldsymbol{\xi}} = N^{-1}\big(\boldsymbol{c} - K^T\hat{\boldsymbol{\lambda}}\big), \tag{5.8a}$$

$$\boldsymbol{\kappa}_0 = K\hat{\boldsymbol{\xi}} = KN^{-1}\boldsymbol{c} - KN^{-1}K^T\hat{\boldsymbol{\lambda}} \tag{5.8b}$$

$$\Rightarrow \hat{\boldsymbol{\lambda}} = -\big(KN^{-1}K^T\big)^{-1}\big(\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}\big), \tag{5.8c}$$

finally leading to the LESS

$$\boxed{\hat{\boldsymbol{\xi}} = N^{-1}\boldsymbol{c} + N^{-1}K^T\big(KN^{-1}K^T\big)^{-1}\big(\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}\big).} \tag{5.8d}$$

The vector difference $\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}$ in (5.8d) is called a *vector of discrepancies*. It shows the mismatch between the vector of specified constants $\boldsymbol{\kappa}_0$ and a linear combination (as generated by the matrix K) of the solution without constraints (i.e., $N^{-1}\boldsymbol{c}$). The estimated vectors $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\lambda}}$ may also be presented in terms of the inverse of the matrix in (5.6), viz.

$$\begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} N^{-1} - N^{-1}K^T\big(KN^{-1}K^T\big)^{-1}KN^{-1} & N^{-1}K^T\big(KN^{-1}K^T\big)^{-1} \\ \big(KN^{-1}K^T\big)^{-1}KN^{-1} & -\big(KN^{-1}K^T\big)^{-1} \end{bmatrix}\begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{\kappa}_0 \end{bmatrix}. \tag{5.9}$$

Also, note that the expectation of the estimated vector of Lagrange multipliers is derived by

$$
\begin{aligned}
E\{\hat{\boldsymbol{\lambda}}\} &= -E\{(KN^{-1}K^T)^{-1}(\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c})\} = \\
&= (KN^{-1}K^T)^{-1}[KN^{-1}A^TPE\{\boldsymbol{y}\} - \boldsymbol{\kappa}_0] = \\
&= (KN^{-1}K^T)^{-1}(K\boldsymbol{\xi} - \boldsymbol{\kappa}_0) = 0.
\end{aligned}
\tag{5.10}
$$

Case 2: N is singular (i.e., not invertible), implying that matrix A does not have full column rank, i.e., $\operatorname{rk} A < m$.

Multiplying equation (5.5b) by K^T and adding the result to (5.5a), leads to

$$
\begin{aligned}
(N + K^TK)\hat{\boldsymbol{\xi}} &= \boldsymbol{c} + K^T(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}) \Rightarrow \\
\hat{\boldsymbol{\xi}} &= (N + K^TK)^{-1}\boldsymbol{c} + (N + K^TK)^{-1}K^T(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}).
\end{aligned}
\tag{5.11}
$$

Then from (5.5b) and (5.11) we have

$$
\begin{aligned}
\boldsymbol{\kappa}_0 = K\hat{\boldsymbol{\xi}} &= K(N + K^TK)^{-1}\boldsymbol{c} + K(N + K^TK)^{-1}K^T(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}) \Rightarrow \\
(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}) &= [K(N + K^TK)^{-1}K^T]^{-1}[\boldsymbol{\kappa}_0 - K(N + K^TK)^{-1}\boldsymbol{c}].
\end{aligned}
\tag{5.12}
$$

Substituting (5.12) into (5.11) leads to the LESS

$$
\boxed{
\begin{aligned}
\hat{\boldsymbol{\xi}} = (N + K^TK)^{-1}\boldsymbol{c} &+ (N + K^TK)^{-1}K^T \cdot \\
&\cdot [K(N + K^TK)^{-1}K^T]^{-1}[\boldsymbol{\kappa}_0 - K(N + K^TK)^{-1}\boldsymbol{c}].
\end{aligned}
}
\tag{5.13}
$$

The form of (5.13) is identical to (5.8d) except that all occurrences of matrix N in (5.8d) have been replaced by $N + K^TK$ in (5.13). Of course, (5.13) can be used for both the singular and nonsingular cases.

Also, note that the expectation of vector difference $\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}$ is derived by

$$
\begin{aligned}
E\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\} &= E\{[K(N + K^TK)^{-1}K^T]^{-1}[\boldsymbol{\kappa}_0 - K(N + K^TK)^{-1}\boldsymbol{c}]\} = \\
&= [K(N + K^TK)^{-1}K^T]^{-1}[\boldsymbol{\kappa}_0 - K(N + K^TK)^{-1}A^TPE\{\boldsymbol{y}\}] = \\
&= [K(N + K^TK)^{-1}K^T]^{-1}K[I_m - (N + K^TK)^{-1}N]\boldsymbol{\xi} = \\
&= [K(N + K^TK)^{-1}K^T]^{-1}K[I_m - (N + K^TK)^{-1}(N + K^TK)]\boldsymbol{\xi} + \\
&\quad + [K(N + K^TK)^{-1}K^T]^{-1}K(N + K^TK)^{-1}K^T \cdot K\boldsymbol{\xi} = K\boldsymbol{\xi} \\
&\Rightarrow E\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\} = K\boldsymbol{\xi} \quad \text{or} \quad E\{\hat{\boldsymbol{\lambda}}\} = \boldsymbol{\kappa}_0 - K\boldsymbol{\xi} \Rightarrow
\end{aligned}
\tag{5.14a}
$$

$$
E\{\hat{\boldsymbol{\lambda}}\} = \boldsymbol{0}.
\tag{5.14b}
$$

5.3 Derivation of Dispersion Matrices

We now compute the formal dispersion matrices for the both the vector of estimated parameters $\hat{\boldsymbol{\xi}}$ and the vector of estimated Lagrange multipliers $\hat{\boldsymbol{\lambda}}$.

Case 1: For case 1, we start with (5.6), from which we have

$$\begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{\kappa}_0 \end{bmatrix}. \tag{5.15}$$

Applying the law of covariance propagation, noting that $\boldsymbol{\kappa}_0$ is a non-random vector, and substituting the matrix from (5.9) implies that

$$D\{\begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix}\} = \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} \cdot D\{\begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{\kappa}_0 \end{bmatrix}\} \cdot \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} =$$

$$= \sigma_0^2 \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} =$$

$$= \sigma_0^2 \begin{bmatrix} N^{-1} - N^{-1}K^T(KN^{-1}K^T)^{-1}KN^{-1} & 0 \\ 0 & (KN^{-1}K^T)^{-1} \end{bmatrix}, \tag{5.16}$$

which, upon comparing to (5.9), reveals the relation

$$\begin{bmatrix} D\{\hat{\boldsymbol{\xi}}\} & X \\ X^T & -D\{\hat{\boldsymbol{\lambda}}\} \end{bmatrix} = \sigma_0^2 \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1}. \tag{5.17}$$

Here the symbol X represents a term of no particular interest; note that it does not represent the covariance $C\{\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\lambda}}\}$, which turns out to be zero.

Case 2: The results for case 2 are slightly different, because we work with a system of equations involving $N + K^T K$ rather than N itself. Thus, rather than the system of equations shown in (5.15), we work with the modified system

$$\begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} N + K^T K & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{c} + K^T \boldsymbol{\kappa}_0 \\ \boldsymbol{\kappa}_0 \end{bmatrix}. \tag{5.18}$$

Note that the matrix in (5.15) has full rank even when matrix N is singular, so it is not necessary to use the modified system (5.18). However, this modified system has its own benefits, and it is consistent with equation (5.13) derived above.

Using the formulas for inverting a partitioned matrix (see (A.14) and (A.15)) and introducing the notation $N_K := (N + K^T K)$ for the sake of brevity, we can write

$$\begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix}^{-1} = \begin{bmatrix} N_K^{-1} - N_K^{-1}K^T(KN_K^{-1}K^T)^{-1}KN_K^{-1} & N_K^{-1}K^T(KN_K^{-1}K^T)^{-1} \\ (KN_K^{-1}K^T)^{-1}KN_K^{-1} & -(KN_K^{-1}K^T)^{-1} \end{bmatrix}. \tag{5.19}$$

Our goal is to express the inverted matrix in (5.19) in terms of the inverted matrix in (5.15). We start by multiplying the former by the inverse of the latter and then carry out some matrix algebra.

$$\begin{bmatrix} N+K^TK & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix} = \begin{bmatrix} N+K^TK & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} N+K^TK-K^TK & K^T \\ K & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} I_m & 0 \\ 0 & I_l \end{bmatrix} - \begin{bmatrix} N+K^TK & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} K^TK & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_l \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I_l \end{bmatrix} \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix} =$$

$$= \left(\begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & I_l \end{bmatrix} \right) \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}. \tag{5.20}$$

Multiplying the first and last products of (5.20) by the inverse of their last terms and considering (5.17) reveals that

$$\begin{bmatrix} N+K^TK & K^T \\ K & 0 \end{bmatrix}^{-1} = \begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & I_l \end{bmatrix} = \begin{bmatrix} \sigma_0^{-2}D\{\hat{\boldsymbol{\xi}}\} & X \\ X^T & -\sigma_0^{-2}D\{\hat{\boldsymbol{\lambda}}\} - I_l \end{bmatrix}, \tag{5.21}$$

and therefore, in consideration of (5.19),

$$-\left[K\left(N+K^TK\right)^{-1}K^T\right]^{-1} = -\sigma_0^{-2}D\{\hat{\boldsymbol{\lambda}}\} - I_l \Rightarrow$$
$$D\{\hat{\boldsymbol{\lambda}}\} = \sigma_0^2\left\{\left[K\left(N+K^TK\right)^{-1}K^T\right]^{-1} - I_l\right\}. \tag{5.22}$$

Alternative derivation of dispersion matrix The following alternative approach to deriving the dispersion matrix for case 2 was recognized by Dru Smith and Kyle Snow during collaborative work, where, again, the abbreviation $N_K := (N + K^TK)$ is used: The law of linear covariance propagation (law of error propagation) allows us to write

$$D\{\begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix}\} = \begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix}^{-1} \cdot D\{\begin{bmatrix} \boldsymbol{c} + K^T\boldsymbol{\kappa}_0 \\ \boldsymbol{\kappa}_0 \end{bmatrix}\} \cdot \begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix}^{-1} =$$

$$= \sigma_0^2 \begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix}^{-1} =: \sigma_0^2 \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \tag{5.23a}$$

where, upon substitution of (5.19), the block matrices Q_{11}, Q_{12}, and Q_{22} turn out to be

$$Q_{22} = \left(KN_K^{-1}K^T\right)^{-1}K\left(N_K^{-1}{\cdot}N{\cdot}N_K^{-1}\right)K^T\left(KN_K^{-1}K^T\right)^{-1}, \tag{5.23b}$$

$$Q_{12} = \left(N_K^{-1}{\cdot}N{\cdot}N_K^{-1}\right)K^T\left(KN_K^{-1}K^T\right)^{-1} - N_K^{-1}K^TQ_{22}, \tag{5.23c}$$

$$Q_{11} = \left[N_K^{-1} - N_K^{-1}K^T\left(KN_K^{-1}K^T\right)^{-1}KN_K^{-1}\right] \cdot NN_K^{-1} - Q_{12}KN_K^{-1}. \quad (5.23\text{d})$$

Now, we wish to reduce these matrices to simpler forms, for which the following relationship is useful

$$N_K^{-1}{\cdot}N{\cdot}N_K^{-1} = N_K^{-1} - N_K^{-1}(K^TK)N_K^{-1}. \quad (5.24)$$

Substituting (5.24) into (5.23b) leads to

$$Q_{22} = \left(KN_K^{-1}K^T\right)^{-1}K\left[N_K^{-1} - N_K^{-1}(K^TK)N_K^{-1}\right]K^T\left(KN_K^{-1}K^T\right)^{-1} =$$
$$= \left(KN_K^{-1}K^T\right)^{-1} - I_l = \sigma_0^{-2}D\{\hat{\boldsymbol{\lambda}}\}. \quad (5.25\text{a})$$

Then, substituting (5.25a) into (5.23c) results in

$$Q_{12} = \left(N_K^{-1}{\cdot}N{\cdot}N_K^{-1}\right)K^T\left(KN_K^{-1}K^T\right)^{-1} - N_K^{-1}K^T\left[\left(KN_K^{-1}K^T\right)^{-1} - I_l\right] =$$
$$= \left[N_K^{-1} - N_K^{-1}(K^TK)N_K^{-1}\right]K^T\left(KN_K^{-1}K^T\right)^{-1} - N_K^{-1}K^T\left(KN_K^{-1}K^T\right)^{-1} + N_K^{-1}K^T =$$
$$= 0 = C\{\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\lambda}}\}, \quad (5.25\text{b})$$

and, therefore, (5.23d) reduces to

$$Q_{11} = \left[N_K^{-1} - N_K^{-1}K^T\left(KN_K^{-1}K^T\right)^{-1}KN_K^{-1}\right] \cdot NN_K^{-1} =$$
$$= N_K^{-1} - N_K^{-1}(K^TK)N_K^{-1} - N_K^{-1}K^T\left(KN_K^{-1}K^T\right)^{-1}K\left[N_K^{-1} - N_K^{-1}(K^TK)N_K^{-1}\right] =$$
$$= N_K^{-1} - N_K^{-1}K^T\left(KN_K^{-1}K^T\right)^{-1}KN_K^{-1} = \sigma_0^{-2}D\{\hat{\boldsymbol{\xi}}\}. \quad (5.25\text{c})$$

Summary of dispersion matrices   For convenience, we summarize the dispersion matrices of the estimated parameters and the estimated Lagrange multipliers for both cases 1 and 2 as follows:

Case 1 (N nonsingular):

$$\boxed{D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2\left[N^{-1} - N^{-1}K^T\left(KN^{-1}K^T\right)^{-1}KN^{-1}\right]} \quad (5.26\text{a})$$

$$\boxed{D\{\hat{\boldsymbol{\lambda}}\} = \sigma_0^2\left(KN^{-1}K^T\right)^{-1}} \quad (5.26\text{b})$$

Case 2 (N singular):

$$\boxed{\begin{aligned}D\{\hat{\boldsymbol{\xi}}\} &= \sigma_0^2\left(N + K^TK\right)^{-1} - \sigma_0^2\left(N + K^TK\right)^{-1}K^T{\cdot}\\ &\quad {\cdot}\left[K\left(N + K^TK\right)^{-1}K^T\right]^{-1}K\left(N + K^TK\right)^{-1}\end{aligned}} \quad (5.27\text{a})$$

$$\boxed{D\{\hat{\boldsymbol{\lambda}}\} = \sigma_0^2\left\{\left[K\left(N + K^TK\right)^{-1}K^T\right]^{-1} - I_l\right\}} \quad (5.27\text{b})$$

Cases 1 and 2:

$$\boxed{C\{\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\lambda}}\} = 0} \quad (5.28)$$

As with the parameter estimates, the dispersion matrices for both cases 1 and 2 have a similar form, with every occurrence of N in case 1 being replaced by $N + K^T K$ in case 2, the exception being the identity matrix I_l appearing in case 2. Also note that the dispersion matrices in (5.26a) and (5.27a) are nothing more than the coefficient matrices multiplying the vector $\boldsymbol{c}$ in (5.8d) and (5.13), respectively, multiplied by the (unknown) variance component σ_0^2. Finally, it is clear from the above that the constraints reduce the dispersion matrix of $\hat{\boldsymbol{\xi}}$ compared to the corresponding dispersion matrix within the GMM (without constraints) derived in Chapter 3 (cf. (3.13)).

5.4   Residuals and Adjusted Observations

For both cases 1 and 2, the residual vector $\tilde{\boldsymbol{e}}$ and vector of adjusted observations $\hat{\boldsymbol{\mu}}_y$ may be obtained in a straightforward way after the estimation of the parameters by use of the formulas

$$\boxed{\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}},} \tag{5.29}$$

and

$$\boxed{\widehat{E\{\boldsymbol{y}\}} = \hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}.} \tag{5.30}$$

Here, $\hat{\boldsymbol{\mu}}_y$ is also interpreted as an estimate of the true, and thus unknown, vector of observables $\boldsymbol{\mu}_y$, where $E\{\boldsymbol{y}\} = \boldsymbol{\mu}_y$.

The dispersion matrix for the residual vector $\tilde{\boldsymbol{e}}$ can be derived from application of the law of covariance propagation as follows: Since

$$D\{\tilde{\boldsymbol{e}}\} = D\{\boldsymbol{y} - A\hat{\boldsymbol{\xi}}\} = D\{\boldsymbol{y}\} + AD\{\hat{\boldsymbol{\xi}}\}A^T - 2C\{\boldsymbol{y}, A\hat{\boldsymbol{\xi}}\}, \tag{5.31}$$

we start by deriving the covariance matrix $C\{\boldsymbol{y}, A\hat{\boldsymbol{\xi}}\}$. For case 1 we have

$$C\{\boldsymbol{y}, A\hat{\boldsymbol{\xi}}\} = I_n \cdot D\{\boldsymbol{y}\} \cdot \left\{ A\left[N^{-1}A^T P - N^{-1}K^T (KN^{-1}K^T)^{-1}KN^{-1}A^T P \right] \right\}^T = \tag{5.32a}$$

$$= \sigma_0^2 P^{-1} \left[PAN^{-1}A^T - PAN^{-1}K^T (KN^{-1}K^T)^{-1}KN^{-1}A^T \right] = \tag{5.32b}$$

$$= \sigma_0^2 A\left[N^{-1} - N^{-1}K^T (KN^{-1}K^T)^{-1}KN^{-1} \right]A^T = \tag{5.32c}$$

$$= AD\{\hat{\boldsymbol{\xi}}\}A^T = D\{A\hat{\boldsymbol{\xi}}\} = C\{\boldsymbol{y}, A\hat{\boldsymbol{\xi}}\}. \tag{5.32d}$$

Then, by substituting (5.32d) into (5.31), we arrive at

$$D\{\tilde{\boldsymbol{e}}\} = D\{\boldsymbol{y}\} - AD\{\hat{\boldsymbol{\xi}}\}A^T \Rightarrow \tag{5.33a}$$

$$\boxed{D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot \left\{ P^{-1} - A\left[N^{-1} - N^{-1}K^T (KN^{-1}K^T)^{-1}KN^{-1} \right]A^T \right\}} \tag{5.33b}$$

and

$$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot \left[P^{-1} - AN^{-1}A^T + AN^{-1}K^T (KN^{-1}K^T)^{-1}KN^{-1}A^T \right]. \tag{5.33c}$$

Note that (5.33c) reveals that the dispersion matrix for the residuals within the GMM with constraints is larger than that for the GMM without constraints (cf. (3.14a)). For case 2, one only needs to replace the matrix N^{-1} with $(N + K^T K)^{-1}$ in formulas (5.32) and (5.33).

Obviously, the dispersion matrix for the adjusted observations is written as

$$\boxed{D\{\hat{\boldsymbol{\mu}}_y\} = D\{\boldsymbol{y} - \tilde{\boldsymbol{e}}\} = D\{A\hat{\boldsymbol{\xi}}\} = AD\{\hat{\boldsymbol{\xi}}\}A^T,} \tag{5.34}$$

implying that the dispersion matrix of the residual vector can be expressed as

$$D\{\tilde{\boldsymbol{e}}\} = D\{\boldsymbol{y}\} - D\{\hat{\boldsymbol{\mu}}_y\}, \tag{5.35}$$

revealing once again that the dispersion matrix of the residuals is smaller than that of the observations.

5.4.1 A Numerical Example

A simple numerical example can be used to verify several of the equations derived above. We borrow our example from Smith et al. (2018), which is a small leveling network depicted in Figure 5.1.
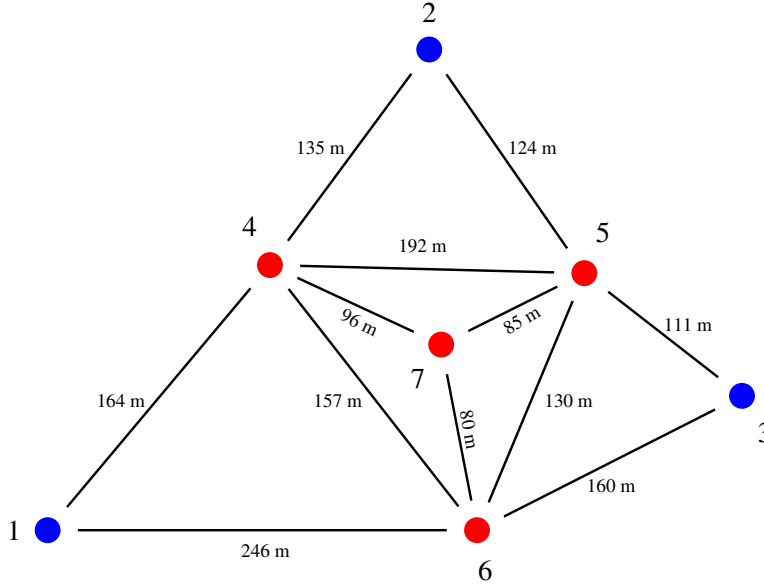
Figure 5.1: Small leveling network copied from Smith et al. (2018)

The matrices of interest are shown below. The unknown parameters (heights of stations) are ordered according to the seven numbered stations in Figure 5.1. The

connection between the observations and the stations is reflected in the coefficient matrix A.

$$
\underset{12\times7}{A} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 & 0
\end{bmatrix}, \quad
\underset{12\times1}{\boldsymbol{y}} = \begin{bmatrix}
0.333557 \\
0.365859 \\
2.850824 \\
-0.948661 \\
-1.040570 \\
-0.824317 \\
-1.989007 \\
-0.528043 \\
2.517497 \\
-1.692892 \\
-0.296337 \\
-0.162582
\end{bmatrix} \text{m},
$$

$$
\underset{3\times7}{K} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
\underset{3\times1}{\boldsymbol{\kappa}_0} = \begin{bmatrix}
68.8569 \\
66.9471 \\
68.1559
\end{bmatrix} \text{m},
$$

$$
P^{-1} = \mathrm{diag}(2.214, 1.440, 1.476, 1.215, 1.116, 0.720, 1.728, 1.170, 1.413,
$$
$$
0.864, 0.765, 0.999) \cdot 10^{-6}\,\text{m}^2
$$

Since the matrix A does not have full column rank, the problem belongs to case 2.

5.5   Estimated Variance Component

The estimated variance component for the GMM with constraints is derived similar to that for the GMM without constraints as shown in Section 3.3. The estimation is based on the principle

$$
\frac{\hat{\sigma}_0^2}{\tilde{e}^T P \tilde{e}} = \frac{\sigma_0^2}{E\{\tilde{e}^T P \tilde{e}\}}, \tag{5.36}
$$

with the assumption $E\{\hat{\sigma}_0^2\} = \sigma_0^2$. Furthermore, for the purpose of validating the constraints via hypothesis testing, we wish to decompose the quadratic form $\tilde{e}^T P \tilde{e}$ into the sum of two quadratic forms, viz. $\tilde{e}^T P \tilde{e} = \Omega + R$, where Ω is the sum of squared residuals (SSR) associated with the LESS within the *GMM without constraints*. In the following, we derive these components for both cases 1 and 2.

5.5.1   Case 1 — Matrix N is invertible $\Rightarrow \mathrm{rk}\,A = m$

$$
\tilde{e}^T P \tilde{e} = \left(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}\right)^T P\left(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}\right) =
$$

$$= \left[(\boldsymbol{y} - AN^{-1}\boldsymbol{c}) - AN^{-1}K^T (KN^{-1}K^T)^{-1} (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}) \right]^T P \cdot$$
$$\cdot \left[(\boldsymbol{y} - AN^{-1}\boldsymbol{c}) - AN^{-1}K^T (KN^{-1}K^T)^{-1} (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}) \right] =$$
$$= (\boldsymbol{y} - AN^{-1}\boldsymbol{c})^T P (\boldsymbol{y} - AN^{-1}\boldsymbol{c}) -$$
$$- (\boldsymbol{y} - AN^{-1}\boldsymbol{c})^T PAN^{-1}K^T (KN^{-1}K^T)^{-1} (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}) -$$
$$- (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c})^T (KN^{-1}K^T)^{-1} KN^{-1}A^T P (\boldsymbol{y} - AN^{-1}\boldsymbol{c}) +$$
$$+ (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c})^T (KN^{-1}K^T)^{-1} KN^{-1} (A^T PA) N^{-1}K^T (KN^{-1}K^T)^{-1} \cdot$$
$$\cdot (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}) =$$

(Note that $A^T P (\boldsymbol{y} - AN^{-1}\boldsymbol{c}) = 0$.)

$$= (\boldsymbol{y} - AN^{-1}\boldsymbol{c})^T P (\boldsymbol{y} - AN^{-1}\boldsymbol{c}) + (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c})^T (KN^{-1}K^T)^{-1} \cdot$$
$$\cdot (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}) =$$
$$= (\boldsymbol{y} - AN^{-1}\boldsymbol{c})^T P (\boldsymbol{y} - AN^{-1}\boldsymbol{c}) + \hat{\boldsymbol{\lambda}}^T (KN^{-1}K^T) \hat{\boldsymbol{\lambda}} = \Omega + R \qquad (5.37)$$

The scalars Ω and R defined as

$$\Omega := (\boldsymbol{y} - AN^{-1}\boldsymbol{c})^T P (\boldsymbol{y} - AN^{-1}\boldsymbol{c}) \qquad (5.38\text{a})$$

and

$$R := (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c})^T (KN^{-1}K^T)^{-1} (\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}), \qquad (5.38\text{b})$$

respectively.

Thus we have decomposed the quadratic form $\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$ into components Ω and R. Obviously, both Ω and R are random numbers since they are both functions of the random vector $\boldsymbol{c}$. It turns out that they are also uncorrelated. The random variable Ω is associated with the LESS within the GMM without constraints, whereas R is due to the addition of the constraints $\boldsymbol{\kappa}_0 = K\boldsymbol{\xi}$. From (5.38b) we see that R is always positive, revealing that the inclusion of constraints increases the value of $\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$. The random variables Ω and R are used for hypothesis testing as discussed in Chapter 9.

We now derive the expectation of $\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$.

$$E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\} = E\{\Omega\} + E\{R\} =$$
$$= (n-m)\sigma_0^2 + E\{\hat{\boldsymbol{\lambda}}^T (KN^{-1}K^T) \hat{\boldsymbol{\lambda}}\} = \qquad \text{using (3.29) for } E\{\Omega\}$$
$$= (n-m)\sigma_0^2 + \text{tr}\left[(KN^{-1}K^T) E\{\hat{\boldsymbol{\lambda}} \, \hat{\boldsymbol{\lambda}}^T\} \right] =$$
$$= (n-m)\sigma_0^2 + \text{tr}\left[(KN^{-1}K^T) (D\{\hat{\boldsymbol{\lambda}}\} + E\{\hat{\boldsymbol{\lambda}}\} E\{\hat{\boldsymbol{\lambda}}\}^T) \right] =$$

$$(\text{with } E\{\hat{\boldsymbol{\lambda}}\} = \boldsymbol{0} \text{ and } D\{\hat{\boldsymbol{\lambda}}\} = \sigma_0^2 (KN^{-1}K^T)^{-1})$$

$$= (n-m)\sigma_0^2 + \text{tr}\left[(KN^{-1}K^T) \sigma_0^2 (KN^{-1}K^T)^{-1} \right] =$$
$$= (n-m+l)\sigma_0^2 \qquad (5.39)$$

Substitution of (5.37) and (5.39) into (5.36) yields the following formula for the estimated variance component:

$$
\hat{\sigma}_0^2 = \frac{\left(\boldsymbol{y} - AN^{-1}\boldsymbol{c}\right)^T P\left(\boldsymbol{y} - AN^{-1}\boldsymbol{c}\right)}{n - m + l} +
$$
$$
+ \frac{\left(\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}\right)^T \left(KN^{-1}K^T\right)^{-1}\left(\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}\right)}{n - m + l}. \tag{5.40}
$$

Other useful forms of $\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$ are derived below starting with (5.37).

$$
\begin{aligned}
\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} &= \left(\boldsymbol{y} - AN^{-1}\boldsymbol{c}\right)^T P\left(\boldsymbol{y} - AN^{-1}\boldsymbol{c}\right) + \hat{\boldsymbol{\lambda}}^T\left(KN^{-1}K^T\right)\hat{\boldsymbol{\lambda}} = \\
&= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T N^{-1}\boldsymbol{c} - \left(\boldsymbol{\kappa}_0^T - \boldsymbol{c}^T N^{-1}K^T\right)\hat{\boldsymbol{\lambda}} = && \text{using (5.8c)} \\
&= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T N^{-1}\left(\boldsymbol{c} - K^T\hat{\boldsymbol{\lambda}}\right) - \boldsymbol{\kappa}_0^T\hat{\boldsymbol{\lambda}} = && \text{using (5.8a)} \\
&= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T\hat{\boldsymbol{\xi}} - \boldsymbol{\kappa}_0^T\hat{\boldsymbol{\lambda}} = \\
&= \boldsymbol{y}^T P\left(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}\right) - \boldsymbol{\kappa}_0^T\hat{\boldsymbol{\lambda}} = \\
&= \boldsymbol{y}^T P \tilde{\boldsymbol{e}} - \boldsymbol{\kappa}_0^T\hat{\boldsymbol{\lambda}} \tag{5.41}
\end{aligned}
$$

5.5.2   Case 2 — Matrix N is singular $\Rightarrow \operatorname{rk} A < m$

$$
\begin{aligned}
\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} &= \\
&= \left\{\boldsymbol{y} - A\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]\right\}^T P\left\{\boldsymbol{y} - A\left(N + K^T K\right)^{-1}\cdot \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \cdot \left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]\right\} = \\
&= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{y}^T PA\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right] - \left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]^T\cdot \\
&\quad \cdot \left(N + K^T K\right)^{-1} A^T P \boldsymbol{y} + \left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]^T\left(N + K^T K\right)^{-1}\cdot \\
&\qquad \cdot \left(A^T PA + K^T K - K^T K\right)\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right] = \\
&= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right] - \left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]^T\cdot \\
&\quad \cdot \left(N + K^T K\right)^{-1}\boldsymbol{c} + \left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]^T\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right] - \\
&\quad - \left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]^T\left(N + K^T K\right)^{-1}K^T K\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right] = \\
&= \boldsymbol{y}^T P \boldsymbol{y} - \cancel{\boldsymbol{c}^T\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]} - \\
&\qquad - \boldsymbol{c}^T\left(N + K^T K\right)^{-1}\boldsymbol{c} - \cancel{\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)^T K\left(N + K^T K\right)^{-1}\boldsymbol{c}} + \\
&\quad + \cancel{\boldsymbol{c}^T\left(N + K^T K\right)^{-1}\left[\boldsymbol{c} + K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)\right]} + \cancel{\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)^T K\left(N + K^T K\right)^{-1}\boldsymbol{c}} + \\
&\qquad + \left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)^T K\left(N + K^T K\right)^{-1}K^T\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right) - \hat{\boldsymbol{\xi}}^T K^T K\hat{\boldsymbol{\xi}} = \\
&= \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T\left(N + K^T K\right)^{-1}\boldsymbol{c} + \left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right)^T\left[K\left(N + K^T K\right)^{-1}K^T\right]\left(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\right) - \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \boldsymbol{\kappa}_0^T\boldsymbol{\kappa}_0 \tag{5.42}
\end{aligned}
$$

Now we compute the expectation for $\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$.

$E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\} =$

$= E\{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T (N + K^T K)^{-1} \boldsymbol{c}\} +$

$\quad + (\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})^T [K(N + K^T K)^{-1} K^T](\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}) - \boldsymbol{\kappa}_0^T \boldsymbol{\kappa}_0\} =$

$= E\{\boldsymbol{y}^T P[\boldsymbol{y} - A(N + K^T K)^{-1} \boldsymbol{c}]\} + E\{(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})^T [K(N + K^T K)^{-1} K^T] \cdot$

$\qquad\qquad\qquad\qquad\qquad\qquad \cdot (\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})\} - E\{\boldsymbol{\kappa}_0^T \boldsymbol{\kappa}_0\} =$

$= \operatorname{tr} P[I_n - A(N + K^T K)^{-1} A^T P] E\{\boldsymbol{y}\boldsymbol{y}^T\} + \operatorname{tr}[K(N + K^T K)^{-1} K^T] \cdot$

$\qquad\qquad\qquad \cdot E\{(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})^T\} - \operatorname{tr} E\{\boldsymbol{\kappa}_0^T \boldsymbol{\kappa}_0\} =$

(Note that $E\{(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})^T\} = D\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\} + E\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\}E\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\}^T$ and
$D\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\} = D\{\hat{\boldsymbol{\lambda}}\} = \sigma_0^2\{[K(N + K^T K)^{-1} K]^{-1} - I_l\}$, and $E\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\} = K\boldsymbol{\xi}$, and
$E\{\boldsymbol{y}\boldsymbol{y}^T\} = D\{\boldsymbol{y}\} + E\{\boldsymbol{y}\}E\{\boldsymbol{y}\}^T = \sigma_0^2 P^{-1} + A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T$).

$= \operatorname{tr} P[I_n - A(N + K^T K)^{-1} A^T P](\sigma_0^2 P^{-1} + A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T) +$

$\quad + \operatorname{tr}[K(N + K^T K)^{-1} K^T][D\{\hat{\boldsymbol{\lambda}}\} + E\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\}E\{\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}\}^T] - \operatorname{tr} K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T =$

$= \operatorname{tr}[\sigma_0^2 I_n + PA\boldsymbol{\xi}\boldsymbol{\xi}^T A^T - PA(N + K^T K)^{-1} A^T \sigma_0^2 - PA(N + K^T K)^{-1} A^T PA \cdot$

$\quad \cdot \boldsymbol{\xi}\boldsymbol{\xi}^T A^T] + \operatorname{tr}[K(N + K^T K)^{-1} K^T](\{[K(N + K^T K)^{-1} K^T]^{-1} - I_l\}\sigma_0^2 +$

$\qquad\qquad\qquad\qquad\qquad\qquad + K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T) - \operatorname{tr} K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T =$

$= \sigma_0^2 n + \operatorname{tr}\boldsymbol{\xi}\boldsymbol{\xi}^T N - \sigma_0^2 \operatorname{tr}(N + K^T K)^{-1} N - \operatorname{tr}(N + K^T K)^{-1} N\boldsymbol{\xi}\boldsymbol{\xi}^T N + \sigma_0^2 l -$

$\quad - \sigma_0^2 \operatorname{tr}[K(N + K^T K)^{-1} K^T] + \operatorname{tr}[(N + K^T K)^{-1} K^T K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T K] -$

$\qquad\qquad\qquad\qquad\qquad\qquad - \operatorname{tr} K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T =$

$= \sigma_0^2 n - \sigma_0^2 \operatorname{tr}(N + K^T K)^{-1}(N + K^T K) + \sigma_0^2 l +$

$\quad + \operatorname{tr}[I_m - (N + K^T K)^{-1} N]\boldsymbol{\xi}\boldsymbol{\xi}^T N - \operatorname{tr}[I_m - (N + K^T K)^{-1} K^T K]\boldsymbol{\xi}\boldsymbol{\xi}^T K^T K =$

$= \sigma_0^2(n - m + l) + \operatorname{tr}(N + K^T K)^{-1} K^T K\boldsymbol{\xi}\boldsymbol{\xi}^T N - \operatorname{tr}(N + K^T K)^{-1} N\boldsymbol{\xi}\boldsymbol{\xi}^T K^T K =$

$= \sigma_0^2(n - m + l) + \operatorname{tr}[(N + K^T K)^{-1} K^T K\boldsymbol{\xi}\boldsymbol{\xi}^T N]^T -$

$\quad - \operatorname{tr} N\boldsymbol{\xi}\boldsymbol{\xi}^T K^T K(N + K^T K)^{-1} = \sigma_0^2(n - m + l)$

$$\Rightarrow \boxed{E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}\} = \sigma_0^2(n - m + l)} \qquad (5.43)$$

Finally, substituting (5.42) and (5.43) into (5.36) yields

$$\boxed{\begin{aligned}\hat{\sigma}_0^2 &= \frac{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T (N + K^T K)^{-1} \boldsymbol{c}}{(n - m + l)} + \\ &\quad + \frac{(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}})^T [K(N + K^T K)^{-1} K^T](\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}) - \boldsymbol{\kappa}_0^T \boldsymbol{\kappa}_0}{(n - m + l)},\end{aligned}} \qquad (5.44a)$$

or, by use of (5.12) and with $N_K := N + K^T K$ for compactness,

$$\hat{\sigma}_0^2 = \frac{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T N_K^{-1} \boldsymbol{c} + \left(\boldsymbol{\kappa}_0 - K N_K^{-1} \boldsymbol{c}\right)^T \left(K N_K^{-1} K^T\right) \left(\boldsymbol{\kappa}_0 - K N_K^{-1} \boldsymbol{c}\right) - \boldsymbol{\kappa}_0^T \boldsymbol{\kappa}_0}{(n - m + l)},$$

(5.44b)

or

$$\hat{\sigma}_0^2 = \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{(n - m + l)}.$$

(5.44c)

We cannot directly identify Ω and R in (5.42) as we could in case 1. Therefore, we define Ω as

$$\Omega = \left(\boldsymbol{y} - A N^- \boldsymbol{c}\right)^T P \left(\boldsymbol{y} - A N^- \boldsymbol{c}\right),$$

(5.45)

and R as

$$R = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} - \Omega,$$

(5.46)

where $\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$ is given in (5.42). The symbol N^- in (5.45) stands for a generalized inverse of the matrix N. While generalized inverses are beyond the scope of these notes, the following generalized inverse is shown so that readers unfamiliar with the topic can still make use of equation (5.45). First, assume that the matrix N and vector $\boldsymbol{c}$ have been partitioned as follows:

$$\underset{m \times m}{N} = \left[\begin{array}{c|c} \underset{q \times q}{N_{11}} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] \quad \text{and} \quad \underset{m \times 1}{\boldsymbol{c}} = \left[\begin{array}{c} \underset{q \times 1}{\boldsymbol{c}_1} \\ \hline \boldsymbol{c}_2 \end{array}\right],$$

(5.47)

where the $q \times q$ submatrix N_{11} has full rank q, i.e., $\operatorname{rk} N_{11} = q := \operatorname{rk} N$. Note that such a partitioning can always be formed, even if the parameters in $\boldsymbol{\xi}$ must be reordered to do so. Then, the $m \times m$ matrix $G := \left[\begin{smallmatrix} N_{11}^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a generalized inverse of N and thus can be used in (5.45) for N^-, which simplifies that equation to

$$\Omega = \boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}_1^T N_{11}^{-1} \boldsymbol{c}_1 \quad \text{if} \quad \operatorname{rk} N_{11} = \operatorname{rk} N.$$

(5.48)

5.6 Hypothesis Test Using the Estimated Variance Component

The following ratio is formed for both cases 1 and 2 for the purposes of hypothesis testing (see Chapter 9 for more details on hypothesis testing):

$$T := \frac{R/(l - m + q)}{\Omega/(n - q)} \sim F(l - m + q, n - q), \quad \text{with} \quad q := \operatorname{rk}(A).$$

(5.49)

The ratio T is called a *Test statistic*. It is assumed to have an F-distribution with $l - m + q$ and $n - q$ degrees of freedom.[1]   The hypothesis test is then stated as

$$H_0 : K\boldsymbol{\xi} = \boldsymbol{\kappa}_0 \text{ versus } H_A : K\boldsymbol{\xi} \neq \boldsymbol{\kappa}_0, \tag{5.50}$$

where H_0 is called the *null hypothesis* and H_A is called the *alternative hypothesis*. For some chosen significance level α,

$$\text{Accept } H_0 : \text{ if } T \leq F_{\alpha, l-m+q, n-q}$$
$$\text{Reject } H_0 : \text{ if } T > F_{\alpha, l-m+q, n-q},$$

where $F_{\alpha, l-m+q, n-q}$ is taken from a table of *critical values* for the F-distribution. The critical values for certain values of r_1 and r_2 are listed in Appendix C. If MATLAB is available, the critical value may be generated by use of the MATLAB command `finv(1 - α,r₁,r₂)`.

Note that the redundancy $r_2 := n - q$ represents the degrees of freedom for the system of equations if no constraints were applied, whereas the redundancy $r_1 := l - m + q$ represents the *increase* in degrees of freedom due to the constraints, i.e.

$$r = r_1 + r_2 = (l - m + q) + (n - q) = n - m + l, \tag{5.51}$$

which agrees with (5.3). In the case that matrix A has full column rank (i.e., $\text{rk } A = q = m$), then the redundancies reduce to $r_1 := l$ and $r_2 := n - m$, respectively.

5.7   Practice Problems

1. Derive the expectation of the vector of estimated parameters $\hat{\boldsymbol{\xi}}$ given in (5.8d). Is $\hat{\boldsymbol{\xi}}$ an unbiased estimator of the vector of unknown parameters $\boldsymbol{\xi}$?

2. Solve the following problems for the data given in Section 5.4.1:

   (a) Confirm that $N = A^T P A$ is rank deficient and that the rank condition (5.1c) is satisfied.

   (b) Compute the vector of estimated parameters $\hat{\boldsymbol{\xi}}$ by (5.13) and confirm that it agrees with that obtained by (5.18).

   (c) Compute the dispersion matrices of $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\lambda}}$ using (5.27a) and (5.27b), respectively, and compare to that obtained by (5.21).

   Note that the matrix to invert in (5.18) may be ill-conditioned (nearly singular) in this case due to the relative magnitude of the elements of matrices N and K. To increase numerical stability, you may need to scale matrix K and vector $\boldsymbol{\kappa}_0$ before using them in (5.18). Try scaling by 10^4. No scaling should be necessary for the other formulas.

---

[1]The assumption of F-distribution is based on an underlying assumption that the residuals are normally distributed, which means that functions of their squares, such as Ω and R, have a χ^2-distribution. The ratio of two independent variables that each have a χ^2-distribution is itself a random variable with F-distribution. Recall that no assumption about the probability density function of the random errors was required for the derivation of their least-squares prediction.

3. With reference to Section 5.4.1, we now simulate case 1 by changing the third element of the last row of matrix A from -1 to 0 and by changing the last element of vector $\boldsymbol{y}$ from -0.162582 to 67.992. All matrices involving N and vectors involving $\boldsymbol{y}$ must be recomputed accordingly.

   (a) Confirm that the revised matrix $N = A^T P A$ has full rank.

   (b) Compute the vector of estimated parameters $\hat{\boldsymbol{\xi}}$ by both (5.8d) and (5.13) and confirm that they are equal.

   (c) Compute the dispersion matrices of $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\lambda}}$ using (5.17) and (5.21) and confirm that they are equal.

   (d) Compute the dispersion matrices using the formulas for case 1, (5.26a) and (5.26b), and confirm that they agree with the respective formulas for case 2, (5.27a) and (5.27b).

   Note that the solution of this problem will not match that of the preceding problem; they are different problems.

4. Using the GMM with constraints, constrain the height of point D in problem 9 of Section 3.6 to $1928.277\,\text{ft}$ and check that the LESS agrees with what you computed in parts (b), (c), and (d) of that problem.

5. By imposing certain constraints upon the unknown parameters, the affine transformation problem presented in Problem 8 of Section 3.6, can be converted to an orthogonality-preserving transformation (only one rotation angle instead of two), or, by a different set of constraints, it can be converted to a similarity transformation (one rotation angle and one scale factor). Using the data from Problem 8, setup the GMM with constraints, state the model redundancy, and compute the LESS for the unknown parameters and variance component in the following two cases:

   (a) Orthogonality-preserving transformation: Impose a constraint on the second rotation angle so that $\epsilon = 0$ via the following:

   $$\xi_4/\xi_6 = \xi_5/\xi_3 \Rightarrow \boxed{\xi_3\xi_4 - \xi_5\xi_6 = 0.}$$

   Note that linearization is required in this case.

   (b) Similarity transformation: Impose the constraints that $\epsilon = 0$ and $\omega_1 = \omega_2$ via the following:

   $$\boxed{\xi_3 - \xi_6 = 0 \ \text{ and } \ \xi_4 - \xi_5 = 0.}$$

6. To monitor deformation, points P_1 and P_2 were established between deforming and non-deforming regions, respectively. Distances were observed from three known points, A, B, and C, to both points P_1 and P_2 (see Figure 5.2) The 2D coordinates of the known points are listed in Table 5.1, and the observations are listed in Table 5.2. The variance of each observation is

$\sigma^2 = (0.005\,]m)^2$. Distances observed from the same point have a correlation coefficient of $\rho = 0.4$. Otherwise the observations are uncorrelated. Suppose the baseline between points P_1 and P_2 is thought to be $251.850\,\mathrm{m}$ (perhaps determined from a previous survey), and it is decided to use this value as a constraint in a least-squares adjustment. Determine the following by use of the LESS within the GMM with constraints:

(a) The redundancy of the model.

(b) The 2D coordinates of points P_1 and P_2 and their dispersion matrix.

(c) The vector of observation residuals and its dispersion matrix.

(d) The estimated variance component.

(e) Setup a hypothesis test with significance level $\alpha = 0.05$ and determine if the constraint is consistent with the observations.
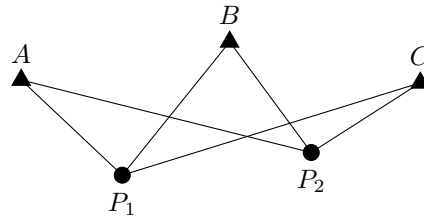
Figure 5.2: Observations from known points A, B, and C

Table 5.1: Coordinates of known points

Station	X [m]	Y [m]
A	456.351	500.897
B	732.112	551.393
C	984.267	497.180

Table 5.2: Observations from known points A, B, and C

From	To	Obs. [m]
A	P_1	183.611
A	P_2	395.462
B	P_1	226.506
B	P_2	181.858
C	P_1	412.766
C	P_2	171.195

7. The data plotted in Figure 3.2 are listed in Table 5.3 below, where the x-coordinates are assumed to be known and the y-coordinates were measured independently and have a common variance of $\sigma^2 = (1\,\text{cm})^2$.

Now suppose a fitted parabola must pass through data point number 5 exactly. Compute the LESS within the GMM with constraints for the three unknown parameters of the parabola and form a hypothesis test to check the validity of the constraint.

Table 5.3: Known x-coordinates and measured y-coordinates plotted in Figure 3.2

No.	x_i [m]	y_i [m]
1	1.001	1.827
2	2.000	1.911
3	3.001	1.953
4	4.000	2.016
5	5.000	2.046
6	6.003	2.056
7	7.003	2.062
8	8.003	2.054
9	9.001	2.042
10	9.998	1.996
11	11.001	1.918
12	12.003	1.867

5.8 Summary Formulas for the Least-Squares Solution Within the Gauss-Markov Model with Constraints

The Gauss-Markov Model with constraints is given by

$$
\begin{aligned}
\underset{n\times 1}{\boldsymbol{y}} &= \underset{n\times m}{A}\,\boldsymbol{\xi} + \boldsymbol{e}, \ \ \boldsymbol{e}\sim(\boldsymbol{0},\sigma_0^2 P^{-1}), \ \ \operatorname{rk}A =: q \le m, \\
\underset{l\times 1}{\boldsymbol{\kappa}_0} &= \underset{l\times m}{K}\,\boldsymbol{\xi}, \ \ \operatorname{rk}K =: l \ge m-q, \ \ \operatorname{rk}\!\big[A^T,\, K^T\big] = m.
\end{aligned}
$$

Table 5.4: Summary formulas for the LESS within the GMM with constraints

Quantity	Formula	Eq.
Model redundancy	$r = n - m + \operatorname{rk}K = n - m + l$	(5.3)
Vector of estimated parameters, when $\operatorname{rk}A = m$	$\hat{\boldsymbol{\xi}} =$ $N^{-1}\boldsymbol{c} + N^{-1}K^T\big(KN^{-1}K^T\big)^{-1}\big(\boldsymbol{\kappa}_0 - KN^{-1}\boldsymbol{c}\big)$	(5.8d)
Dispersion matrix for estimated parameters, when $\operatorname{rk}A = m$	$D\{\hat{\boldsymbol{\xi}}\} =$ $\sigma_0^2\!\cdot\!\big[N^{-1} - N^{-1}K^T\big(KN^{-1}K^T\big)^{-1}KN^{-1}\big]$	(5.26a)
Vector of estimated parameters, when $\operatorname{rk}A < m$	$\hat{\boldsymbol{\xi}} = \big(N + K^TK\big)^{-1}\boldsymbol{c} + \big(N + K^TK\big)^{-1}K^T\big[K\big(N + K^TK\big)^{-1}K^T\big]^{-1}\big[\boldsymbol{\kappa}_0 - K\big(N + K^TK\big)^{-1}\boldsymbol{c}\big]$	(5.13)
Dispersion matrix for estimated parameters, when $\operatorname{rk}A < m$	$D\{\hat{\boldsymbol{\xi}}\} =$ $\sigma_0^2\!\cdot\!\big[N_K^{-1} - N_K^{-1}K^T\big(KN_K^{-1}K^T\big)^{-1}KN_K^{-1}\big]$ with $N_K := N + K^TK$	(5.27a)
Vector of predicted residuals	$\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}}$	(5.29)
Dispersion matrix for residuals, when $\operatorname{rk}A = m$	$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2\!\cdot\!\big\{P^{-1} - A\big[N^{-1} - N^{-1}K^T\big(KN^{-1}K^T\big)^{-1}KN^{-1}\big]A^T\big\}$	(5.33b)

Quantity	Formula	Eq.
Dispersion matrix for residuals, when $\operatorname{rk} A < m$	$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot \big(P^{-1} - A\big\{(N + K^T K)^{-1} - (N + K^T K)^{-1} K^T [K(N + K^T K)^{-1} K^T]^{-1} K(N + K^T K)^{-1}\big\} A^T\big)$	(5.33b)
Sum of squared residuals (SSR)	$\text{SSR} = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$	(5.41) (5.42)
Estimated variance component	$\hat{\sigma}_0^2 = (\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}})/r$	(5.44c)
Vector of adjusted observations	$\hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}$	(5.30)
Dispersion matrix for adjusted observations	$D\{\hat{\boldsymbol{\mu}}_y\} = A \cdot D\{\hat{\boldsymbol{\xi}}\} \cdot A^T$	(5.34)

Continued from previous page

Chapter 6

The Gauss-Markov Model
with Stochastic Constraints

6.1   Model Definition

The Gauss-Markov Model (GMM) with stochastic constraints is similar in form
to the GMM with constraints shown in Chapter 5, with one important difference:
the constraints in the stochastic case are specified with some level of uncertainty,
expressed in the form of a given weight matrix P_0, or an associated cofactor matrix
$Q_0 := P_0^{-1}$. The model reads

$$\underset{n \times 1}{\boldsymbol{y}} = \underset{n \times m}{A} \boldsymbol{\xi} + \boldsymbol{e}, \quad \operatorname{rk} A =: q \leq \min\{m, n\}, \tag{6.1a}$$

$$\underset{l \times 1}{\boldsymbol{z}_0} = \underset{l \times m}{K} \boldsymbol{\xi} + \boldsymbol{e}_0, \quad \operatorname{rk} K =: l \geq m - q, \tag{6.1b}$$

$$\begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e}_0 \end{bmatrix} \sim (\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & P_0^{-1} \end{bmatrix}). \tag{6.1c}$$

Note that in this model there is no correlation between the random error vectors $\boldsymbol{e}$
and $\boldsymbol{e}_0$. Also, the unknown variance component σ_0^2 is common to both cofactor
matrices P^{-1} and P_0^{-1}. However, there may be correlations within one or both of
the cofactor matrices, just not between them. Depending on the application, the
data in the vector $\boldsymbol{y}$ can be thought of as new information, while the constraint
information in the vector $\boldsymbol{z}_0$ can be thought of as prior information (for example,
$\boldsymbol{z}_0$ could contain coordinates estimated from a previous adjustment, now considered
as prior information).

It is required that the column space of the augmented matrix $[A^T \,|\, K^T]$ spans
$\mathbb{R}^m$, which holds when the rank condition

$$\operatorname{rk}[A^T \,|\, K^T] = m \tag{6.2}$$

is satisfied. The redundancy of the model is given by

$$\boxed{r = n - m + l.} \tag{6.3}$$

In words, we can say that the redundancy is the number of observation equations minus the number of parameters to estimate plus the number of constraint equations.

6.1.1   The Addition Theory of Normal Equations

Consider the case of two given data sets that, in part, depend on a set of common, unknown model parameters. Starting with their associated systems of normal equations, eliminate all parameters from each system that are not common to both, for instance by means of a "reduced weight matrix." The two resulting sets of "reduced normal equations" will then only contain common parameters. If there are any remaining common parameters that are of no interest (so-called nuisance parameters) these may likewise be eliminated from both systems. The two systems of reduced normal equations can now be added to each other in order to find the least-squares solution for the entire system. Obviously, this procedure works for any number of data sets, not just for two.

6.2   Least-Squares Solution

According to Schaffrin (1995), the LEast-Squares Solution (LESS) for the unknown parameters $\boldsymbol{\xi}$ within model (6.1) may be derived by minimizing the Lagrange target function

$$\Phi(\boldsymbol{e}, \boldsymbol{e}_0, \boldsymbol{\xi}, \boldsymbol{\lambda}, \boldsymbol{\lambda}_0) = \boldsymbol{e}^T P \boldsymbol{e} + \boldsymbol{e}_0^T P_0 \boldsymbol{e}_0 + 2\big[\boldsymbol{\lambda}^T,\ \boldsymbol{\lambda}_0^T\big]\left(\begin{bmatrix} A \\ K \end{bmatrix}\boldsymbol{\xi} + \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e}_0 \end{bmatrix} - \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}\right) =$$
$$= \underset{\boldsymbol{e}, \boldsymbol{e}_0, \boldsymbol{\xi}, \boldsymbol{\lambda}, \boldsymbol{\lambda}_0}{\text{stationary.}} \quad (6.4)$$

Here we simply consider (6.1) as an extended GMM and apply *the addition theory of normal equations*, stated in Section 6.1.1, as follows:

$$\begin{bmatrix} A^T & K^T \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & P_0 \end{bmatrix}\begin{bmatrix} A \\ K \end{bmatrix} \cdot \hat{\boldsymbol{\xi}} = \begin{bmatrix} A^T & K^T \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & P_0 \end{bmatrix}\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z}_0 \end{bmatrix} \qquad (6.5a)$$

or

$$\big(N + K^T P_0 K\big)\hat{\boldsymbol{\xi}} = \boldsymbol{c} + K^T P_0 \boldsymbol{z}_0, \qquad (6.5b)$$

where

$$\big[N, \boldsymbol{c}\big] := A^T P\big[A, \boldsymbol{y}\big]. \qquad (6.6)$$

In the case where the matrix N is invertible, the Sherman-Morrison-Woodbury-Schur formula (A.6a) may be used to invert the matrix on the left side of (6.5b) as in the following:

$$\boxed{\hat{\boldsymbol{\xi}} = \big(N + K^T P_0 K\big)^{-1}\big(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\big) =} \qquad (6.7a)$$

$$= \left[N^{-1} - N^{-1}K^T \left(P_0^{-1} + KN^{-1}K^T \right)^{-1} KN^{-1} \right] \left(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0 \right) =$$

$$= N^{-1}\boldsymbol{c} + N^{-1}K^T P_0 \boldsymbol{z}_0 + N^{-1}K^T \left(P_0^{-1} + KN^{-1}K^T \right)^{-1} \cdot$$

$$\cdot \left(-KN^{-1}\boldsymbol{c} - KN^{-1}K^T P_0 \boldsymbol{z}_0 \right) =$$

$$= N^{-1}\boldsymbol{c} + N^{-1}K^T \left(P_0^{-1} + KN^{-1}K^T \right)^{-1} \left[\left(P_0^{-1} + KN^{-1}K^T \right) P_0 \boldsymbol{z}_0 - KN^{-1}\boldsymbol{c} - \right.$$

$$\left. - KN^{-1}K^T P_0 \boldsymbol{z}_0 \right] \Rightarrow$$

$$\boxed{\hat{\boldsymbol{\xi}} = N^{-1}\boldsymbol{c} + N^{-1}K^T \left(P_0^{-1} + KN^{-1}K^T \right)^{-1} \left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c} \right).} \tag{6.7b}$$

Thus, the LESS (6.7b) can be viewed as a weighted average between the prior and the new information. The vector $\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}$ is referred to as a *vector of discrepancies*. The solution can also be recognized as an update to the solution $\hat{\boldsymbol{\xi}} = N^{-1}\boldsymbol{c}$ within the GMM (3.1). It is also interesting to express it as an update to the LESS within the GMM with "fixed" constraints (5.1). This can be done by changing the symbols $\hat{\boldsymbol{\xi}}$ and $\boldsymbol{\kappa}_0$ in (5.8d) to $\hat{\boldsymbol{\xi}}_K$ and $\boldsymbol{z}_0$, respectively, solving for $N^{-1}\boldsymbol{c}$ in terms of these renamed variables, and substituting into (6.7b), which yields the following:

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}_K + N^{-1}K^T \left[\left(P_0^{-1} + KN^{-1}K^T \right)^{-1} - \left(KN^{-1}K^T \right)^{-1} \right] \left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c} \right). \tag{6.8}$$

Note that as $P_0^{-1} = Q_0 \to 0$, $\hat{\boldsymbol{\xi}} \to \hat{\boldsymbol{\xi}}_K$.

By applying the laws of covariance propagation to (6.7a), the dispersion matrix for the vector of estimated parameters $\hat{\boldsymbol{\xi}}$ is computed as follows:

$$D\{\hat{\boldsymbol{\xi}}\} = \left(N + K^T P_0 K \right)^{-1} D\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\} \left(N + K^T P_0 K \right)^{-1} =$$

$$= \sigma_0^2 \left(N + K^T P_0 K \right)^{-1} \left(N + K^T P_0 K \right) \left(N + K^T P_0 K \right)^{-1} \Rightarrow$$

$$\boxed{D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 \left(N + K^T P_0 K \right)^{-1} = \sigma_0^2 \left[N^{-1} - N^{-1}K^T \left(P_0^{-1} + KN^{-1}K^T \right)^{-1} KN^{-1} \right].}$$
$$\tag{6.9}$$

The subtraction in (6.9) implies that our knowledge of the parameters has improved (variance decreased) by supplying the additional prior information, provided the estimated variance component $\hat{\sigma}_0^2$ does not change much in doing so. Indeed, if the new data, $\boldsymbol{y}$, is consistent with the old, $\boldsymbol{z}_0$, then $\hat{\sigma}_0^2$ is not expected to change very much when the data are combined. In contrast, $\hat{\sigma}_0^2$ is expected to increase if there is inconsistency between the old and new information. In such a case, it may be advisable to introduce a second variance component, one associated with the weight matrix P and the other with P_0. This is the purpose of the variance component model, which is introduced in Part II.

We now present the residual vectors $\tilde{\boldsymbol{e}}$ and $\tilde{\boldsymbol{e}}_0$ (also called predicted random error vectors). The residual vector $\tilde{\boldsymbol{e}}$ for the observations $\boldsymbol{y}$ is computed by

$$\boxed{\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}}.} \tag{6.10}$$

The residual vector $\tilde{\boldsymbol{e}}_0$ associated with the prior information $\boldsymbol{z}_0$ is

$$\boxed{\tilde{\boldsymbol{e}}_0 = \boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}} =} \tag{6.11a}$$

$$= \left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}\right) - \left(KN^{-1}K^T + P_0^{-1} - P_0^{-1}\right)\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\cdot$$
$$\cdot \left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}\right) =$$
$$= \left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}\right) - \Big[\left(KN^{-1}K^T + P_0^{-1}\right)\left(P_0^{-1} + KN^{-1}K^T\right)^{-1} -$$
$$- P_0^{-1}\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\Big]\left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}\right) =$$
$$= \left\{I_l - \left[I_l - P_0^{-1}\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\right]\right\}\left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}\right) =$$
$$= P_0^{-1}\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}\right) \Rightarrow$$

$$\boxed{\tilde{\boldsymbol{e}}_0 = \left(I_l + KN^{-1}K^T P_0\right)^{-1}\left(\boldsymbol{z}_0 - KN^{-1}\boldsymbol{c}\right).} \qquad (6.11b)$$

The dispersion matrix of the residual vectors is derived as follows (see also Practice Problem 2 in Section 6.6):

$$D\left\{\begin{bmatrix}\tilde{\boldsymbol{e}}\\\tilde{\boldsymbol{e}}_0\end{bmatrix}\right\} = D\left\{\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{z}_0\end{bmatrix}\right\} + D\left\{\begin{bmatrix}A\\K\end{bmatrix}\hat{\boldsymbol{\xi}}\right\} - 2C\left\{\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{z}_0\end{bmatrix}, \begin{bmatrix}A\\K\end{bmatrix}\hat{\boldsymbol{\xi}}\right\} = D\left\{\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{z}_0\end{bmatrix}\right\} - D\left\{\begin{bmatrix}A\\K\end{bmatrix}\hat{\boldsymbol{\xi}}\right\} =$$
$$\tag{6.12a}$$

$$= \sigma_0^2\begin{bmatrix}P^{-1} & 0\\0 & P_0^{-1}\end{bmatrix} - \sigma_0^2\begin{bmatrix}A\\K\end{bmatrix}\left[N^{-1} - N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}KN^{-1}\right]\cdot$$
$$\cdot \begin{bmatrix}A^T & K^T\end{bmatrix} =$$

$$= \sigma_0^2\begin{bmatrix}P^{-1} - AN^{-1}A^T & -AN^{-1}K^T\\-KN^{-1}A^T & P_0^{-1} - KN^{-1}K^T\end{bmatrix} +$$

$$+ \sigma_0^2\begin{bmatrix}AN^{-1}K^T\\KN^{-1}K^T\end{bmatrix}\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\begin{bmatrix}KN^{-1}A^T & KN^{-1}K^T\end{bmatrix}. \qquad (6.12b)$$

From (6.12b), we can write the dispersion matrices for the residual vectors individually as

$$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2\left(P^{-1} - AN^{-1}A^T\right) + \sigma_0^2 AN^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}KN^{-1}A^T \Rightarrow$$
$$\tag{6.13a}$$

$$\boxed{D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2\left[P^{-1} - A\left(N + K^T P_0 K\right)^{-1}A^T\right],} \qquad (6.13b)$$

and

$$
\begin{aligned}
D\{\tilde{e}_0\} &= \sigma_0^2 P_0^{-1} - \sigma_0^2 KN^{-1}K^T + \sigma_0^2 KN^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}KN^{-1}K^T = \\
&= \sigma_0^2 P_0^{-1} - \sigma_0^2 KN^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\cdot \\
&\qquad\qquad \cdot\left(P_0^{-1} + KN^{-1}K^T - KN^{-1}K^T\right) = \\
&= \sigma_0^2 P_0^{-1} - \sigma_0^2 KN^{-1}K^T\left(I_l + P_0 KN^{-1}K^T\right)^{-1} = \\
&= \sigma_0^2 P_0^{-1}\left(I_l + P_0 KN^{-1}K^T\right)\left(I_l + P_0 KN^{-1}K^T\right)^{-1} - \sigma_0^2 KN^{-1}K^T\cdot \\
&\qquad\qquad\qquad\qquad\qquad \cdot\left(I_l + P_0 KN^{-1}K^T\right)^{-1} = \\
&= \sigma_0^2 P_0^{-1}\left(I_l + P_0 KN^{-1}K^T\right)^{-1} + \sigma_0^2 KN^{-1}K^T\left(I_l + P_0 KN^{-1}K^T\right)^{-1} - \\
&\qquad\qquad\qquad - \sigma_0^2 KN^{-1}K^T\left(I_l + P_0 KN^{-1}K^T\right)^{-1} \Rightarrow
\end{aligned}
$$

$$
\boxed{D\{\tilde{e}_0\} = \sigma_0^2 P_0^{-1}\left(I_l + P_0 KN^{-1}K^T\right)^{-1}.} \tag{6.14}
$$

We summarize by listing a few equivalent formulas for $D\{\tilde{e}_0\}$.

$$
\begin{aligned}
D\{\tilde{e}_0\} &= \sigma_0^2 P_0^{-1}\left(I_l + P_0 KN^{-1}K^T\right)^{-1} = \tag{6.15a} \\
&= \sigma_0^2\left(I_l + KN^{-1}K^T P_0\right)^{-1}P_0^{-1} = \tag{6.15b} \\
&= \sigma_0^2 P_0^{-1}\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}P_0^{-1} = \tag{6.15c} \\
&= \sigma_0^2\left(P_0 + P_0 KN^{-1}K^T P_0\right)^{-1} = \tag{6.15d} \\
&= \sigma_0^2\left[P_0^{-1} - K\left(N + K^T P_0 K\right)^{-1}K^T\right] \tag{6.15e}
\end{aligned}
$$

The symmetry of the matrix $D\{\tilde{e}_0\}$ has been exploited to get from (6.15a) to (6.15b), using the rule for the transpose of a matrix product (A.1) and the rule for the transpose of an inverse (A.2). Also (A.3) has been used in the above.

Now it remains to write a succinct form for the covariance matrix $C\{\tilde{e}, \tilde{e}_0\}$, beginning with the off-diagonal element of (6.12b).

$$
C\{\tilde{e}, \tilde{e}_0\} = -\sigma_0^2 AN^{-1}K^T + \sigma_0^2 AN^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}KN^{-1}K^T = \tag{6.16a}
$$

$$
= -\sigma_0^2 AN^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\left(P_0^{-1} + KN^{-1}K^T - KN^{-1}K^T\right) = \tag{6.16b}
$$

$$
= -\sigma_0^2 AN^{-1}K^T\left(I_l + P_0 KN^{-1}K^T\right)^{-1} = \tag{6.16c}
$$

$$
= -\sigma_0^2 A\left(I_m + N^{-1}K^T P_0 K\right)^{-1}N^{-1}K^T = \tag{6.16d}
$$

$$
= -\sigma_0^2 A\left(N + K^T P_0 K\right)^{-1}K^T \tag{6.16e}
$$

The line following (6.16c) is based on relations shown in equations (A.8). To see how these equations are used, compare what follows the term $-\sigma_0^2 A$ in (6.16c) and (6.16d), with the first and last lines in (A.8).

We also note that in the GMM with stochastic constraints, the predicted residual vector $\tilde{e} = y - A\hat{\xi}$ by itself is no longer a projection of y onto the range space of A.

However, the vector $\begin{bmatrix} \tilde{e}^T, \tilde{e}_0^T \end{bmatrix}^T$ does represent a projection of $\begin{bmatrix} y^T, z_0^T \end{bmatrix}^T$ onto the range space of $[A^T, K^T]^T$, since

$$\begin{bmatrix} \tilde{e} \\ \tilde{e}_0 \end{bmatrix} = \begin{bmatrix} y - A\hat{\xi} \\ z_0 - K\hat{\xi} \end{bmatrix} = \left\{ \begin{bmatrix} I_n & 0 \\ 0 & I_l \end{bmatrix} - \begin{bmatrix} A \\ K \end{bmatrix} \left(N + K^T P_0 K \right)^{-1} \begin{bmatrix} A^T P & K^T P_0 \end{bmatrix} \right\} \begin{bmatrix} y \\ z_0 \end{bmatrix},$$
(6.17)

and the matrix in braces is idempotent, which can be verified by application of (1.7a).

The adjusted observations and adjusted constraint values are easily computed by

$$\boxed{\hat{\mu}_y = y - \tilde{e} = A\hat{\xi},}$$
(6.18)

and

$$\boxed{\hat{\mu}_{z_0} = z_0 - \tilde{e}_0 = K\hat{\xi}.}$$
(6.19)

Their respective dispersion matrices are derived by simple application of variance propagation as follows:

$$\boxed{D\{\hat{\mu}_y\} = D\{A\hat{\xi}\} = A \cdot D\{\hat{\xi}\} \cdot A^T = \sigma_0^2 \cdot A\left(N + K^T P_0 K\right)^{-1} A^T,}$$
(6.20)

$$\boxed{D\{\hat{\mu}_{z_0}\} = D\{K\hat{\xi}\} = K \cdot D\{\hat{\xi}\} \cdot K^T = \sigma_0^2 \cdot K\left(N + K^T P_0 K\right)^{-1} K^T.}$$
(6.21)

Here, $\hat{\mu}_y$ is also interpreted as an estimate of the true, and thus unknown, vector of observables μ_y, where $E\{y\} = \mu_y$; likewise, $E\{z_0\} = \mu_{z_0}$.

With the help of (6.20) and (6.21), the dispersion matrices of the residual vectors can by expressed as

$$D\{\tilde{e}\} = D\{y\} - D\{\hat{\mu}_y\},$$
(6.22a)

$$D\{\tilde{e}_0\} = D\{z_0\} - D\{\hat{\mu}_{z_0}\}.$$
(6.22b)

Alternative derivation of normal equations   Starting with the Lagrange target function (6.4), the vector of random errors e and the vector of Lagrange multipliers λ can be eliminated by substitution of $y - A\xi$ for e. Furthermore, by introducing

$$-P_0^{-1}\lambda_0 = e_0 = z_0 - K\xi,$$
(6.23a)

as in Schaffrin (1995), the target function can be expressed equivalently as

$$\Phi(\xi, \lambda_0) = (y - A\xi)^T P(y - A\xi) + 2\lambda_0^T\left(K\xi - z_0\right) - \lambda_0^T P_0^{-1}\lambda_0 = \underset{\xi, \lambda_0}{\text{stationary.}}$$
(6.23b)

Minimizing the above target function leads to the following system of normal equations:

$$\begin{bmatrix} N & K^T \\ K & -P_0^{-1} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{z}_0 \end{bmatrix}. \tag{6.24}$$

Using (6.1b) and (6.24), we can express the predicted residual vector $\tilde{\boldsymbol{e}}_0$ as a function of the vector of Lagrange multipliers $\hat{\boldsymbol{\lambda}}_0$ as follows:

$$\boldsymbol{z}_0 = K\hat{\boldsymbol{\xi}} + \tilde{\boldsymbol{e}}_0 = K\hat{\boldsymbol{\xi}} - P_0^{-1}\hat{\boldsymbol{\lambda}}_0 \Rightarrow \tilde{\boldsymbol{e}}_0 = -P_0^{-1}\hat{\boldsymbol{\lambda}}_0. \tag{6.25}$$

Therefore, the dispersion of $\tilde{\boldsymbol{e}}_0$ is given also by

$$D\{\tilde{\boldsymbol{e}}_0\} = P_0^{-1}D\{\hat{\boldsymbol{\lambda}}_0\}P_0^{-1}. \tag{6.26}$$

Assuming matrix N is invertible, from (6.24) we see that the dispersion of $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\lambda}}_0$ can be found from

$$D\{\begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}}_0 \end{bmatrix}\} = \begin{bmatrix} N & K^T \\ K & -P_0^{-1} \end{bmatrix}^{-1} D\{\begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{z}_0 \end{bmatrix}\} \begin{bmatrix} N & K^T \\ K & -P_0^{-1} \end{bmatrix}^{-1} =$$

$$= \sigma_0^2 \begin{bmatrix} N & K^T \\ K & -P_0^{-1} \end{bmatrix}^{-1} \begin{bmatrix} N & 0 \\ 0 & P_0^{-1} \end{bmatrix} \begin{bmatrix} N & K^T \\ K & -P_0^{-1} \end{bmatrix}^{-1} =$$

$$= \sigma_0^2 \begin{bmatrix} N & K^T \\ K & -P_0^{-1} \end{bmatrix}^{-1} \begin{bmatrix} N^{-1} & 0 \\ 0 & P_0 \end{bmatrix}^{-1} \begin{bmatrix} N & K^T \\ K & -P_0^{-1} \end{bmatrix}^{-1} =$$

$$= \sigma_0^2 \left[\begin{array}{c|c} N + K^T P_0 K & 0 \\ \hline 0 & P_0^{-1} + KN^{-1}K^T \end{array} \right]^{-1}. \tag{6.27}$$

The last line was reached by successively applying the rule for the product of two inverses (A.3). From (6.27) we see that

$$D\{\hat{\boldsymbol{\lambda}}_0\} = \sigma_0^2 \left(P_0^{-1} + KN^{-1}K^T\right)^{-1} = \sigma_0^2 \left[P_0 - P_0 K\left(N + K^T P_0 K\right)^{-1} K^T P_0\right]. \tag{6.28}$$

Finally, substituting (6.28) into (6.26) and applying the product-of-inverses rule, we can write

$$D\{\tilde{\boldsymbol{e}}_0\} = \sigma_0^2 P_0^{-1}\left(P_0^{-1} + KN^{-1}K^T\right)^{-1} P_0^{-1} = \sigma_0^2 \left(P_0 + P_0 KN^{-1}K^T P_0\right)^{-1}. \tag{6.29}$$

Also, the off-diagonal blocks of (6.27) reveal that $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\lambda}}_0$ are uncorrelated, viz.

$$C(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\lambda}}_0) = 0. \tag{6.30}$$

6.3   Variance Component Estimate

The derivation of the variance component estimate is shown here in detail. The trace operator is employed analogously to what was done in Section 3.3. We also make use of the following expectation and dispersion relationships:

$$E\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\} = \begin{bmatrix} A^T P & K^T P_0 \end{bmatrix} E\{\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z}_0 \end{bmatrix}\} =$$

$$= \begin{bmatrix} A^T P & K^T P_0 \end{bmatrix} \begin{bmatrix} A \\ K \end{bmatrix} \boldsymbol{\xi} = (N + K^T P_0 K)\boldsymbol{\xi}, \quad (6.31)$$

$$D\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\} = D\{\begin{bmatrix} A^T P & K^T P_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z}_0 \end{bmatrix}\} =$$

$$= \sigma_0^2 \begin{bmatrix} A^T P & K^T P_0 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & P_0^{-1} \end{bmatrix} \begin{bmatrix} PA \\ P_0 K \end{bmatrix} = \sigma_0^2 (N + K^T P_0 K), \quad (6.32)$$

as well as

$$E\{(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)^T\} = D\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\} + \tag{6.33a}$$
$$+ E\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\} E\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\}^T,$$

$$E\{\boldsymbol{y}\boldsymbol{y}^T\} = D\{\boldsymbol{y}\} + E\{\boldsymbol{y}\}E\{\boldsymbol{y}\}^T = \sigma_0^2 P^{-1} + A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T, \tag{6.33b}$$

$$E\{\boldsymbol{z}_0 \boldsymbol{z}_0^T\} = D\{\boldsymbol{z}_0\} + E\{\boldsymbol{z}_0\}E\{\boldsymbol{z}_0\}^T = \sigma_0^2 P_0^{-1} + K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T. \tag{6.33c}$$

The estimated variance component is derived from the expectation of the combined quadratic forms of the residual vectors, $\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0$, based on the principle

$$\frac{\hat{\sigma}_0^2}{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0} = \frac{\sigma_0^2}{E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0\}}. \tag{6.34}$$

The derivation proceeds as follows:

$$E\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0\} =$$

$$= E\{(\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z}_0 \end{bmatrix} - \begin{bmatrix} A \\ K \end{bmatrix} \hat{\boldsymbol{\xi}})^T \begin{bmatrix} P & 0 \\ 0 & P_0 \end{bmatrix} (\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z}_0 \end{bmatrix} - \begin{bmatrix} A \\ K \end{bmatrix} \hat{\boldsymbol{\xi}})\} =$$

$$= E\{\boldsymbol{y}^T P \boldsymbol{y} + \boldsymbol{z}_0^T P_0 \boldsymbol{z}_0 - 2\hat{\boldsymbol{\xi}}^T (\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0) + \hat{\boldsymbol{\xi}}^T (N + K^T P_0 K)\hat{\boldsymbol{\xi}}\} =$$

$$= E\{\boldsymbol{y}^T P \boldsymbol{y} + \boldsymbol{z}_0^T P_0 \boldsymbol{z}_0 - 2\hat{\boldsymbol{\xi}}^T (\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0) + \hat{\boldsymbol{\xi}}^T (\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)\} =$$

$$= E\{\boldsymbol{y}^T P \boldsymbol{y} + \boldsymbol{z}_0^T P_0 \boldsymbol{z}_0 - \hat{\boldsymbol{\xi}}^T (\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)\} =$$

$$= E\{\boldsymbol{y}^T P \boldsymbol{y} + \boldsymbol{z}_0^T P_0 \boldsymbol{z}_0 - (\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)^T (N + K^T P_0 K)^{-1} (\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)\} =$$

$$= E\{\text{tr}(\boldsymbol{y}^T P \boldsymbol{y}) + \text{tr}(\boldsymbol{z}_0^T P_0 \boldsymbol{z}_0) - \text{tr}[(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)^T (N + K^T P_0 K)^{-1} \cdot$$
$$\cdot (\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)]\} =$$

$$= E\{\mathrm{tr}(P\boldsymbol{y}\boldsymbol{y}^T) + \mathrm{tr}(P_0 \boldsymbol{z}_0 \boldsymbol{z}_0^T) - \mathrm{tr}[(N + K^T P_0 K)^{-1}(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)\cdot$$
$$\cdot\,(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)^T]\} =$$
$$= \mathrm{tr}(PE\{\boldsymbol{y}\boldsymbol{y}^T\}) + \mathrm{tr}(P_0 E\{\boldsymbol{z}_0 \boldsymbol{z}_0^T\}) - \mathrm{tr}[(N + K^T P_0 K)^{-1} E\{(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)\cdot$$
$$\cdot\,(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0)^T\}] =$$
$$= \mathrm{tr}(PE\{\boldsymbol{y}\boldsymbol{y}^T\}) + \mathrm{tr}(P_0 E\{\boldsymbol{z}_0 \boldsymbol{z}_0^T\}) - \mathrm{tr}[(N + K^T P_0 K)^{-1} D\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\}] -$$
$$- \mathrm{tr}[(N + K^T P_0 K)^{-1} E\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\} E\{\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\}^T] =$$
$$= \mathrm{tr}[P(\sigma_0^2 P^{-1} + A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T)] + \mathrm{tr}[P_0(\sigma_0^2 P_0^{-1} + K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T)] -$$
$$- \sigma_0^2 \mathrm{tr}[(N + K^T P_0 K)^{-1}(N + K^T P_0 K)] -$$
$$- \mathrm{tr}[(N + K^T P_0 K)^{-1}(N + K^T P_0 K)\boldsymbol{\xi}\boldsymbol{\xi}^T(N + K^T P_0 K)] =$$
$$= \sigma_0^2 \mathrm{tr}(PP^{-1}) + \mathrm{tr}(PA\boldsymbol{\xi}\boldsymbol{\xi}^T A^T) + \sigma_0^2 \mathrm{tr}(P_0 P_0^{-1}) + \mathrm{tr}(P_0 K\boldsymbol{\xi}\boldsymbol{\xi}^T K^T) -$$
$$- \sigma_0^2 \mathrm{tr}(I_m) - \mathrm{tr}(\boldsymbol{\xi}\boldsymbol{\xi}^T N + \boldsymbol{\xi}\boldsymbol{\xi}^T K^T P_0 K) =$$
$$= \sigma_0^2 \mathrm{tr}(I_n) + \mathrm{tr}(\boldsymbol{\xi}^T N\boldsymbol{\xi}) + \sigma_0^2 \mathrm{tr}(I_l) + \mathrm{tr}(\boldsymbol{\xi}^T K^T P_0 K\boldsymbol{\xi}) -$$
$$- \sigma_0^2 \mathrm{tr}(I_m) - \mathrm{tr}(\boldsymbol{\xi}^T N\boldsymbol{\xi}) - \mathrm{tr}(\boldsymbol{\xi}^T K^T P_0 K\boldsymbol{\xi}) =$$
$$= \sigma_0^2(n + l - m)$$
$$\Rightarrow \sigma_0^2 = (n + l - m)^{-1}\cdot E\{\tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0\}$$

From the preceding derivation, it follows that

$$\boxed{\hat{\sigma}_0^2 = \frac{\tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0}{n - m + l}} \tag{6.35}$$

provides an unbiased estimate of the variance component σ_0^2. Here, the numerator contains the sum of squared residuals

$$\boxed{\mathrm{SSR} : \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0,} \tag{6.36}$$

while the denominator contains the model redundancy, $r = n - m + l$, as specified in (6.3).

6.4 Hypothesis Test Using the Estimated Variance Component

Hypothesis testing can be used to validate that the least-squares solution is consistent with the stochastic constraints in the model (6.1). The test statistic to be computed is comprised of a ratio of two estimated, and therefore random, variances and thus has an F-distribution (see Section 9.4). The idea is to extract from the sum of the quadratic products in (6.36) the associated sum of squared residuals that would have been computed for the LESS within the unconstrained GMM solution, viz. $\hat{\boldsymbol{\xi}}_u = N^{-1}\boldsymbol{c}$, had it been estimated. We label this quantity Ω. What

remains after extracting Ω from (6.36) is a quantity that depends on the weight matrix P_0. We denote this remaining portion as $R(P_0)$ to indicate that it is a function of P_0. Both Ω and $R(P_0)$ are scalars, and both have random properties. These two variables, which are used to form the test statistic, are defined as follows:

$$\Omega := \big(\boldsymbol{y} - AN^{-1}\boldsymbol{c}\big)^T P\big(\boldsymbol{y} - AN^{-1}\boldsymbol{c}\big) = \boldsymbol{y}^T P\boldsymbol{y} - \boldsymbol{c}^T N^{-1}\boldsymbol{c}, \tag{6.37a}$$

$$R(P_0) := \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0\tilde{\boldsymbol{e}}_0 - \Omega. \tag{6.37b}$$

Note: If the matrix N in (6.37a) is singular, than N^{-1} can be replaced with any generalized inverse of N as discussed on page 118.

Again we note that $\hat{\boldsymbol{\xi}}_u = N^{-1}\boldsymbol{c}$ represents the least-squares solution within model (6.1) had the stochastic constraints been omitted. In the following derivations, we also make use of (6.7b), (6.11a), and (6.11b) to write formulas for $\tilde{\boldsymbol{e}}_0$ and $\hat{\boldsymbol{\xi}}$ in terms of $\hat{\boldsymbol{\xi}}_u$ as follows:

$$\tilde{\boldsymbol{e}}_0 = \boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}} = \big(I_l + KN^{-1}K^T P_0\big)^{-1}\big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big), \tag{6.38}$$

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}_u + N^{-1}K^T P_0\tilde{\boldsymbol{e}}_0. \tag{6.39}$$

As long as N is non-singular (matrix A has full-column rank), we can determine a formula for $R(P_0)$ independent of Ω. To do so, we begin with the quadratic form for the full predicted residual vector appearing in (6.36) (also called sum of squared residuals, SSR) and decompose it into Ω and $R(P_0)$. Note that the crossed-out vector in the first line below is neglected since its contribution vanishes in the quadratic product.

$$\tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0\tilde{\boldsymbol{e}}_0 = (\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{z}_0\end{bmatrix} - \begin{bmatrix}\cancel{A}\\\cancel{K}\end{bmatrix}\hat{\boldsymbol{\xi}})^T \begin{bmatrix}P & 0\\0 & P_0\end{bmatrix} (\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{z}_0\end{bmatrix} - \begin{bmatrix}A\\K\end{bmatrix}\hat{\boldsymbol{\xi}}) =$$

$$= \boldsymbol{y}^T P\boldsymbol{y} - \boldsymbol{y}^T PA\hat{\boldsymbol{\xi}} + \boldsymbol{z}_0^T P_0\boldsymbol{z}_0 - \boldsymbol{z}_0^T P_0 K\hat{\boldsymbol{\xi}} =$$

$$= \boldsymbol{y}^T P\boldsymbol{y} - \boldsymbol{y}^T PA\big(\hat{\boldsymbol{\xi}}_u + N^{-1}K^T P_0\tilde{\boldsymbol{e}}_0\big) + \boldsymbol{z}_0^T P_0\boldsymbol{z}_0 - \boldsymbol{z}_0^T P_0 K\big(\hat{\boldsymbol{\xi}}_u + N^{-1}K^T P_0\tilde{\boldsymbol{e}}_0\big) =$$

$$= \underbrace{\big(\boldsymbol{y}^T P\boldsymbol{y} - \boldsymbol{y}^T PA\hat{\boldsymbol{\xi}}_u\big)}_{\Omega} + \boldsymbol{z}_0^T P_0 \underbrace{\big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big)}_{(I_l+KN^{-1}K^T P_0)\tilde{\boldsymbol{e}}_0} - \underbrace{\big(\boldsymbol{c} + K^T P_0\boldsymbol{z}_0\big)^T}_{\hat{\boldsymbol{\xi}}^T(N+K^T P_0 K)} N^{-1}K^T P_0\tilde{\boldsymbol{e}}_0 =$$

$$= \Omega + \boldsymbol{z}_0^T P_0\big(I_l + KN^{-1}K^T P_0\big)\tilde{\boldsymbol{e}}_0 - \hat{\boldsymbol{\xi}}^T\big(N + K^T P_0 K\big)N^{-1}K^T P_0\tilde{\boldsymbol{e}}_0 =$$

$$= \Omega + \boldsymbol{z}_0^T\big(I_l + P_0 KN^{-1}K^T\big)P_0\tilde{\boldsymbol{e}}_0 - \big(K\hat{\boldsymbol{\xi}}\big)^T\big(I_l + P_0 KN^{-1}K^T\big)P_0\tilde{\boldsymbol{e}}_0 =$$

$$= \Omega + \big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}\big)^T\big(I_l + P_0 KN^{-1}K^T\big)P_0\tilde{\boldsymbol{e}}_0 =$$

$$= \Omega + \big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big)^T\big(I_l + P_0 KN^{-1}K^T\big)^{-1}\big(I_l + P_0 KN^{-1}K^T\big)\cdot$$
$$\cdot\big(P_0^{-1} + KN^{-1}K^T\big)^{-1}\big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big) =$$

$$= \Omega + \big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big)^T\big(P_0^{-1} + KN^{-1}K^T\big)^{-1}\big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big) =$$
$$= \Omega + R(P_0)$$

Thus, $R(P_0)$ is defined as

$$R(P_0) := \big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big)^T\big(P_0^{-1} + KN^{-1}K^T\big)^{-1}\big(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_u\big), \tag{6.40}$$

with $\hat{\boldsymbol{\xi}}_u := N^{-1}\boldsymbol{c}$ and assuming the inverse of N exists, in which case (6.37b) and (6.40) should yield identical results.

Finally, the test statistic T can be expressed as a ratio of $R(P_0)$ to Ω, viz.

$$T = \frac{(\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0 - \Omega)/(l - m + q)}{\Omega/(n - q)} =$$
$$= \frac{R(P_0)/(l - m + q)}{\Omega/(n - q)} \sim F(l - m + q, n - q). \qquad (6.41)$$

Recall from (6.2) that $l := \mathrm{rk}(K)$ and $q := \mathrm{rk}(A)$.

The following hypothesis test can now be performed, where $\mathcal{N}$ stands for the normal distribution and $\boldsymbol{z}_0$ is an unknown quantity:

$$H_0 : \boldsymbol{z}_0 \sim \mathcal{N}\big(K\boldsymbol{\xi}, \sigma_0^2 P_0^{-1}\big) \ \text{ against } H_A : \boldsymbol{z}_0 \sim \mathcal{N}\big(\boldsymbol{z}_0 \neq K\boldsymbol{\xi}, \sigma_0^2 P_0^{-1}\big). \qquad (6.42)$$

The term H_0 is called the null hypothesis, and H_A is the alternative hypothesis. After choosing a level of significance α and taking $F_{\alpha, l-m+q, n-q}$ from a table of critical values for the F-distribution, the following logic can be applied:

$$\text{If } T \leq F_{\alpha, l-m+q, n-q} \ \text{ accept } H_0; \text{ else reject } H_0. \qquad (6.43)$$

If MATLAB is available, the critical value may be generated by use of the MATLAB command `finv(1 - α, l - m + q, n - q)`.

6.5 Some Comments on Reproducing Estimators

In this section we briefly discuss two estimators within the Gauss-Markov Model with stochastic constraints (6.1) that leave the constrained parameters unchanged, i.e., unchanged from the values specified in $\boldsymbol{z}_0$. Such estimators are called *reproducing estimators*. For example, in a network adjustment problem the a priori coordinates of a station might need to be left unchanged by the adjustment.

For simplicity, we restrict the discussion to models of full rank, i.e., $\mathrm{rk}\,A = m$, where m is the number of columns of matrix A and also the number of parameters to estimate.

One approach that is sometimes taken to obtain a reproducing estimator is to simply adopt the estimator within the Gauss-Markov Model with fixed constraints shown in (5.8d), which is optimal for that model. Two points should be made regarding the use of that estimator within the model (6.1). First, it is not an optimal estimator within model (6.1), and, second, its dispersion matrix shown in (5.26a) and (5.27a) is not correct within model (6.1). In the following, we show the proper dispersion matrix for the reproducing estimator within model (6.1). First, we introduce different subscripts to denote various linear estimators for $\boldsymbol{\xi}$.

$\hat{\boldsymbol{\xi}}_U$ denotes the unconstrained estimator $\hat{\boldsymbol{\xi}}_U = N^{-1}\boldsymbol{c}$, which is not optimal within model (6.1).

$\hat{\boldsymbol{\xi}}_K$ denotes the reproducing estimator from equation (5.8d), which is not optimal within model (6.1).

$\hat{\boldsymbol{\xi}}_S$ denotes the estimator from equation (6.7a), which is optimal within model (6.1).

Next we express the estimator $\hat{\boldsymbol{\xi}}_K$ as a function of the optimal estimator $\hat{\boldsymbol{\xi}}_S$. Using (6.5b), we can write

$$\left(N + K^T P_0 K\right)^{-1} \boldsymbol{c} = \hat{\boldsymbol{\xi}}_S - \left(N + K^T P_0 K\right)^{-1} K^T P_0 \boldsymbol{z}_0. \tag{6.44}$$

We then repeat (5.8d) for the estimator $\hat{\boldsymbol{\xi}}_K$ with N replaced by $\left(N + K^T P_0 K\right)$ and $\boldsymbol{\kappa}_0$ replaced by $\boldsymbol{z}_0$ according to the model (6.1). This is our starting point.

$$\hat{\boldsymbol{\xi}}_K = \left(N + K^T P_0 K\right)^{-1} \boldsymbol{c} +$$
$$+ \left(N + K^T P_0 K\right)^{-1} K^T \left[K\left(N + K^T P_0 K\right)^{-1} K^T\right]^{-1} \left[\boldsymbol{z}_0 - K\left(N + K^T P_0 K\right)^{-1} \boldsymbol{c}\right] \tag{6.45}$$

Now using (6.44) in (6.45), we can write

$$\hat{\boldsymbol{\xi}}_K = \hat{\boldsymbol{\xi}}_S - \left(N + K^T P_0 K\right)^{-1} K^T P_0 \boldsymbol{z}_0 +$$
$$+ \left(N + K^T P_0 K\right)^{-1} K^T \left[K\left(N + K^T P_0 K\right)^{-1} K^T\right]^{-1} \left[\boldsymbol{z}_0 - K\left(N + K^T P_0 K\right)^{-1} \boldsymbol{c}\right].$$

Factoring out $-\left(N + K^T P_0 K\right)^{-1} K^T \left[K\left(N + K^T P_0 K\right)^{-1} K^T\right]^{-1}$ yields

$$\hat{\boldsymbol{\xi}}_K = \hat{\boldsymbol{\xi}}_S - \left(N + K^T P_0 K\right)^{-1} K^T \left[K\left(N + K^T P_0 K\right)^{-1} K^T\right]^{-1} \cdot$$
$$\cdot \left\{\left[K\left(N + K^T P_0 K\right)^{-1} K^T\right] P_0 \boldsymbol{z}_0 - \boldsymbol{z}_0 + K\left(N + K^T P_0 K\right)^{-1} \boldsymbol{c}\right\}.$$

Now, from (6.7a) we recognize $K\hat{\boldsymbol{\xi}}_S$ in the above line; thus we write:

$$\hat{\boldsymbol{\xi}}_K = \hat{\boldsymbol{\xi}}_S + \left(N + K^T P_0 K\right)^{-1} K^T \left[K\left(N + K^T P_0 K\right)^{-1} K^T\right]^{-1} \left(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_S\right). \tag{6.46}$$

We now have the fixed-constraint estimator $\hat{\boldsymbol{\xi}}_K$ expressed as a function of the optimal estimator for model (6.1), namely $\hat{\boldsymbol{\xi}}_S$. Using a familiar formula for $\left(N + K^T P_0 K\right)^{-1}$ and noting that

$$\left(N + K^T P_0 K\right)^{-1} K^T P_0 = N^{-1} K^T \left(P_0^{-1} + K N^{-1} K^T\right)^{-1},$$

we can rewrite (6.46) as:

$$\hat{\boldsymbol{\xi}}_K = \hat{\boldsymbol{\xi}}_S + \left[N^{-1} - N^{-1} K^T \left(P_0^{-1} + K N^{-1} K^T\right)^{-1} K N^{-1}\right] K^T \cdot$$
$$\cdot \left[K N^{-1} K^T \left(P_0^{-1} + K N^{-1} K^T\right)^{-1} P_0^{-1}\right]^{-1} \left(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_S\right). \tag{6.47}$$

Note the following useful relations:

$$\left[N^{-1} - N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}KN^{-1}\right]K^T =$$
$$= N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}P_0^{-1} \quad (6.48)$$

and

$$\left[KN^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}P_0^{-1}\right]^{-1} =$$
$$= P_0\left(P_0^{-1} + KN^{-1}K^T\right)\left(KN^{-1}K^T\right)^{-1}. \quad (6.49)$$

Equation (6.48) is derived as follows:

$$\left[N^{-1} - N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}KN^{-1}\right]K^T =$$
$$= N^{-1}K^T - N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\left(P_0^{-1} + KN^{-1}K^T - P_0^{-1}\right) =$$
$$= N^{-1}K^T - N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\left(P_0^{-1} + KN^{-1}K^T\right) -$$
$$- N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}\left(-P_0^{-1}\right) =$$
$$= N^{-1}K^T - N^{-1}K^T + N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}P_0^{-1} =$$
$$= N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}P_0^{-1}.$$

Successive application of the rule for the product of inverted matrices was used in equation (6.49). Substituting (6.48) and (6.49) into (6.47) yields:

$$\hat{\boldsymbol{\xi}}_K = \hat{\boldsymbol{\xi}}_S + N^{-1}K^T\left(P_0^{-1} + KN^{-1}K^T\right)^{-1}P_0^{-1}P_0\left(P_0^{-1} + KN^{-1}K^T\right) \cdot$$
$$\cdot\left(KN^{-1}K^T\right)^{-1}\left(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_S\right) =$$
$$= \hat{\boldsymbol{\xi}}_S + N^{-1}K^T\left(KN^{-1}K^T\right)^{-1}\left(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_S\right). \quad (6.50)$$

Equation (6.50) gives an elegant expression of the fixed-constraint estimator $\hat{\boldsymbol{\xi}}_K$ in terms of the optimal estimator $\hat{\boldsymbol{\xi}}_S$. Realizing that the model with stochastic constraints (6.1) becomes the model with fixed constraints (5.1) when P_0^{-1} is replaced by zero, we can replace (6.50) with (6.51) below, which is also obvious from our starting equation (6.45). This also makes the appropriate dispersion matrix $D\{\hat{\boldsymbol{\xi}}_K\}$ under model (6.1) easier to compute.

$$\hat{\boldsymbol{\xi}}_K = \hat{\boldsymbol{\xi}}_U + N^{-1}K^T\left(KN^{-1}K^T\right)^{-1}\left(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_U\right) \quad (6.51)$$

Note that $C\{\boldsymbol{z}_0, \boldsymbol{y}\} = 0$, which allows us to apply the dispersion operator to (6.51) as follows:

$$D\{\hat{\boldsymbol{\xi}}_K\} = D\{\hat{\boldsymbol{\xi}}_U - N^{-1}K^T\left(KN^{-1}K^T\right)^{-1}K\hat{\boldsymbol{\xi}}_U\} +$$
$$+ D\{N^{-1}K^T\left(KN^{-1}K^T\right)^{-1}\boldsymbol{z}_0\} \Rightarrow$$
$$D\{\hat{\boldsymbol{\xi}}_S \to \hat{\boldsymbol{\xi}}_K\} = \sigma_0^2 N^{-1} - \sigma_0^2 N^{-1}K^T\left(KN^{-1}K^T\right)^{-1}KN^{-1} +$$
$$+ \sigma_0^2 N^{-1}K^T\left(KN^{-1}K^T P_0 KN^{-1}K^T\right)^{-1}KN^{-1}. \quad (6.52)$$

The notation $D\{\hat{\boldsymbol{\xi}}_S \to \hat{\boldsymbol{\xi}}_K\}$ reflects that the dispersion is computed within model (6.1) and that the optimal estimator $\hat{\boldsymbol{\xi}}_S$ within that model is replaced by the re-producing estimator $\hat{\boldsymbol{\xi}}_K$. Compare (6.52) to (5.16) to see that $D\{\hat{\boldsymbol{\xi}}_K\}$ increases by

$$\sigma_0^2 N^{-1} K^T \left(KN^{-1}K^T P_0 KN^{-1}K^T\right)^{-1} KN^{-1}$$

in this case.

We already noted that $\hat{\boldsymbol{\xi}}_K$ is a sub-optimal (reproducing) estimator within model (6.1). We now give the optimal reproducing estimator without derivation (for details see Schaffrin (1997a)).

$$\hat{\boldsymbol{\xi}}_{opt-rep} = \hat{\boldsymbol{\xi}}_S + K^T \left(KK^T\right)^{-1} \left(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_S\right) \tag{6.53}$$

The symbol $\hat{\boldsymbol{\xi}}_S$ on the right side of (6.53) represents the optimal ("non-reproducing") estimator. Equation (6.53) is identical to (6.50) when N^{-1} is replaced by I.

The dispersion matrix is given by

$$D\{\hat{\boldsymbol{\xi}}_{opt-rep}\} = D\{\hat{\boldsymbol{\xi}}_S\} + D\{K^T \left(KK^T\right)^{-1} \left(\boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}_S\right)\} =$$
$$= \sigma_0^2 N^{-1} - \sigma_0^2 N^{-1} K^T \left(P_0^{-1} + KN^{-1}K^T\right)^{-1} KN^{-1} +$$
$$+ \sigma_0^2 K^T \left(KK^T\right)^{-1} P_0^{-1} \left(P_0^{-1} + KN^{-1}K^T\right)^{-1} P_0^{-1} \left(KK^T\right)^{-1} K. \tag{6.54}$$

Also note that

$$E\{\hat{\boldsymbol{\xi}}_{opt-rep}\} = \boldsymbol{\xi}, \tag{6.55a}$$
$$\boldsymbol{z}_0 - K\boldsymbol{\xi}_{opt-rep} = \boldsymbol{0}, \tag{6.55b}$$
$$D\{K\hat{\boldsymbol{\xi}}_{opt-rep}\} = D\{\boldsymbol{z}_0\} = \sigma_0^2 P_0^{-1}. \tag{6.55c}$$

6.6   Practice Problems

1. Given the target function

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\lambda}_0) = (\boldsymbol{y} - A\boldsymbol{\xi})^T P(\boldsymbol{y} - A\boldsymbol{\xi}) + 2\boldsymbol{\lambda}_0^T (K\boldsymbol{\xi} - \boldsymbol{z}_0) - \boldsymbol{\lambda}_0^T P_0^{-1} \boldsymbol{\lambda}_0$$

   from (6.23b), complete the following:

   (a) With the help of (6.23a), show that equations (6.4) and (6.23b) are equivalent.

   (b) Formulate the Euler-Lagrange necessary conditions for the least-squares solution of the unknown parameters $\boldsymbol{\xi}$ and the unknown vector of La-grange multipliers $\boldsymbol{\lambda}_0$.

   (c) Show how the sufficient condition for minimization is satisfied.

   (d) Using the Euler-Lagrange necessary conditions that you formulated in item 1.b, derive the vector of estimated parameters $\hat{\boldsymbol{\xi}}$ and check that it agrees with (6.7b).

2. Confirm that (6.12a) is correct by showing that

$$C\{\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z}_0 \end{bmatrix}, \begin{bmatrix} A \\ K \end{bmatrix} \hat{\boldsymbol{\xi}}\} = D\{\begin{bmatrix} A \\ K \end{bmatrix} \hat{\boldsymbol{\xi}}\}.$$

3. Repeat Problem 4 of Section 5.7, this time using the following constraints:

   (a) Use 1928.277 ft as $\boldsymbol{z}_0$ and $\sigma^2 = (0.005\,\text{ft})^2$ for its variance. Compare your answers to those of Problem 4. Are they the same? If so, what is your explanation for that? Can a hypothesis test be formulated as described in Section 6.4?

   (b) Now add another constraint that requires the height of point D to be 248.750 ft greater than the height of point A, with variance $\sigma^2 = 2(0.005^2)$ ft^2. Form a hypothesis test to check the consistency of the observation equations and the constraint equations.

4. Repeat Problem 6 of Section 5.7. This time use 251.850 m as $\boldsymbol{z}_0$ and $\sigma^2 = (0.005\,\text{m})^2$ for its variance. Compare your answers to those of Problem 6. What changes, what stays the same? Form a hypothesis test to check the consistency of the observation equations and the constraint equations.

5. Repeat Problem 7 of Section 5.7. This time use 2.046 m as $\boldsymbol{z}_0$ and $\sigma^2 = (1\,\text{cm})^2$ for its variance. Compare your answers to those of Problem 7. What changes, what stays the same? Form a hypothesis test to check the consistency of the observation equations and the constraint equations.

6. Referring to the example problem in Section 5.4.1, set the vector $\boldsymbol{\kappa}_0$ shown there equal to $\boldsymbol{z}_0$. Use the following matrix for P_0^{-1}:

$$P_0^{-1} = \begin{bmatrix} 2.84067584875257 & 0.533989733139618 & 0.535740019844372 \\ 0.533989733139618 & 2.14132575448909 & 0.531530384522843 \\ 0.535740019844372 & 0.531530384522843 & 2.19379908268108 \end{bmatrix} \cdot$$
$$\cdot (10^{-6})\,\text{m}^2.$$

   In addition, multiply the cofactor matrix P^{-1} by 0.017381 and the cofactor matrix P_0^{-1} by 8.709801 to account for the variance components estimated in Smith et al. (2018), which should result in a solution that agrees with the results shown therein.

   Complete the following:

   (a) Estimate the heights of all points.

   (b) Form a hypothesis test to check the consistency of the observation equations and the constraint equations.

7. Show that the total residual vector $[\tilde{\boldsymbol{e}}^T, \tilde{\boldsymbol{e}}_0^T]^T$ results from a projection of $[\boldsymbol{y}^T, \boldsymbol{z}_0^T]^T$ onto the range space of $[A^T, K^T]^T$. Hint: see equation (6.17).

6.7 Summary Formulas for the Least-Squares Solution Within the Gauss-Markov Model with Stochastic Constraints

The Gauss-Markov Model with stochastic constraints is given by

$$
\begin{aligned}
\underset{n\times 1}{\boldsymbol{y}} &= \underset{n\times m}{A}\,\boldsymbol{\xi} + \boldsymbol{e}, \\
\underset{l\times 1}{\boldsymbol{z}_0} &= \underset{l\times m}{K}\,\boldsymbol{\xi} + \boldsymbol{e}_0, \\
\begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e}_0 \end{bmatrix} &\sim (\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & P_0^{-1} \end{bmatrix}).
\end{aligned}
$$

Table 6.1: Summary formulas for the LESS within the Gauss-Markov Model with stochastic constraints

Quantity	Formula	Eq.
Model redundancy	$r = n - m + l$	(6.3)
Vector of estimated parameters	$\hat{\boldsymbol{\xi}} = \left(N + K^T P_0 K\right)^{-1}\left(\boldsymbol{c} + K^T P_0 \boldsymbol{z}_0\right)$	(6.7a)
Dispersion matrix for estimated parameters	$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 \cdot \left(N + K^T P_0 K\right)^{-1}$	(6.9)
Vector of predicted residuals	$\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}}$	(6.10)
Dispersion matrix for residuals	$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \cdot \left[P^{-1} - A\left(N + K^T P_0 K\right)^{-1} A^T\right]$	(6.13b)
Vector of residuals of prior information	$\tilde{\boldsymbol{e}}_0 = \boldsymbol{z}_0 - K\hat{\boldsymbol{\xi}}$	(6.11a)
Dispersion matrix for residuals of prior information	$D\{\tilde{\boldsymbol{e}}_0\} = \sigma_0^2 \cdot P_0^{-1}\left(I_l + P_0 K N^{-1} K^T\right)^{-1}$	(6.14)
Sum of squared residuals (SSR)	$\Omega + R(P_0) = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0$	(6.36), (6.37b)

Continued on next page

Quantity	Formula	Eq.
Estimated variance component	$\hat{\sigma}_0^2 = (\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}_0^T P_0 \tilde{\boldsymbol{e}}_0)/(n - m + l)$	(6.35)
Vector of adjusted observations	$\hat{\boldsymbol{\mu}}_y = \boldsymbol{y} - \tilde{\boldsymbol{e}}$	(6.18)
Dispersion matrix for adjusted observations	$D\{\hat{\boldsymbol{\mu}}_y\} = \sigma_0^2 \cdot A (N + K^T P_0 K)^{-1} A^T$	(6.20)
Vector of adjusted constraints	$\hat{\boldsymbol{\mu}}_{z_0} = \boldsymbol{z}_0 - \tilde{\boldsymbol{e}}_0$	(6.19)
Dispersion matrix for adjusted constraints	$D\{\hat{\boldsymbol{\mu}}_{z_0}\} = \sigma_0^2 \cdot K (N + K^T P_0 K)^{-1} K^T$	(6.21)

Continued from previous page

Chapter 7

Sequential Adjustments

A sequential adjustment might be called for when two successive data sets must be combined to estimate a single set of parameters. This type of adjustment is especially useful when only the parameter estimates and their dispersion matrix, but not the associated observations, are available from the first data set. Then, update formulas can be used that allow the second data set to be adjusted in a way that depends on the estimates from the first data set, with the results being equivalent to what would have been computed from a simultaneous adjustment of both data sets. Though we refer to the two data sets respectively as *first* and *second*, they could be any two successive data sets that must be treated by a sequential adjustment, e.g., they could be the ninth and tenth.

7.1 Model Definition

The data model for sequential adjustments is based on *two data sets*, denoted below by subscripts 1 and 2, respectively. The first data set is comprised of n_1 observations, and the second is comprised of n_2. It is assumed that the observations from the first data set, $\boldsymbol{y}_1$, are *uncorrelated* with those from the second, $\boldsymbol{y}_2$, i.e., $C\{\boldsymbol{y}_1, \boldsymbol{y}_2\} = 0$. Moreover, all parameters associated with the second data set are also associated with the first data set. Thus, the data model is written as

$$\underset{n_1 \times 1}{\boldsymbol{y}_1} = \underset{n_1 \times m}{A_1} \boldsymbol{\xi} + \boldsymbol{e}_1, \tag{7.1a}$$

$$\underset{n_2 \times 1}{\boldsymbol{y}_2} = \underset{n_2 \times m}{A_2} \boldsymbol{\xi} + \boldsymbol{e}_2, \tag{7.1b}$$

$$\underset{n \times 1}{\begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{bmatrix}} \sim \left(\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{bmatrix} \right). \tag{7.1c}$$

The ranks of the coefficient (design) matrices A_1 and A_2 are such that

$$\operatorname{rk} A_1 = \operatorname{rk} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = m, \text{ with } \operatorname{rk} A_2 \leq m. \tag{7.2}$$

Equations (7.1) and (7.2) show that the coefficient matrix A_1 has full column rank m, that there is *no correlation* between the random error vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, and that both data sets share a *common variance component* σ_0^2. Also, we define the total number of observations, n, from both data-sets 1 and 2 as

$$n := n_1 + n_2. \tag{7.3}$$

The following notation is adopted for normal-equation variables used in Sections 7.1 to 7.3:

$$[N_{ii}, \boldsymbol{c}_i] = A_i^T P_i [A_i, \boldsymbol{y}_i], \quad i \in \{1, 2\}, \tag{7.4a}$$

so that

$$N_{11} = A_1^T P_1 A_1, \; N_{22} = A_2^T P_2 A_2, \; \boldsymbol{c}_1 = A_1^T P_1 \boldsymbol{y}_1, \; \text{and} \; \boldsymbol{c}_2 = A_2^T P_2 \boldsymbol{y}_2. \tag{7.4b}$$

> Subscripts on N and $\boldsymbol{c}$ have somewhat different meanings for sections that follow Section 7.3; careful attention should be paid to their definitions given there.

We use a single hat to denote estimates that are based only on the first data set and a double hat to denote estimates that are based on *both* data sets. For example, the estimated parameter vector $\hat{\boldsymbol{\xi}}$ is based only on the first data set, whereas the estimate $\hat{\hat{\boldsymbol{\xi}}}$ is based on both data sets. This makes it convenient to show estimates based on both data sets as an update to estimates based on only the first data set.

We recognize a structural similarity between the data model shown in (7.1) and the Gauss-Markov Model with stochastic constraints shown in (6.1). Given this similarity, we may immediately write down a least-squares solution for $\boldsymbol{\xi}$, and its dispersion matrix, in the form of (6.7b) and (6.9), respectively, viewing the second data set as analogous to stochastic constraints.

$$\hat{\hat{\boldsymbol{\xi}}} = \hat{\boldsymbol{\xi}} + N_{11}^{-1} A_2^T \left(P_2^{-1} + A_2 N_{11}^{-1} A_2^T \right)^{-1} \left(\boldsymbol{y}_2 - A_2 \hat{\boldsymbol{\xi}} \right) = \tag{7.5}$$

$$= \hat{\boldsymbol{\xi}} + \left(N_{11} + A_2^T P_2 A_2 \right)^{-1} A_2^T P_2 \left(\boldsymbol{y}_2 - A_2 \hat{\boldsymbol{\xi}} \right) \tag{7.6}$$

$$D\{\hat{\hat{\boldsymbol{\xi}}}\} = D\{\hat{\boldsymbol{\xi}}\} - \sigma_0^2 N_{11}^{-1} A_2^T \left(P_2^{-1} + A_2 N_{11}^{-1} A_2^T \right)^{-1} A_2 N_{11}^{-1} \tag{7.7}$$

Equation (A.8a) was used in going from (7.5) to (7.6). It is important to note that the matrix $\left(P_2^{-1} + A_2 N_{11}^{-1} A_2^T \right)$ is of size $n_2 \times n_2$; whereas the size of matrix $\left(N_{11} + A_2^T P_2 A_2 \right)$ is $m \times m$. Therefore, if the second data set has only one observation then $n_2 = 1$, and the update via (7.5) is very fast! This may be the case, for example, in a real-time application where one new observation is added at each epoch in time.

It is also noted that the matrix subtracted in (7.7) is positive definite, which means that regardless of the precision of the second data set reflected in P_2, the dispersion of the parameters estimated from both data sets will be smaller than that estimated from only the first data set.

7.2   Verification of the Sequential Adjustment

In this section we discuss verification of the sequential adjustment, the aim of which is to confirm that the adjustment based on both data sets is consistent with an

adjustment based only on the first data set. By consistent we mean that both the first data set only and the combined data sets fit the model well, implying that the residuals from an adjustment of the first data set would not change much in a sequential adjustment of both data sets.

We can make use of the work done in Chapter 6 to write the estimated variance component $\hat{\sigma}_0^2$ in a form composed of the sum of squared residuals (SSR) Ω based on an adjustment of the first data set only and an update $R(P_2)$ for the contribution to the SSR from the second data set, analogous to the derivation of (6.40). This facilitates hypothesis testing for the purpose of determining if the combined adjustment is consistent with an adjustment based only on the first data set. The decomposition of $\hat{\sigma}_0^2$ into Ω and $R(P_2)$ is expressed as follows:

$$\hat{\sigma}_0^2(n - m) = \Omega + R(P_2); \quad \text{with} \quad \Omega = \hat{\sigma}_0^2(n_1 - m) \tag{7.8a}$$

and where

$$R(P_2) = -\big(\boldsymbol{y}_2 - A_2\hat{\boldsymbol{\xi}}\big)^T \hat{\hat{\boldsymbol{\lambda}}} \quad \text{with} \quad \hat{\hat{\boldsymbol{\lambda}}} := -\big(P_2^{-1} + A_2 N_{11}^{-1} A_2^T\big)^{-1}\big(\boldsymbol{y}_2 - A_2\hat{\boldsymbol{\xi}}\big). \tag{7.8b}$$

Therefore, we can rewrite (7.8a) as

$$\boxed{\hat{\sigma}_0^2(n - m) = \Omega + \big(\boldsymbol{y}_2 - A_2\hat{\boldsymbol{\xi}}\big)^T\big(P_2^{-1} + A_2 N_{11}^{-1} A_2^T\big)^{-1}\big(\boldsymbol{y}_2 - A_2\hat{\boldsymbol{\xi}}\big),} \tag{7.8c}$$

where the form of $R(P_2)$ is obviously similar to that of $R(P_0)$ in (6.40).

Then, the test statistic

$$T = \frac{R/n_2}{\Omega/(n_1 - m)} \sim F(n_2, n_1 - m) \tag{7.9}$$

can be computed to verify the sequential adjustment, i.e., that both the first data set and the combined first and second data sets fit the model well. The test statistic has an F-distribution with n_2 and $n_1 - m$ degrees of freedom. For some specified significance level α, we may claim that the observations from the second data set are consistent with those from the first if $T \leq F_{\alpha,n_2,n_1-m}$. See Chapter 9 for more on hypothesis testing.

7.3 Alternative Solution for the Normal Equations

Using the *addition theory of normal equations*, stated in Section 6.1.1, we may find a matrix representation of the normal equations as follows, where again the double hats above $\boldsymbol{\xi}$ refer to an estimate based on both data sets:

$$\big(A_1^T P_1 A_1 + A_2^T P_2 A_2\big)\hat{\hat{\boldsymbol{\xi}}} = \big(A_1^T P_1 \boldsymbol{y}_1 + A_2^T P_2 \boldsymbol{y}_2\big), \tag{7.10a}$$

or

$$\big(N_{11} + N_{22}\big)\hat{\hat{\boldsymbol{\xi}}} = \big(\boldsymbol{c}_1 + \boldsymbol{c}_2\big). \tag{7.10b}$$

These normal equations lead to

$$N_{11}\hat{\hat{\boldsymbol{\xi}}} + N_{22}\hat{\hat{\boldsymbol{\xi}}} - \boldsymbol{c}_2 = \boldsymbol{c}_1 \Rightarrow \tag{7.11a}$$

$$N_{11}\hat{\hat{\boldsymbol{\xi}}} + A_2^T\hat{\hat{\boldsymbol{\lambda}}}_2 = \boldsymbol{c}_1, \ \text{ with } \hat{\hat{\boldsymbol{\lambda}}} = P_2\big(A_2\hat{\hat{\boldsymbol{\xi}}} - \boldsymbol{y}_2\big) \Rightarrow \tag{7.11b}$$

$$\boldsymbol{y}_2 = A_2\hat{\hat{\boldsymbol{\xi}}} - P_2^{-1}\hat{\hat{\boldsymbol{\lambda}}}. \tag{7.11c}$$

Then, from (7.11b) and (7.11c), we can write the following system of least-squares normal equations:

$$\begin{bmatrix} N_{11} & A_2^T \\ A_2 & -P_2^{-1} \end{bmatrix} \begin{bmatrix} \hat{\hat{\boldsymbol{\xi}}} \\ \hat{\hat{\boldsymbol{\lambda}}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{y}_2 \end{bmatrix}. \tag{7.12}$$

From the first row of (7.12) we get

$$\hat{\hat{\boldsymbol{\xi}}} = N_{11}^{-1}\boldsymbol{c}_1 - N_{11}^{-1}A_2^T\hat{\hat{\boldsymbol{\lambda}}} = \tag{7.13a}$$

$$= \hat{\boldsymbol{\xi}} - N_{11}^{-1}A_2^T\hat{\hat{\boldsymbol{\lambda}}}. \tag{7.13b}$$

Equation (7.13b) is an update formula as a function of the vector of estimated Lagrange multipliers $\hat{\hat{\boldsymbol{\lambda}}}$. Without further derivation, we can compare (7.13b) to (7.5) to get an expression for the estimated vector of Lagrange-multiplier as

$$\hat{\hat{\boldsymbol{\lambda}}} = -\big(P_2^{-1} + A_2 N_{11}^{-1} A_2^T\big)^{-1}\big(\boldsymbol{y}_2 - A_2\hat{\boldsymbol{\xi}}\big), \tag{7.14}$$

which agrees with (7.8b). Applying covariance propagation to (7.13b), we find the dispersion matrix of $\hat{\hat{\boldsymbol{\xi}}}$ to be

$$\boxed{D\{\hat{\hat{\boldsymbol{\xi}}}\} = D\{\hat{\boldsymbol{\xi}}\} - \sigma_0^2 N_{11}^{-1} A_2^T \big(P_2^{-1} + A_2 N_{11}^{-1} A_2^T\big)^{-1} A_2 N_{11}^{-1},} \tag{7.15}$$

where we used the fact that $C\{\boldsymbol{y}_2, \hat{\boldsymbol{\xi}}\} = 0$, which indicates that the observations from the second data set are uncorrelated with the estimated parameters based on the first data set only.

7.4 Sequential Adjustment, Rank-Deficient Case

7.4.1 First Data Set Only

Suppose matrix A_1 does not have full column rank, i.e. $\operatorname{rk} A_1 =: q_1 < m$. Then we may introduce a datum by further splitting the system of equations as was done in Section 4.5. Let us split A_1 into an $n_1 \times q_1$ part denoted A_{11} and an $n_1 \times (m - q_1)$ part denoted A_{12}. Accordingly, we also split the parameter vector $\boldsymbol{\xi}$ into a $q_1 \times 1$ part $\boldsymbol{\xi}_1$ and a $(m - q_1) \times 1$ part $\boldsymbol{\xi}_2$. Thus, we have

$$A_1 = [A_{11},\, A_{12}], \ \ \operatorname{rk} A_{11} =: q_1, \ \text{ and } \boldsymbol{\xi} = \big[\boldsymbol{\xi}_1^T,\, \boldsymbol{\xi}_2^T\big]^T. \tag{7.16a}$$

With this splitting, we introduce new terms for the normal equations, where it is stressed that the subscripts on matrix N and vector $\boldsymbol{c}$ are used differently than in the preceding sections. Most notably, the subscript 2 does not refer to the second data set, but only to the location of a block in the matrix N.

The terms N_{11}, N_{12}, and $\boldsymbol{c}_1$ are defined as follows:

$$\begin{bmatrix} A_{11}^T \\ A_{12}^T \end{bmatrix} P_1 \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} = \left[\begin{array}{c|c} A_{11}^T P_1 A_{11} & A_{11}^T P_1 A_{12} \\ \hline A_{12}^T P_1 A_{11} & A_{12}^T P_1 A_{12} \end{array} \right] = \left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline A_{12}^T P_1 A_{11} & A_{12}^T P_1 A_{12} \end{array} \right],$$

$$(7.16\text{b})$$

and

$$\boldsymbol{c}_1 = A_{11}^T P_1 \boldsymbol{y}_1. \tag{7.16c}$$

Note that we purposely did not use symbols N_{21} and $\boldsymbol{c}_2$ here, because they will be defined in a different way in the next section.

Next we introduce given datum information $\boldsymbol{\xi}_2^0$ for $\boldsymbol{\xi}_2$, such that $\boldsymbol{\xi}_2 \to \boldsymbol{\xi}_2^0$, where the subscript 2 now obviously refers to the datum (the second part of $\boldsymbol{\xi}$), rather than a second data set. The formulas for the estimated parameters and their dispersion matrix based on the first data set only can be copied from (3.43b) and (3.44), respectively.

$$\boxed{\hat{\boldsymbol{\xi}}_1 = N_{11}^{-1}\big(\boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0\big)} \tag{7.17a}$$

$$\boxed{D\{\hat{\boldsymbol{\xi}}_1\} = \sigma_0^2 N_{11}^{-1}} \tag{7.17b}$$

The estimated variance component $\hat{\sigma}_0^2$ is slightly different from that of (3.49) and (3.52) and is given by the formula

$$\boxed{\hat{\sigma}_0^2 = \frac{\boldsymbol{y}_1^T P_1\big(\boldsymbol{y}_1 - A_{11}\hat{\boldsymbol{\xi}}_1 - A_{12}\boldsymbol{\xi}_2^0\big)}{(n_1 - q_1)}} \tag{7.17c}$$

or, equivalently,

$$\boxed{\hat{\sigma}_0^2 = \frac{\big(\boldsymbol{y}_1^T P_1 \boldsymbol{y}_1 - \boldsymbol{c}_1^T N_{11}^{-1} \boldsymbol{c}_1\big)}{(n_1 - q_1)}.} \tag{7.17d}$$

Note that the steps taken from (3.49) to (3.52) can be used to go from (7.17c) to (7.17d).

7.4.2 Both First and Second Data Sets

Now we introduce the second data set with a splitting analogous to the first, viz.

$$\boldsymbol{y}_2 = A_{21}\boldsymbol{\xi}_1 + A_{22}\boldsymbol{\xi}_2 + \boldsymbol{e}_2, \quad \boldsymbol{e}_2 \sim \big(\boldsymbol{0}, \sigma_0^2 P_2^{-1}\big). \tag{7.18}$$

The matrix A_{21} is of size $n_2 \times q_1$, and A_{22} is of size $n_2 \times (m - q_1)$. No information in the second data set refers to the datum choice; it only adds to the redundancy provided by the first data set. Thus, the rank of the normal equation matrix is unchanged, which is true also for the 2×2-block coefficient matrix, i.e.,

$$\text{rk} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =: q = q_1. \tag{7.19}$$

To be clear, the first subscript refers to the data set, 1 or 2, and the second subscript refers to the part of the parameter vector $\boldsymbol{\xi}$, first or second. The full least-squares normal equations are then written as

$$
\left[\begin{array}{c|c} A_{11}^T P_1 A_{11} + A_{21}^T P_2 A_{21} & A_{11}^T P_1 A_{12} + A_{21}^T P_2 A_{22} \\ \hline A_{12}^T P_1 A_{11} + A_{22}^T P_2 A_{21} & A_{12}^T P_1 A_{12} + A_{22}^T P_2 A_{22} \end{array} \right] \left[\begin{array}{c} \hat{\boldsymbol{\xi}}_1 \\ \boldsymbol{\xi}_2^0 \end{array} \right] =
$$
$$
= \left[\begin{array}{c|c} A_{11}^T P_1 & A_{21}^T P_2 \\ \hline A_{12}^T P_1 & A_{22}^T P_2 \end{array} \right] \left[\begin{array}{c} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{array} \right]. \quad (7.20)
$$

From the first row of (7.20), we may write the least-squares solution for $\hat{\hat{\boldsymbol{\xi}}}_1$ directly, followed by its dispersion matrix, as

$$
\boxed{
\begin{aligned}
\hat{\hat{\boldsymbol{\xi}}}_1 &= \left(A_{11}^T P_1 A_{11} + A_{21}^T P_2 A_{21} \right)^{-1} \cdot \\
&\quad \cdot \left[\left(A_{11}^T P_1 \boldsymbol{y}_1 + A_{21}^T P_2 \boldsymbol{y}_2 \right) - \left(A_{11}^T P_1 A_{12} + A_{21}^T P_2 A_{22} \right) \boldsymbol{\xi}_2^0 \right], \\
D\{\hat{\hat{\boldsymbol{\xi}}}_1\} &= \sigma_0^2 \left(A_{11}^T P_1 A_{11} + A_{21}^T P_2 A_{21} \right)^{-1}.
\end{aligned}
}
$$
$$
\begin{aligned}
& (7.21) \\
& (7.22)
\end{aligned}
$$

In order to derive update formulas, it is helpful to introduce an alternative expression for the normal equations analogous to what was done in (7.11a) through (7.12). From (7.17a), we can write

$$
\left(A_{11}^T P_1 A_{11} \right) \hat{\boldsymbol{\xi}}_1 = \left(A_{11}^T P_1 \boldsymbol{y}_1 \right) - \left(A_{11}^T P_1 A_{12} \right) \boldsymbol{\xi}_2^0, \quad (7.23a)
$$
$$
\text{or } N_{11} \hat{\boldsymbol{\xi}}_1 = \boldsymbol{c}_1 - N_{12} \boldsymbol{\xi}_2^0, \quad (7.23b)
$$

which, when subtracted from the first row of (7.20), leaves

$$
\left(A_{21}^T P_2 A_{21} \right) \hat{\boldsymbol{\xi}}_1 = \left(A_{21}^T P_2 \boldsymbol{y}_2 \right) - \left(A_{21}^T P_2 A_{22} \right) \boldsymbol{\xi}_2^0, \quad (7.23c)
$$
$$
\text{or } N_{21} \hat{\boldsymbol{\xi}}_1 = \boldsymbol{c}_2 - N_{22} \boldsymbol{\xi}_2^0. \quad (7.23d)
$$

> Note that the symbols N_{11} and N_{12} are still being used as defined in (7.16b), whereas the definition of N_{22} and N_{21} becomes apparent by comparing (7.23c) to (7.23d).

Together, (7.23b) and (7.23d) comprise the first row of (7.20). Recombining (7.23b) and (7.23d) gives

$$
\left(N_{11} + N_{21} \right) \hat{\hat{\boldsymbol{\xi}}}_1 = \boldsymbol{c}_1 + \boldsymbol{c}_2 - \left(N_{12} + N_{22} \right) \boldsymbol{\xi}_2^0, \quad (7.24a)
$$

implying that

$$
N_{11} \hat{\hat{\boldsymbol{\xi}}}_1 + A_{21}^T \hat{\hat{\boldsymbol{\lambda}}} = \boldsymbol{c}_1 - N_{12} \boldsymbol{\xi}_2^0, \text{ with } \hat{\hat{\boldsymbol{\lambda}}} := P_2 \left(A_{21} \hat{\hat{\boldsymbol{\xi}}}_1 - \boldsymbol{y}_2 + A_{22} \boldsymbol{\xi}_2^0 \right). \quad (7.24b)
$$

Note that in (7.23a)–(7.23d) a single hat was used for the estimate of $\boldsymbol{\xi}_1$ since each respective equation represents only one set of data. The double hat in (7.24a)

denotes the estimate of $\boldsymbol{\xi}_1$ based on both data sets. From (7.24b) we can write the system of normal equations in matrix form as follows:

$$
\begin{bmatrix} N_{11} & A_{21}^T \\ A_{21} & -P_2^{-1} \end{bmatrix} \begin{bmatrix} \hat{\hat{\boldsymbol{\xi}}}_1 \\ \hat{\hat{\boldsymbol{\lambda}}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0 \\ \boldsymbol{y}_2 - A_{22}\boldsymbol{\xi}_2^0 \end{bmatrix} . \tag{7.25}
$$

The solution of (7.25) can be obtained by applying the inversion formula for a partitioned matrix as shown in (A.15), resulting in

$$
\begin{bmatrix} \hat{\hat{\boldsymbol{\xi}}}_1 \\ \hat{\hat{\boldsymbol{\lambda}}} \end{bmatrix} = \begin{bmatrix} N_{11} & A_{21}^T \\ A_{21} & -P_2^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0 \\ \boldsymbol{y}_2 - A_{22}\boldsymbol{\xi}_2^0 \end{bmatrix} =
$$
$$
= \left[\begin{array}{c|c} N_{11}^{-1} - N_{11}^{-1} A_{21}^T S_2^{-1} A_{21} N_{11}^{-1} & N_{11}^{-1} A_{21}^T S_2^{-1} \\ \hline S_2^{-1} A_{21} N_{11}^{-1} & -S_2^{-1} \end{array} \right] \begin{bmatrix} \boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0 \\ \boldsymbol{y}_2 - A_{22}\boldsymbol{\xi}_2^0 \end{bmatrix} , \tag{7.26}
$$

where

$$
S_2 := P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T. \tag{7.27}
$$

Finally, the estimated parameters and Lagrange multipliers are expressed as

$$
\hat{\hat{\boldsymbol{\xi}}}_1 = N_{11}^{-1}\big(\boldsymbol{c}_1 - N_{12}\boldsymbol{\xi}_2^0\big) +
$$
$$
+ N_{11}^{-1}A_{21}^T\big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}\big[A_{21}N_{11}^{-1}\big(-\boldsymbol{c}_1 + N_{12}\boldsymbol{\xi}_2^0\big) + \boldsymbol{y}_2 - A_{22}\boldsymbol{\xi}_2^0\big] \Rightarrow \tag{7.28a}
$$

$$
\boxed{\hat{\hat{\boldsymbol{\xi}}}_1 = \hat{\boldsymbol{\xi}}_1 + N_{11}^{-1}A_{21}^T\big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}\big(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1 - A_{22}\boldsymbol{\xi}_2^0\big),} \tag{7.28b}
$$

$$
\hat{\hat{\boldsymbol{\lambda}}} = -\big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}\big(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1 - A_{22}\boldsymbol{\xi}_2^0\big). \tag{7.28c}
$$

The dispersion matrix of the estimated vector of Lagrange multipliers is

$$
D\{\hat{\hat{\boldsymbol{\lambda}}}\} = \big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}D\{\boldsymbol{y} - A_{21}\hat{\boldsymbol{\xi}}_1\}\big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}, \tag{7.29}
$$

since $D\{\boldsymbol{\xi}_2^0\} = 0$. The following relations also hold:

$$
C\{\boldsymbol{y}_2, \hat{\boldsymbol{\xi}}_1\} = 0, \tag{7.30a}
$$
$$
D\{\boldsymbol{y} - A_{21}\hat{\boldsymbol{\xi}}_1\} = \sigma_0^2\big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big), \tag{7.30b}
$$
$$
D\{\hat{\hat{\boldsymbol{\lambda}}}\} = \sigma_0^2\big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}, \tag{7.30c}
$$
$$
D\{\hat{\hat{\boldsymbol{\xi}}}_1\} = D\{\hat{\boldsymbol{\xi}}_1\} - \sigma_0^2 N_{11}^{-1}A_{21}^T\big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}A_{21}N_{11}^{-1}. \tag{7.30d}
$$

The estimated variance component is expressed as follows:

$$
\hat{\sigma}_0^2(n - q) = \hat{\sigma}_0^2(n_1 - q_1) + \big(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1 - A_{22}\boldsymbol{\xi}_2^0\big)^T \cdot
$$
$$
\cdot \big(P_2^{-1} + A_{21}N_{11}^{-1}A_{21}^T\big)^{-1}\big(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1 - A_{22}\boldsymbol{\xi}_2^0\big) \Rightarrow \tag{7.31a}
$$

$$\boxed{\hat{\hat{\sigma}}_0^2(n - q) = \hat{\sigma}_0^2(n_1 - q_1) - \hat{\hat{\boldsymbol{\lambda}}}^T(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1 - A_{22}\boldsymbol{\xi}_2^0).}$$ (7.31b)

Once again, we note that we have used the definition $N_{11} \coloneqq A_{11}^T P_1 A_{11}$ in this section.

7.5 Sequential Adjustment with New Parameters

In this section we consider the case where the second data set refers to at least some of the parameters involved in the first data set plus some additional new parameters that were not involved in the first data set. Thus we speak of m_1 parameters associated with the first data set and an additional m_2 introduced with the second data set, so that the total number of parameters involved in the combination of both data sets is given by $m = m_1 + m_2$. In the double subscripts used below, the first one refers to the data set, while the second one refers to the matrix splitting. For example, A_{21} is that part of the design matrix from the second data set that refers to the original parameters, whereas A_{22} is associated with the new parameters involved in the second data set. We could have adopted a new symbol to denote a "preprocessed observation" vector that includes datum information, e.g. $\bar{\boldsymbol{y}}$. However, we have elected to continue using $\boldsymbol{y}$ and simply note that it could include datum information in addition to the observations. The data model that follows implies that we have assumed there are no correlations between the observations of data-set 1 and those of data-set 2; it also implies that both sets of observations share a common variance component σ_0^2.

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{bmatrix} \sim (\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{bmatrix})$$ (7.32)

The size of the system of equations is implied by the following:

$$\boldsymbol{y}_1 \in \mathbb{R}^{n_1}, \;\; \boldsymbol{y}_2 \in \mathbb{R}^{n_2}, \;\; \boldsymbol{\xi}_1 \in \mathbb{R}^{m_1}, \;\; \boldsymbol{\xi}_2 \in \mathbb{R}^{m_2}, \;\; [\boldsymbol{\xi}_1^T, \boldsymbol{\xi}_2^T]^T \in \mathbb{R}^m,$$ (7.33a)

$$n = n_1 + n_2, \;\; m = m_1 + m_2.$$ (7.33b)

Now, using the *addition theory of normal equations*, stated in Section 6.1.1, we can write

$$\begin{bmatrix} A_{11}^T & A_{21}^T \\ 0 & A_{22}^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{\hat{\boldsymbol{\xi}}}_1 \\ \hat{\hat{\boldsymbol{\xi}}}_2 \end{bmatrix} = \begin{bmatrix} A_{11}^T P_1 & A_{21}^T P_2 \\ 0 & A_{22}^T P_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} \Rightarrow$$ (7.34a)

$$\left[\begin{array}{c|c} A_{11}^T P_1 A_{11} + A_{21}^T P_2 A_{21} & A_{21}^T P_2 A_{22} \\ \hline A_{22}^T P_2 A_{21} & A_{22}^T P_2 A_{22} \end{array} \right] \begin{bmatrix} \hat{\hat{\boldsymbol{\xi}}}_1 \\ \hat{\hat{\boldsymbol{\xi}}}_2 \end{bmatrix} = \begin{bmatrix} A_{11}^T P_1 \boldsymbol{y}_1 + A_{21}^T P_2 \boldsymbol{y}_2 \\ A_{22}^T P_2 \boldsymbol{y}_2 \end{bmatrix}.$$ (7.34b)

Here again, the double-hats refer to estimates based on both data sets.

Now, the first data set may no longer be available, rather we may have only the estimates from the first adjustment. In this case we can use the bottom row of

(7.34b) to solve for the estimates of the new parameters in terms of only the second set of observations, leading to

$$\boxed{\hat{\hat{\boldsymbol{\xi}}}_2 = \left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2 \left(\boldsymbol{y}_2 - A_{21}\hat{\hat{\boldsymbol{\xi}}}_1\right).} \tag{7.35}$$

Then, from the normal equations based solely on the first data set, we may substitute

$$A_{11}^T P_1 \boldsymbol{y}_1 = \left(A_{11}^T P_1 A_{11}\right)\hat{\boldsymbol{\xi}}_1 \tag{7.36}$$

into the top row of the right side of (7.34b) and invert the normal-equation matrix on the left to solve for the parameter estimates. For convenience, we introduce the following symbols to use in the inverted matrix:

$$S_1 := A_{11}^T P_1 A_{11} + A_{21}^T P_2 A_{21} - A_{21}^T P_2 A_{22}\left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2 A_{21} = \tag{7.37a}$$

$$= A_{11}^T P_1 A_{11} + A_{21}^T \bar{P}_2 A_{21}, \tag{7.37b}$$

$$\bar{P}_2 := P_2 - P_2 A_{22}\left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2, \tag{7.37c}$$

$$N_{22} = A_{22}^T P_2 A_{22}. \tag{7.37d}$$

We refer to $\bar{P}_2$ as a *reduced weight matrix*. Upon inverting the normal-equations matrix from (7.34b) (see (A.15) for the inverse of a partitioned matrix), we find the following solution for $\hat{\hat{\boldsymbol{\xi}}}_1$ and $\hat{\hat{\boldsymbol{\xi}}}_2$:

$$\begin{bmatrix} \hat{\hat{\boldsymbol{\xi}}}_1 \\ \hat{\hat{\boldsymbol{\xi}}}_2 \end{bmatrix} = \left[\begin{array}{c|c} S_1^{-1} & -S_1^{-1}\left(A_{21}^T P_2 A_{22}\right)N_{22}^{-1} \\ \hline -N_{22}^{-1}\left(A_{22}^T P_2 A_{21}\right)S_1^{-1} & N_{22}^{-1} + N_{22}^{-1}\left(A_{22}^T P_2 A_{21}\right)S_1^{-1}\left(A_{21}^T P_2 A_{22}\right)N_{22}^{-1} \end{array} \right] \cdot$$

$$\cdot \begin{bmatrix} \left(A_{11}^T P_1 A_{11}\right)\hat{\boldsymbol{\xi}}_1 + A_{21}^T P_2 \boldsymbol{y}_2 \\ A_{22}^T P_2 \boldsymbol{y}_2 \end{bmatrix}. \tag{7.38}$$

We can continue by using (7.37b) and (7.37c) with the first row of (7.38) to arrive at

$$\hat{\hat{\boldsymbol{\xi}}}_1 = S_1^{-1}\left[\left(A_{11}^T P_1 A_{11}\right)\hat{\boldsymbol{\xi}}_1 + A_{21}^T P_2 \boldsymbol{y}_2 - \left(A_{21}^T P_2 A_{22}\right)N_{22}^{-1} A_{22}^T P_2 \boldsymbol{y}_2\right] = \tag{7.39a}$$

$$= S_1^{-1}\left\{\left[\left(A_{11}^T P_1 A_{11}\right)\hat{\boldsymbol{\xi}}_1 + A_{21}^T \bar{P}_2 \boldsymbol{y}_2\right] + \left[\left(A_{21}^T \bar{P}_2 A_{21}\right) - \left(A_{21}^T \bar{P}_2 A_{21}\right)\right]\hat{\boldsymbol{\xi}}_1\right\} = \tag{7.39b}$$

$$= S_1^{-1} A_{21}^T \bar{P}_2 \left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right) + S_1^{-1}\left(A_{11}^T P_1 A_{11} + A_{21}^T \bar{P}_2 A_{21}\right)\hat{\boldsymbol{\xi}}_1 = \tag{7.39c}$$

$$= S_1^{-1} A_{21}^T \bar{P}_2 \left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right) + \hat{\boldsymbol{\xi}}_1 \Rightarrow \tag{7.39d}$$

$$\boxed{\hat{\hat{\boldsymbol{\xi}}}_1 - \hat{\boldsymbol{\xi}}_1 = S_1^{-1} A_{21}^T \bar{P}_2 \left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right),} \tag{7.39e}$$

where (7.39e) is in the form of an *update formula*.

We assume that P_2 is invertible, as implied in the given model (7.32). We now wish to check the rank of the reduced weight matrix $\bar{P}_2$. It is easy to check that the product $P_2^{-1}\bar{P}_2$ is idempotent. Then using (1.7c) and (A.4) we find

$$\operatorname{rk}\bar{P}_2 = \operatorname{rk}\left(P_2^{-1}\bar{P}_2\right) = \operatorname{tr}\left(P_2^{-1}\bar{P}_2\right) = \operatorname{tr}\left(I_{n_2} - A_{22}\left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2\right) = \tag{7.40a}$$

$$= n_2 - \mathrm{tr}\big[A_{22}\big(A_{22}^T P_2 A_{22}\big)^{-1} A_{22}^T P_2\big] = n_2 - \mathrm{tr}\big[\big(A_{22}^T P_2 A_{22}\big)^{-1} A_{22}^T P_2 A_{22}\big] = \tag{7.40b}$$

$$= n_2 - m_2 < n_2. \tag{7.40c}$$

Thus there is a rank reduction that comes from modifying the original weight matrix P_2 to obtain $\bar{P}_2$. Moreover, we find that matrix $\bar{P}_2$ is singular.

The dispersion matrices for the estimated parameters, i.e., $D\{\hat{\hat{\boldsymbol{\xi}}}_1\}$ and $D\{\hat{\hat{\boldsymbol{\xi}}}_2\}$, are shown at the end of the next section.

7.6   Sequential Adjustment with New Parameters and Small Second Data Set

In (7.39e) we must invert the $m_1 \times m_1$ matrix S_1 to solve the system of equations. However, in some applications, the number of observations n_2 in the second data set may be significantly less than m_1. In this case we would like to reformulate the solution in (7.39e) so that only a matrix of size $n_2 \times n_2$ needs to be inverted.

We have an alternative expression for matrix S_1 in (7.37b), the inverse of which can be derived as follows:

$$S_1^{-1} = \big[\big(A_{11}^T P_1 A_{11}\big) + \big(A_{21}^T \bar{P}_2 A_{21}\big)\big]^{-1} = \tag{7.41a}$$

$$= \big\{\big[I_{m_1} + \big(A_{21}^T \bar{P}_2 A_{21}\big)\big(A_{11}^T P_1 A_{11}\big)^{-1}\big]\big(A_{11}^T P_1 A_{11}\big)\big\}^{-1} = \tag{7.41b}$$

$$= \big(A_{11}^T P_1 A_{11}\big)^{-1}\big[I_{m_1} + \big(A_{21}^T \bar{P}_2 A_{21}\big)\big(A_{11}^T P_1 A_{11}\big)^{-1}\big]^{-1}. \tag{7.41c}$$

Using (7.41c), we may rewrite (7.39e) as

$$\boxed{\begin{aligned} \hat{\hat{\boldsymbol{\xi}}}_1 - \hat{\boldsymbol{\xi}}_1 = \\ = \big(A_{11}^T P_1 A_{11}\big)^{-1}\big[I_{m_1} + \big(A_{21}^T \bar{P}_2 A_{21}\big)\big(A_{11}^T P_1 A_{11}\big)^{-1}\big]^{-1} A_{21}^T \bar{P}_2 \big(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\big) = \\ = \big(A_{11}^T P_1 A_{11}\big)^{-1} A_{21}^T \bar{P}_2\big[I_{n_2} + A_{21}\big(A_{11}^T P_1 A_{11}\big)^{-1} A_{21}^T \bar{P}_2\big]^{-1}\big(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\big). \end{aligned}} \quad \begin{aligned} (7.42a) \\ (7.42b) \end{aligned}$$

Here, we have made use of (A.8a) in the step from (7.42a) to (7.42b), with matrices A and D in (A.8a) set to identity. Note that the matrix to invert inside the square brackets is of size $m_1 \times m_1$ in (7.42a) but is size $n_2 \times n_2$ in (7.42b). The choice of which equation to use will usually be determined by the smaller of m_1 and n_2. Also, we have the relation

$$-\hat{\boldsymbol{\lambda}} = \big[I_{n_2} + A_{21}\big(A_{11}^T P_1 A_{11}\big)^{-1} A_{21}^T \bar{P}_2\big]^{-1}\big(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\big), \tag{7.43}$$

which means that the solution for the first subset of parameters may also be expressed as

$$\boxed{\hat{\hat{\boldsymbol{\xi}}}_1 - \hat{\boldsymbol{\xi}}_1 = -\big(A_{11}^T P_1 A_{11}\big)^{-1} A_{21}^T \bar{P}_2 \hat{\boldsymbol{\lambda}}.} \tag{7.44}$$

Now we begin with (7.35), and substitute (7.42b), to find a solution for the parameters $\hat{\hat{\boldsymbol{\xi}}}_2$ in terms of the Lagrange multipliers $\hat{\hat{\boldsymbol{\lambda}}}$:

$$\hat{\hat{\boldsymbol{\xi}}}_2 = \left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2 \left(\boldsymbol{y}_2 - A_{21}\hat{\hat{\boldsymbol{\xi}}}_1\right) = \tag{7.45a}$$

$$= \left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2 \cdot \left\{\left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right) - A_{21}\left(A_{11}^T P_1 A_{11}\right)^{-1} A_{21}^T \bar{P}_2 \cdot \right.$$
$$\left. \cdot \left[I_{n_2} + A_{21}\left(A_{11}^T P_1 A_{11}\right)^{-1} A_{21}^T \bar{P}_2\right]^{-1}\left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right)\right\} = \tag{7.45b}$$

$$= \left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2 \left[I_{n_2} + A_{21}\left(A_{11}^T P_1 A_{11}\right)^{-1} A_{21}^T \bar{P}_2\right]^{-1}\left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right) \tag{7.45c}$$

$$\boxed{\Rightarrow \hat{\hat{\boldsymbol{\xi}}}_2 = -\left(A_{22}^T P_2 A_{22}\right)^{-1} A_{22}^T P_2 \hat{\hat{\boldsymbol{\lambda}}}.} \tag{7.45d}$$

The inverse formula of (A.6a) was used to go from (7.45b) to (7.45c), with matrices T, W, and V in (A.6a) set to identity matrices of appropriate sizes.

To facilitate computing the parameter dispersion matrix we write the following system of normal equations, noting that (7.46b) is in the form of an update solution (cf. (7.34b) and (7.36)):

$$\left[\begin{array}{c|c} A_{11}^T P_1 A_{11} + A_{21}^T P_2 A_{21} & A_{21}^T P_2 A_{22} \\ \hline A_{22}^T P_2 A_{21} & A_{22}^T P_2 A_{22} \end{array}\right]\left[\begin{array}{c} \hat{\hat{\boldsymbol{\xi}}}_1 \\ \hat{\hat{\boldsymbol{\xi}}}_2 \end{array}\right] = \left[\begin{array}{c} \left(A_{11}^T P_1 A_{11}\right)\hat{\boldsymbol{\xi}}_1 + A_{21}^T P_2 \boldsymbol{y}_2 \\ A_{22}^T P_2 \boldsymbol{y}_2 \end{array}\right], \tag{7.46a}$$

or

$$\left[\begin{array}{c|c} A_{11}^T P_1 A_{11} + A_{21}^T P_2 A_{21} & A_{21}^T P_2 A_{22} \\ \hline A_{22}^T P_2 A_{21} & A_{22}^T P_2 A_{22} \end{array}\right]\left[\begin{array}{c} \hat{\hat{\boldsymbol{\xi}}}_1 - \hat{\boldsymbol{\xi}}_1 \\ \hat{\hat{\boldsymbol{\xi}}}_2 \end{array}\right] = \left[\begin{array}{c} A_{21}^T P_2 \left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right) \\ A_{22}^T P_2 \left(\boldsymbol{y}_2 - A_{21}\hat{\boldsymbol{\xi}}_1\right) \end{array}\right]. \tag{7.46b}$$

Note that (7.46a) is equivalent to (7.34b) shown earlier.

We have already inverted the normal-equation matrix in (7.38). Taking elements from (7.38), we may write the parameter dispersion and covariance matrices as follows:

$$\boxed{\begin{aligned} D\{\hat{\hat{\boldsymbol{\xi}}}_1\} &= \sigma_0^2 S_1^{-1} = \sigma_0^2 \left(A_{11}^T P_1 A_{11} + A_{21}^T \bar{P}_2 A_{21}\right)^{-1}, & (7.47a) \\ C\{\hat{\hat{\boldsymbol{\xi}}}_1, \hat{\hat{\boldsymbol{\xi}}}_2\} &= -D\{\hat{\hat{\boldsymbol{\xi}}}_1\}\left(A_{21}^T P_2 A_{22}\right)\left(A_{22}^T P_2 A_{22}\right)^{-1}, & (7.47b) \\ D\{\hat{\hat{\boldsymbol{\xi}}}_2\} &= \sigma_0^2 \left(A_{22}^T P_2 A_{22}\right)^{-1} - \left(A_{22}^T P_2 A_{22}\right)^{-1}\left(A_{22}^T P_2 A_{21}\right)C\{\hat{\hat{\boldsymbol{\xi}}}_1, \hat{\hat{\boldsymbol{\xi}}}_2\}. & (7.47c) \end{aligned}}$$

Each of the above covariance matrices (7.47a) through (7.47c) include the matrix S_1^{-1}, which implies that a matrix of size $m_1 \times m_1$ must be inverted. However, with the insertion of I_{n_2} into (7.47a), and with appropriate matrix groupings, we may apply the inversion formula (A.6a) to find an inverse of smaller dimension as shown in the following:

$$D\{\hat{\hat{\boldsymbol{\xi}}}_1\} = \sigma_0^2 \left[\left(A_{11}^T P_1 A_{11}\right) + \left(A_{21}^T \bar{P}_2\right)I_{n_2} A_{21}\right]^{-1} = \tag{7.48a}$$

$$= \sigma_0^2 N_{11}^{-1} - \sigma_0^2 N_{11}^{-1} A_{21}^T \bar{P}_2 \left(I_{n_2} + A_{21} N_{11}^{-1} A_{21}^T \bar{P}_2 \right)^{-1} A_{21} N_{11}^{-1}. \qquad (7.48\text{b})$$

Here again, we have used $N_{11} := A_{11}^T P_1 A_{11}$ for compactness. The parenthetical term that must be inverted in equation (7.48b) is an $n_2 \times n_2$ matrix, which, again, may be much smaller than an $m_1 \times m_1$ matrix, depending on the application. Of course, the matrix $(A_{11}^T P_1 A_{11})^{-1}$ is also size $m_1 \times m_1$, but it is assumed that this inverse had already been performed in the adjustment of the first data set and was saved for subsequent use.

The estimated variance component is expressed as

$$\hat{\sigma}_0^2 (n - m) = \hat{\sigma}_0^2 (n_1 - m_1) - \left(\boldsymbol{y}_2 - A_{21} \hat{\boldsymbol{\xi}}_1 \right)^T \bar{P}_2 \hat{\hat{\boldsymbol{\lambda}}}. \qquad (7.49\text{a})$$

Then, substituting (7.43) results in

$$\hat{\sigma}_0^2 (n - m) = \hat{\sigma}_0^2 (n_1 - m_1) +$$
$$+ \left(\boldsymbol{y}_2 - A_{21} \hat{\boldsymbol{\xi}}_1 \right)^T \bar{P}_2 \left[I_{n_2} + A_{21} \left(A_{11}^T P_1 A_{11} \right)^{-1} A_{21}^T \bar{P}_2 \right]^{-1} \left(\boldsymbol{y}_2 - A_{21} \hat{\boldsymbol{\xi}}_1 \right). \quad (7.49\text{b})$$

7.7   Practice Problems

1. Considering Problem 9 of Section 3.6, assume that a second observation campaign has been conducted, where the original observation scheme was repeated, except that the final three observations from the first campaign were not repeated in the second one. Both data sets are listed in Table 7.1, and a diagram of the leveling network is shown in Figure 3.5. Furthermore, assume that the weight of each observation in both data sets is defined as the distance in miles associated with the observation divided by 100. Introduce datum information so that the height of point D is fixed at $1928.277\,\text{ft}$.

   (a) Compute estimates for $\hat{\boldsymbol{\xi}}$, along with its cofactor matrix, and the estimated variance component $\hat{\sigma}_0^2$ based only on the first data set.

   (b) Using the results of the previous step, compute estimates for $\hat{\hat{\boldsymbol{\xi}}}$, $D\{\hat{\hat{\boldsymbol{\xi}}}\}$, and the estimated variance component $\hat{\hat{\sigma}}_0^2$ using update formulas that do not directly depend on the observations from the first data set.

Table 7.1: Leveling data from Rainsford (1968) as $\boldsymbol{y}_I$ and simulated second data set as $\boldsymbol{y}_{II}$. d stands for distance between stations.

From	To	No.	$\boldsymbol{y}_I$ [ft]	$\boldsymbol{y}_{II}$, [ft]	d [miles]
A	B	1	+124.632	+124.659	68
B	C	2	+217.168	+217.260	40
C	D	3	−92.791	−92.904	56
A	D	4	+248.754	+248.797	171
A	F	5	−11.418	−11.402	76
F	E	6	−161.107	−161.172	105
E	D	7	+421.234		80
B	F	8	−135.876		42
C	E	9	−513.895		66

2. Now consider the case where one new station, G, was added to the network during the second observation campaign as depicted in Figure 7.1. The data for the first observation campaign can be taken from Table 7.1. The data from the second observation campaign are listed in Table 7.2.

   Use (7.39e) followed by (7.35) to compute $\hat{\hat{\boldsymbol{\xi}}}_1$ and $\hat{\hat{\boldsymbol{\xi}}}_2$, respectively, or, instead, use (7.42b) for $\hat{\hat{\boldsymbol{\xi}}}_1$.

   Hint: Because of the network datum deficiency of one, you can modify the observation vector from the second data set by subtracting out the datum value of 1928.277 ft from observations y_3 and y_7 and then remove the parameter for the height of station D from the parameter vector. You may check your answers by combining both data sets into one, and then solve for the unknown parameters according to Section 3.5.
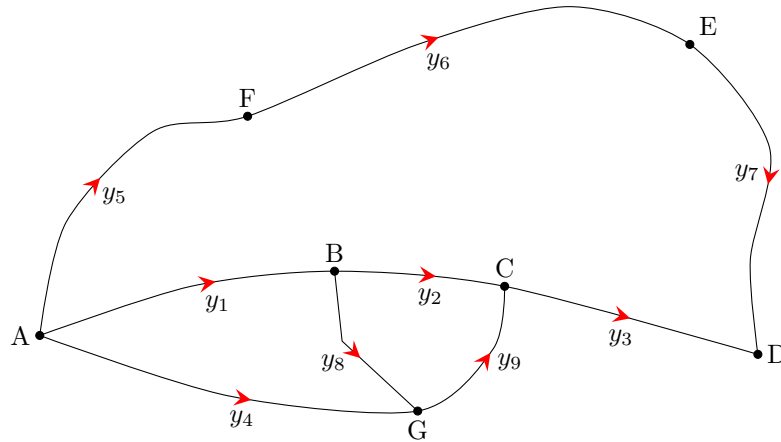
Figure 7.1: Simulated extension of a leveling network by Rainsford (1968)

Table 7.2: Leveling data for simulated second data set as $\boldsymbol{y}_{II}$. d stands for distance between stations.

From	To	No.	$\boldsymbol{y}_{II}$, [ft]	d [miles]
A	B	1	$+124.659$	68
B	C	2	$+217.260$	40
C	D	3	-92.904	56
A	G	4	$+178.852$	85
A	F	5	-11.402	76
F	E	6	-161.172	105
E	D	7	$+421.212$	80
B	G	8	$+54.113$	45
G	C	9	$+162.992$	45

Chapter 8

Condition Equations with Parameters: the Gauss-Helmert Model

8.1 Model Definition

Data models introduced prior to this chapter have either admitted observation equations with unknown parameters *or* condition equations without parameters, but not both. In contrast, the Gauss-Helmert Model (GHM) allows both condition equations *and* equations involving unknown parameters to be combined in the same model. Thus, the GHM can be viewed as being more flexible (or more general) than either the Gauss-Markov Model (GMM) (Chapter 3) or the Model of Condition Equations (Chapter 4), since it combines aspects of both. In some cases, the GHM might be useful for dealing with complicated observation equations, for example when multiple observations are related functionally to one or more parameters via specified (possibly nonlinear) equations.

In other cases, the LEast-Squares Solution (LESS) within the GHM is equivalent to that of orthogonal regression, or, more generally, to a total least-squares (TLS) solution. Such solutions are sought within models that have both independent and dependent random data variables. Examples are line and curve fitting in 2D when both x- and y-coordinates are measured. Coordinate transformation problems also fall in this category when the coordinates from both the source and target systems are measured quantities. We will learn how to treat these problems in this chapter.

8.1.1 An example Gauss-Helmert Model

We begin our discussion of the GHM with a leveling-network example in order to contrast the GMM with the Model of Condition Equations and to show how the GHM combines the information used in those two models. The diagram in Figure 8.1 shows a leveling network with four points (P_1, P_2, P_3, P_4) that has been observed

in two closed loops comprised of a total of five observations $(y_1, y_2, y_3, y_4, y_5)$. First we present a (rank deficient) partitioned GMM as

$$\boldsymbol{y} = A_1\boldsymbol{\xi}_1 + A_2\boldsymbol{\xi}_2 + \boldsymbol{e}, \tag{8.1a}$$

$$\boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 P^{-1}\right), \tag{8.1b}$$

$$\operatorname{rk} A_1 = \operatorname{rk}\left[A_1 \mid A_2\right] =: q < m, \tag{8.1c}$$

where the coefficient matrix A and the vector of unknown parameters $\boldsymbol{\xi}$ have been partitioned, respectively, as

$$A = \left[\begin{array}{c|c} A_1 \\ n \times q \end{array}\middle|\begin{array}{c} A_2 \\ n \times (m-q) \end{array}\right] \text{ and } \boldsymbol{\xi} = \left[\begin{array}{c|c} \boldsymbol{\xi}_1^T \\ 1 \times q \end{array}\middle|\begin{array}{c} \boldsymbol{\xi}_2^T \\ 1 \times (m-q) \end{array}\right]^T. \tag{8.2}$$
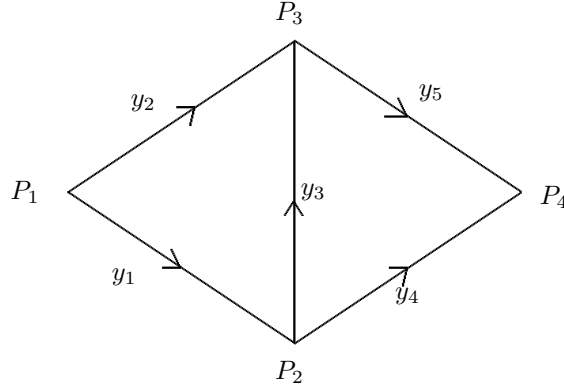
Figure 8.1: Leveling network. Arrows point in the direction of the level runs.

In this example, the number of unknown parameters is $m = 4$ (heights of four points). Since leveled height-differences supply no information about the height datum, we can only estimate the heights of three of the points with respect to the remaining fourth one. That explains why $\operatorname{rk} A =: q = 3 < m$, implying a *datum deficiency* of $m - q = 1$. Thus, the model has been partitioned so that $\boldsymbol{\xi}_1$ contains three estimable heights, and $\boldsymbol{\xi}_2$ is a single non-estimable height, which must be assigned a "datum value." In this example, we arbitrarily chose point P_4 for the non-estimable height. As was stated in Section 3.5, we have the relationship $A_2 = A_1 L$ for some $q \times (m - q)$ matrix L, which means that matrix A_2 is a linear combination of the columns of matrix A_1, reflecting the rank deficiency of matrix $A = [A_1 \mid A_2]$.

The problem could also be solved within the Model of Condition Equations introduced in Chapter 4, which reads

$$B\boldsymbol{y} = B\boldsymbol{e}, \quad \boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 P^{-1}\right), \tag{8.3a}$$

with the orthogonality condition

$$\text{i. } B \cdot \left[A_1 \mid A_2\right] = 0, \tag{8.3b}$$

and the rank condition

$$\text{ii. } \operatorname{rk} B = r = n - \operatorname{rk} A_1. \tag{8.3c}$$

These two conditions ensure equivalent least-squares solutions within the models of (8.1) and (8.3a) as discussed in Section 4.3.

We have the following design (coefficient) matrices and parameter vectors for the example leveling network, for which it is easy to verify that both conditions i and ii are satisfied:

$$A_1 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \mid A_2 \end{bmatrix},$$

$$\tag{8.4a}$$

$$B = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix}, \quad \boldsymbol{\xi}_1 = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}, \quad \boldsymbol{\xi}_2 = \begin{bmatrix} h_4 \end{bmatrix},$$

with

$$q := \operatorname{rk} A_1 = \operatorname{rk} A = 3, \quad r := \operatorname{rk} B = 2 = n - \operatorname{rk} A_1 = 5 - 3 = 2, \text{ and } B{\cdot}A = 0. \tag{8.4b}$$

Here, h_i represents the height of point P_i.

Now we wish to introduce a new coefficient matrix B that does not contain matrix A in its nullspace, so that we can form a Model of Condition Equations with parameters. For now we use the symbol $\bar{B}$ in order to distinguish it from the coefficient matrix B used in the Model of Condition Equations, which does contain matrix A in its nullspace (i.e., $BA = 0$, but $\bar{B}A \neq 0$). Similarly, we introduce other bar-terms to form the following GHM:

$$\bar{\boldsymbol{y}} = \bar{B}\boldsymbol{y} = \bar{\boldsymbol{w}} = \bar{B}A_1\boldsymbol{\xi}_1 + \bar{B}A_2\boldsymbol{\xi}_2 + \bar{B}\boldsymbol{e}, \tag{8.5a}$$

$$\bar{B}\boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 \bar{B}P^{-1}\bar{B}^T\right), \tag{8.5b}$$

$$\operatorname{rk}\left(\bar{B}\right) =: \bar{r}. \tag{8.5c}$$

The size of $\bar{B}$ is $\bar{r} \times n$, implying that $\bar{B}$ has full row rank. The GHM in (8.5) is equivalent to the GMM in (8.1) if, and only if,

iii. $\bar{B}A_1$ has $n - \bar{r}$ columns of zeros, and

iv. $\operatorname{rk}(\bar{B}A_1) + r = \bar{r} \Leftrightarrow n = \bar{r} + q - \operatorname{rk}(\bar{B}A_1) = \operatorname{rk}\bar{B} + \operatorname{rk}A - \operatorname{rk}(\bar{B}A_1)$

Note that, through the matrix $\bar{B}$, one observation is eliminated for each eliminated parameter. Referring to the level network example, we may wish to eliminate the height of point P_3 from the parameter list (perhaps it is a temporary benchmark

of no particular interest). This can be done by introducing the following example matrix $\bar{B}$:

$$\bar{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \bar{B}A_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{B}A_1 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

With these example matrices we have $n = 5$, $r = 2$, $\bar{r} = \operatorname{rk}\bar{B} = 4$, $q = \operatorname{rk}A_1 = 3$, and $\operatorname{rk}(\bar{B}A_1) = 2$. Since $n - \bar{r} = 1$, the single column of zeros in $\bar{B}A_1$ satisfies condition iii. Also, condition iv is satisfied since $n = 5 = \operatorname{rk}\bar{B} + \operatorname{rk}A - \operatorname{rk}(\bar{B}A_1) = 4 + 3 - 2$.

As an aside, we note that it is also possible to remove l estimable parameters via the splitting of the constraint equation introduced in (5.1), i.e.

$$\boldsymbol{\kappa}_0 = \underset{l \times m}{K} \boldsymbol{\xi} = \begin{bmatrix} K_1, & K_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} \Rightarrow \tag{8.6a}$$

$$\boldsymbol{\xi}_1 = K_1^{-1}\boldsymbol{\kappa}_0 - K_1^{-1}K_2\boldsymbol{\xi}_2. \tag{8.6b}$$

Here, K_1 is a $l \times l$ invertible matrix, and K_2 is of size $l \times (m - l)$. Upon substitution for $\boldsymbol{\xi}_1$ of (8.6b) into (8.1), we find the following modified system of observation equations with l parameters eliminated:

$$\boldsymbol{y} = A_1\boldsymbol{\xi}_1 + A_2\boldsymbol{\xi}_2 + \boldsymbol{e} = A_1K_1^{-1}\boldsymbol{\kappa}_0 + (A_2 - A_1K_1^{-1}K_2)\boldsymbol{\xi}_2 + \boldsymbol{e}. \tag{8.7}$$

The $l \times 1$ vector $\boldsymbol{\xi}_1$ has vanished on the right side of (8.7). While this technique is possible, it might not be used frequently in practice.

8.2 Least-Squares Solution

We could derive the solution for $\boldsymbol{\xi}$ within the GHM (8.5) from statistical principles via BLUUE (Best Linear Uniformly Unbiased Estimate), but here we use the equivalent principle of LESS (LEast-Squares Solution) as was done in Sections 2.2, 3.2 and 4.3, etc. In the following, we recombine coefficient matrices A_1 and A_2 back into the single matrix A and recombine the partitioned parameter vector back into a single vector $\boldsymbol{\xi} = [\boldsymbol{\xi}_1^T, \boldsymbol{\xi}_2^T]^T$. Accordingly, we can rewrite (8.5) as

$$\bar{\boldsymbol{w}} = \bar{B}A_1\boldsymbol{\xi}_1 + \bar{B}A_2\boldsymbol{\xi}_2 + \bar{B}\boldsymbol{e} = \bar{A}\boldsymbol{\xi} + \bar{B}\boldsymbol{e}, \tag{8.8}$$

where another bar-symbol was introduced for convenience, viz. $\bar{A} := \bar{B}A$.

Our target function should minimize a quadratic form in the random error vector $\boldsymbol{e}$ itself, rather than $B\boldsymbol{e}$; i.e., we minimize $\boldsymbol{e}^T P\boldsymbol{e}$. Then, the Lagrange target function is written as

$$\Phi(\boldsymbol{e}, \boldsymbol{\xi}, \boldsymbol{\lambda}) =: \boldsymbol{e}^T P\boldsymbol{e} + 2\boldsymbol{\lambda}^T(\bar{B}\boldsymbol{e} + \bar{A}\boldsymbol{\xi} - \bar{\boldsymbol{w}}), \tag{8.9}$$

which must be made stationary with respect to the unknown vectors $\boldsymbol{e}$, $\boldsymbol{\xi}$, and $\boldsymbol{\lambda}$. This is done by imposing the Euler-Lagrange necessary conditions, which results in the following system of equations:

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{e}} = P\tilde{\boldsymbol{e}} + \bar{B}^T\hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \tag{8.10a}$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\xi}} = \bar{A}^T\hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \tag{8.10b}$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\lambda}} = \bar{B}\tilde{\boldsymbol{e}} + \bar{A}\hat{\boldsymbol{\xi}} - \bar{\boldsymbol{w}} \doteq \boldsymbol{0}. \tag{8.10c}$$

The vectors of predicted random errors (residuals) and estimated parameters are then solved for as follows:

$$\tilde{\boldsymbol{e}} = -\big(P^{-1}\bar{B}^T\big)\hat{\boldsymbol{\lambda}} \Rightarrow \qquad\qquad \text{from equation (8.10a)}$$

$$-\big(\bar{B}P^{-1}\bar{B}^T\big)\hat{\boldsymbol{\lambda}} = \bar{\boldsymbol{w}} - \bar{A}\hat{\boldsymbol{\xi}} \Rightarrow \qquad \text{multiplying by } \bar{B} \text{ and using (8.10c)}$$

$$-\hat{\boldsymbol{\lambda}} = \big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\big(\bar{\boldsymbol{w}} - \bar{A}\hat{\boldsymbol{\xi}}\big) \Rightarrow \qquad \big(\bar{B}P^{-1}\bar{B}^T\big) \text{ is invertible}$$

$$-\bar{A}^T\hat{\boldsymbol{\lambda}} = \bar{A}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\big(\bar{\boldsymbol{w}} - \bar{A}\hat{\boldsymbol{\xi}}\big) = \boldsymbol{0} \Rightarrow \quad \text{mult. by } \bar{A}^T \text{ and using (8.10b)}$$

$$\bar{A}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\bar{A}\hat{\boldsymbol{\xi}} = \bar{A}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\bar{\boldsymbol{w}}$$

Finally, we arrive at

$$\boxed{\hat{\boldsymbol{\xi}} = \big[\bar{A}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\bar{A}\big]^{-1}\bar{A}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\bar{\boldsymbol{w}}} \tag{8.11a}$$

and

$$\boxed{\tilde{\boldsymbol{e}} = P^{-1}\bar{B}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\big(\bar{\boldsymbol{w}} - \bar{A}\hat{\boldsymbol{\xi}}\big)} \tag{8.11b}$$

for the estimated parameters and predicted residuals, respectively. Equation (8.11a) has the same form as the LESS derived within the GMM in Section 3.2, and (8.11b) looks much like formula (4.5d) for the residual vector within the Model of Condition Equations.

Note that matrix $\bar{A}$ would need to have full column rank to use (8.11a). Thus, in the example problem in the preceding section, the datum deficiency would need to be handled first. This could be done, for example, by modifying ("pre-processing") the observation vector as mentioned in Section 7.5 and in Problem 2 of Section 7.7. In the remainder of this chapter, we will assume that there are no rank deficiencies in the the data models.

The dispersion matrix for the estimated parameter vector $\hat{\boldsymbol{\xi}}$ is expressed by

$$\boxed{D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2\big[\bar{A}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\bar{A}\big]^{-1}.} \tag{8.12}$$

And the dispersion matrix for the residual vector reads

$$\boxed{\begin{aligned}D\{\tilde{\boldsymbol{e}}\} = P^{-1}\bar{B}^T\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\big[\bar{B}\cdot D\{\boldsymbol{e}\}\cdot\bar{B}^T - \\ -\bar{A}\cdot D\{\hat{\boldsymbol{\xi}}\}\cdot\bar{A}^T\big]\big(\bar{B}P^{-1}\bar{B}^T\big)^{-1}\bar{B}P^{-1},\end{aligned}} \tag{8.13}$$

with $D\{\boldsymbol{e}\} = \sigma_0^2 P^{-1}$ as stated in the model (8.1).

Notation change:　For the remainder of the chapter we drop the bars from the symbols as a matter of convenience. Recall that the bars were introduced in the first place to distinguish between the matrix B introduced in (8.5) and that used in Chapter 4 for the Model of Condition Equations. Dropping the bars means that $\bar{B} \to B$, $\bar{\boldsymbol{w}} \to \boldsymbol{w}$, $\bar{A} \to BA$.

We make one more notation change by replacing the matrix product BA that was used in the derivations above with the symbol A itself to represents a more general form of the model. Recall that the matrix BA above included the coefficient matrix A from a Gauss-Markov Model (GMM). However the more general formulation of the least-squares adjustment within the GHM would not necessarily reference quantities used in a GMM.

With these simplified notations, we rewrite the solution (8.11a) as follows:

$$\hat{\boldsymbol{\xi}} = \left[A^T \left(BP^{-1}B^T\right)^{-1} A\right]^{-1} A^T \left(BP^{-1}B^T\right)^{-1} \boldsymbol{w}. \qquad (8.14)$$

The dispersion of $\hat{\boldsymbol{\xi}}$ is derived in parts as follows:

$$D\{A^T\left(BP^{-1}B^T\right)^{-1}\boldsymbol{w}\} = A^T\left(BP^{-1}B^T\right)^{-1} D\{\boldsymbol{w}\}\left(BP^{-1}B^T\right)^{-1}A =$$
$$= A^T\left(BP^{-1}B^T\right)^{-1} B \cdot D\{\boldsymbol{y}\} \cdot B^T\left(BP^{-1}B^T\right)^{-1}A =$$
$$= \left(BP^{-1}B^T\right)^{-1}\left(\sigma_0^2 A^T BP^{-1}B^T\right)\left(BP^{-1}B^T\right)^{-1}A =$$
$$= \sigma_0^2 A^T\left(BP^{-1}B^T\right)^{-1}A;$$

therefore

$$D\{\hat{\boldsymbol{\xi}}\} = \left[A^T\left(BP^{-1}B^T\right)^{-1}A\right]^{-1} \cdot D\{A^T\left(BP^{-1}B^T\right)^{-1}\boldsymbol{w}\} \cdot \left[A^T\left(BP^{-1}B^T\right)^{-1}A\right]^{-1} =$$
$$= \left[A^T\left(BP^{-1}B^T\right)^{-1}A\right]^{-1}\left[\sigma_0^2 A^T\left(BP^{-1}B^T\right)^{-1}A\right]\left[A^T\left(BP^{-1}B^T\right)^{-1}A\right]^{-1},$$

finally resulting in

$$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2\left[A^T\left(BP^{-1}B^T\right)^{-1}A\right]^{-1}. \qquad (8.15)$$

8.3　Iteratively Linearized Gauss-Helmert Model

In this section we present the Gauss-Helmert Model (GHM) as an iteratively linearized model, showing how to form both the model and the least-squares solution within the model at each step of an iteration scheme. The developed algorithm is useful for a wide range of problems encountered in geodetic science and other disciplines.

The reader should be clear that the coefficient matrices A and B used in this section (and those that follow) are not the same as the coefficient matrices A and B used in the GMM and Model of Condition Equations, respectively. This should be obvious from the development that follows.

Suppose we are given a nonlinear functional model that relates n observations $\boldsymbol{y}$ to m unknown parameters $\boldsymbol{\Xi}$ among $m + r$ nonlinear condition equations $\boldsymbol{b}$ such

that

$$\boldsymbol{b}(\underbrace{\boldsymbol{y}-\boldsymbol{e}}_{n\times 1},\underbrace{\boldsymbol{\Xi}}_{m\times 1}) = \boldsymbol{0}, \ \ \boldsymbol{b} \in \mathbb{R}^{m+r}, \ \ \boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 P_{n\times n}^{-1}), \tag{8.16}$$

representing also a mapping $\boldsymbol{b} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+r}$. Equation (8.16) is a nonlinear Gauss-Helmert Model with redundancy r.

By introducing the "true" $n \times 1$ vector of observables $\boldsymbol{\mu}$ as

$$\boldsymbol{\mu} := \boldsymbol{y} - \boldsymbol{e} = E\{\boldsymbol{y}\}, \tag{8.17}$$

the least-squares objective for model (8.16) is then defined by

$$\boldsymbol{e}^T P \boldsymbol{e} = \min, \ \ \text{subject to} \ \ \boldsymbol{b}(\boldsymbol{\mu}, \boldsymbol{\Xi}) = \boldsymbol{0}. \tag{8.18}$$

An iterative linearization of (8.16), together with the least-squares estimation of the unknown parameters $\boldsymbol{\Xi}$ and prediction of the unknown random errors $\boldsymbol{e}$, can be formed as follows.

Begin by assigning initial values $\boldsymbol{\mu}_0$ and $\boldsymbol{\Xi}_0$ to the unknowns $\boldsymbol{\mu}$ and $\boldsymbol{\Xi}$, respectively, e.g., $\boldsymbol{\mu}_0 = \boldsymbol{y} - \boldsymbol{0}$ and $\boldsymbol{\Xi}_0$ by some approximate method (perhaps using LESS within the GMM if linearization would not be required for that solution). Then execute the following conditional loop:

While

$$\delta < \|\hat{\boldsymbol{\xi}}_j\| \ \ \text{or} \ \ \epsilon < \|\tilde{\boldsymbol{e}}^{(j)} - \tilde{\boldsymbol{e}}^{(j-1)}\| \tag{8.19}$$

for chosen thresholds δ and ϵ, and $j \in \mathbb{N}$, perform the following steps:

(i) Use the truncated Taylor series about expansion point $(\boldsymbol{\mu}_j, \boldsymbol{\Xi}_j)$:

$$\left[\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\mu}^T}\Big|_{\boldsymbol{\mu}_j, \boldsymbol{\Xi}_j}, \ \ \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\Xi}^T}\Big|_{\boldsymbol{\mu}_j, \boldsymbol{\Xi}_j} \right] \cdot \begin{bmatrix} \boldsymbol{\mu} - \boldsymbol{\mu}_j \\ \boldsymbol{\Xi} - \boldsymbol{\Xi}_j \end{bmatrix} + \boldsymbol{b}(\boldsymbol{\mu}_j, \boldsymbol{\Xi}_j) = \boldsymbol{0}, \tag{8.20a}$$

and replace $\boldsymbol{\mu}$ with $\boldsymbol{y} - \boldsymbol{e}$ in accordance with (8.17), to introduce

$$\underset{m\times 1}{\boldsymbol{\xi}_{j+1}} := \boldsymbol{\Xi} - \boldsymbol{\Xi}_j, \ \ \underset{(m+r)\times m}{A^{(j)}} := -\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\Xi}^T}\Big|_{\boldsymbol{\mu}_j, \boldsymbol{\Xi}_j}, \ \ \underset{(m+r)\times n}{B^{(j)}} := \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\mu}^T}\Big|_{\boldsymbol{\mu}_j, \boldsymbol{\Xi}_j}, \tag{8.20b}$$

$$\underset{(m+r)\times 1}{\boldsymbol{w}_j} := \boldsymbol{b}(\boldsymbol{\mu}_j, \boldsymbol{\Xi}_j) + B^{(j)} \cdot (\boldsymbol{y} - \boldsymbol{\mu}_j), \tag{8.20c}$$

and to form the linearized Gauss-Helmert Model

$$\boldsymbol{w}_j = A^{(j)} \hat{\boldsymbol{\xi}}_{j+1} + B^{(j)} \boldsymbol{e}, \ \ \boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}). \tag{8.20d}$$

(ii) Produce the $(j+1)$th LEast-Squares Solution (LESS) for (8.20d), viz.

$$\hat{\boldsymbol{\xi}}_{j+1} = \left\{ (A^{(j)})^T \left[(B^{(j)}) P^{-1} (B^{(j)})^T \right]^{-1} (A^{(j)}) \right\}^{-1} \cdot$$
$$\cdot (A^{(j)})^T \left[(B^{(j)}) P^{-1} (B^{(j)})^T \right]^{-1} \boldsymbol{w}_j, \tag{8.20e}$$

$$\tilde{\boldsymbol{e}}^{(j+1)} = P^{-1} (B^{(j)})^T \left[(B^{(j)}) P^{-1} (B^{(j)})^T \right]^{-1} (\boldsymbol{w}_j - (A^{(j)}) \hat{\boldsymbol{\xi}}_{j+1}). \tag{8.20f}$$

(iii) Obtain new approximate values (non-random) through

$$\boldsymbol{\Xi}_{j+1} := \hat{\boldsymbol{\Xi}}^{(j+1)} - \underset{\sim}{\mathbf{0}} = \boldsymbol{\Xi}_j + \hat{\boldsymbol{\xi}}_{j+1} - \underset{\sim}{\mathbf{0}}, \tag{8.20g}$$

$$\boldsymbol{\mu}_{j+1} := \hat{\boldsymbol{\mu}}^{(j+1)} - \underset{\sim}{\mathbf{0}} = \boldsymbol{y} - \tilde{\boldsymbol{e}}^{(j+1)} - \underset{\sim}{\mathbf{0}}, \tag{8.20h}$$

where $\underset{\sim}{\mathbf{0}}$ denotes a "random zero vector" of suitable size (following Harville, 1986). This means that the jth (approximate) estimates are stripped of their randomness while keeping their numerical values. The use of $\underset{\sim}{\mathbf{0}}$ is formally required to avoid the assignment of random values to a non-random quantity; however, its use is of no consequence in practice, since it does not affect the numerical results.

Repeat the cycle until convergence is reached.

As already suggested, the initial approximate values for $\boldsymbol{\mu}$ might be taken from the observation vector $\boldsymbol{y}$ via $\boldsymbol{\mu}_0 := \boldsymbol{y} - \underset{\sim}{\mathbf{0}}$. Unfortunately, this has occasionally led to the misunderstanding that the so-called "misclosure vector" $\boldsymbol{w}_i$, in the ith iteration cycle, ought to be updated by $\boldsymbol{b}(\boldsymbol{\mu}_i, \boldsymbol{\Xi}_i)$ when, in fact, the *correct update* is described by (8.20c). Also, the expression for $\boldsymbol{w}_j$ in (8.20c) is approximately equal to $\boldsymbol{b}(\boldsymbol{y}, \boldsymbol{\Xi}_j)$ and sometimes may turn out to be precisely equal to it; however, in some cases its usage may lead to convergence to an inaccurate solution. An excellent treatment of potential pitfalls for solving nonlinear least-squares problems, can be found in Pope (1972), which the reader is encouraged to read. See Schaffrin and Snow (2010) for a more detailed discussion of this topic.

8.4   Estimated Variance Component

The P-weighted norm of the residual vector $\tilde{\boldsymbol{e}}$ is defined as

$$\boxed{\Omega := \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} =} \tag{8.21a}$$

$$= \big(\hat{\boldsymbol{\lambda}}^T B P^{-1}\big) P \big(P^{-1} B^T \hat{\boldsymbol{\lambda}}\big) = \tag{8.21b}$$

$$= \big[-\big(\boldsymbol{w} - A\hat{\boldsymbol{\xi}}\big)^T \big(B P^{-1} B^T\big)^{-1}\big]\big(B P^{-1} B^T\big)\hat{\boldsymbol{\lambda}} = \tag{8.21c}$$

$$= \big(\boldsymbol{w} - A\hat{\boldsymbol{\xi}}\big)^T \big(B P^{-1} B^T\big)^{-1}\big(\boldsymbol{w} - A\hat{\boldsymbol{\xi}}\big) = \tag{8.21d}$$

$$= \big(B\tilde{\boldsymbol{e}}\big)^T \big(B P^{-1} B^T\big)^{-1}\big(B\tilde{\boldsymbol{e}}\big). \tag{8.21e}$$

Thus it follows that, the *uniformly unbiased estimate* of the variance component σ_0^2 is given by

$$\boxed{\hat{\sigma}_0^2 = \frac{\big(B\tilde{\boldsymbol{e}}\big)^T \big(B P^{-1} B^T\big)^{-1}\big(B\tilde{\boldsymbol{e}}\big)}{r} = \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{r} = \frac{-\boldsymbol{w}^T \hat{\boldsymbol{\lambda}}}{r},} \tag{8.22}$$

where the redundancy r is defined as

$$\boxed{r := \operatorname{rk} B - \operatorname{rk} A,} \tag{8.23}$$

which is the number of rows of B minus the number of columns of A, assuming matrix B has full row rank and matrix A has full column rank.

8.5   Equivalent Normal Equations

From (8.10b) and the second equation following (8.10c), and considering the notation changed described on Page 162, we can recognize the following system of normal equations:

$$
\begin{bmatrix} BP^{-1}B^T & -A \\ -A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\xi}} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{w} \\ \boldsymbol{0} \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\xi}} \end{bmatrix} = \left[\begin{array}{c|c} BP^{-1}B^T & -A \\ \hline -A^T & 0 \end{array} \right]^{-1} \begin{bmatrix} -\boldsymbol{w} \\ \boldsymbol{0} \end{bmatrix}. \tag{8.24}
$$

We want to show that the solution to this system yields the same $\hat{\boldsymbol{\xi}}$ as that of (8.14). The formula for the inverse of a partitioned matrix (see (A.15)) leads to the following solution:

$$
\begin{bmatrix} \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\xi}} \end{bmatrix} = \left[\begin{array}{c|c} X_1 & X_2 \\ \hline -W^{-1}A^T(BP^{-1}B^T)^{-1} & (0-W)^{-1} \end{array} \right] \begin{bmatrix} -\boldsymbol{w} \\ \boldsymbol{0} \end{bmatrix},
$$

with $W := A^T(BP^{-1}B^T)^{-1}A$, and finally to

$$
\begin{bmatrix} \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\xi}} \end{bmatrix} = \left[\begin{array}{c} -X_1\boldsymbol{w} \\ \left[A^T(BP^{-1}B^T)^{-1}A\right]^{-1}A^T(BP^{-1}B^T)^{-1}\boldsymbol{w} \end{array} \right]. \tag{8.25}
$$

Here the symbols X_1 and X_2 represent quantities of no interest. We see that the solution for the parameters $\hat{\boldsymbol{\xi}}$ is the same in (8.14).

8.6   Example Problems

The following example problems are meant to help illustrate the use of the Gauss-Helmert Model (GHM).

8.6.1   Example — Fitting a Parabola When Both x- and y-Coordinates are Observed

In this example, we show how the GHM can be used to fit a parabola when both the x- and y-coordinates have been observed. This is in contrast to the problem treated in Section 3.2.1 with the GMM, where only the dependent variables (y-coordinates) could be considered as measurements. Here, the observation vector $\boldsymbol{y}$ is comprised of all pairs of the $n/2$ measured points. For example, $\boldsymbol{y}$ could be defined as

$$
\underset{n \times 1}{\boldsymbol{y}} = \left[x_1, x_2, \ldots, x_{n/2}, y_1, y_2, \ldots, y_{n/2} \right]^T. \tag{8.26}
$$

Alternatively, the elements of $\boldsymbol{y}$ could be ordered by coordinate pairs, i.e., $\boldsymbol{y} = [x_1, y_1, \ldots, x_{n/2}, y_{n/2}]^T$. The key is that consistency of ordering must be maintained for the coefficient matrix B, the random error vector $\boldsymbol{e}$, and the observation cofactor matrices, too.

Denoting the true (unknown) variables as μ_{x_i} and μ_{y_i}, $i = 1, 2, \ldots, n/2$, the following equations can be written for the ith pair of observed variables (x_i, y_i):

$$x_i = \mu_{x_i} + e_{x_i}, \ \ E\{e_{x_i}\} = 0 \Rightarrow E\{x_i\} = \mu_{x_i}, \tag{8.27a}$$

$$y_i = \mu_{y_i} + e_{y_i}, \ \ E\{e_{y_i}\} = 0 \Rightarrow E\{y_i\} = \mu_{y_i}. \tag{8.27b}$$

For this example, we assume that the measurement errors are iid. Collecting the random error terms in vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$, respectively, their stochastic nature can then be expressed succinctly as

$$\underset{n \times 1}{\boldsymbol{e}} = \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix} \sim \left(\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} I_{n/2} & 0 \\ 0 & I_{n/2} \end{bmatrix} \right). \tag{8.28}$$

The (nonlinear) function that relates the ith pair of variables (μ_{x_i}, μ_{y_i}) to the non-random parameters $\boldsymbol{\Xi} = [\Xi_1, \Xi_2, \Xi_3]^T$ is given by

$$b_i(\Xi_1, \Xi_2, \Xi_3, \mu_{x_i}, \mu_{y_i}) = \mu_{y_i} - \mu_{x_i}^2 \Xi_1 - \mu_{x_i} \Xi_2 - \Xi_3 = 0, \ \ i \in \{1, 2, \ldots, n/2\}, \tag{8.29a}$$

which can be linearized about $(\boldsymbol{u}_{i0}, \boldsymbol{\Xi}_0)$ by

$$b_i^0 + \mathrm{d}\mu_{y_i} - (2\mu_{x_i}^0 \Xi_1^0 + \Xi_2^0)\, \mathrm{d}\mu_{x_i} - (\mu_{x_i}^2)^0\, \mathrm{d}\Xi_1 - \mu_{x_i}^0\, \mathrm{d}\Xi_2 - \mathrm{d}\Xi_3 = 0, \tag{8.29b}$$

where higher order terms have been neglected. Here the superscript 0 denotes the expansion point for the variables and parameters that the derivatives are evaluated at, viz. $\boldsymbol{u}_{i0} = [\mu_{x_i}^0, \mu_{y_i}^0]^T$ and $\boldsymbol{\Xi}_0 = [\Xi_1^0, \Xi_2^0, \Xi_3^0]^T$. The argument list for b_i has been dropped for the sake of brevity. Now define $n/2$ equations with:

$$\boldsymbol{\xi} = \begin{bmatrix} \mathrm{d}\Xi_1, \mathrm{d}\Xi_2, \mathrm{d}\Xi_3 \end{bmatrix}^T = \boldsymbol{\Xi} - \boldsymbol{\Xi}_0, \tag{8.30a}$$

$$-A_i = \begin{bmatrix} -(\mu_{x_i}^0)^2, -\mu_{x_i}^0, -1 \end{bmatrix}, \tag{8.30b}$$

$$B_i = \begin{bmatrix} -2\mu_{x_i}^0 \Xi_1^0 - \Xi_2^0, 1 \end{bmatrix}, \tag{8.30c}$$

where A_i is the ith row of an $(m + r) \times m$ matrix A (with $m = 3$ and r being the redundancy of the model, and $n = 2(m+r)$ in this example). In contrast, B_i shows only the non-zero elements of a row of an $(m+r) \times n$ matrix B. Those two elements go in the ith and $2i$th columns, respectively, of the ith full row of B (assuming the ordering of observations shown in (8.26)). Further define

$$\mathrm{d}\mu_{x_i} = \mu_{x_i} - \mu_{x_i}^0 = x_i - \mu_{x_i}^0 - e_{x_i} \ \text{ and } \ \mathrm{d}\mu_{y_i} = \mu_{y_i} - \mu_{y_i}^0 = y_i - \mu_{y_i}^0 - e_{y_i}, \tag{8.30d}$$

along with vectors

$$\boldsymbol{e}_i = \begin{bmatrix} e_{x_i}, e_{y_i} \end{bmatrix}^T, \ \text{ and } \ w_i = b_i^0 + B_i \begin{bmatrix} x_i - \mu_{x_i}^0, y_i - \mu_{y_i}^0 \end{bmatrix}^T \tag{8.30e}$$

so that (8.29b) can be rewritten for the ith observed coordinate pair as

$$-A_i \boldsymbol{\xi} - B_i \boldsymbol{e}_i + w_i = 0. \tag{8.31}$$

Then the complete set of $n/2 = m + r$ equations can be expressed as

$$\boldsymbol{w} = A\boldsymbol{\xi} + B\boldsymbol{e}, \tag{8.32}$$

which is obviously in the form of a GHM, within which a least-squares solution can be computed using the algorithm described in Section 8.3. In order to avoid non-convergence or convergence to the wrong solution, one must pay careful attention to the comments in the last paragraph of Section 8.3. In practice, they mean that all the terms A, B, and $\boldsymbol{w}$ must be updated at each iteration, using numerical value computed from the previous iteration.

The preceding formulation can be applied to the fitting of many different kinds of functions in 2D- and 3D-space, including lines, planes, quadratic surfaces, etc. When the data are iid, these adjustments amount to solving so-called "orthogonal regression" problems, since the residuals pairs $(\tilde{e}_{x_i}, \tilde{e}_{y_i})$ define vectors that are orthogonal to the fitted curve (or surface in 3D). In the case of a general weight matrix P, we might prefer to say "P-weighted orthogonal regression," since the weights will influence the direction of the 2D and 3D residual vectors.

8.6.2 Example — Fitting a Ellipse When Both x- and y-Coordinates are Observed

An equation for an ellipse can be written as a function of its center point (z_1, z_2), the length of its semi-major axis a, the length of its semi-minor axis b, and the counter-clockwise angle α between the z_1-axis and the semi-major axis (see Figure 8.2). Accordingly, a (nonlinear) function that relates the ith pair of $n/2$ pairs of random variables (μ_{x_i}, μ_{y_i}) to the unknown (but non-random) parameters $(\mu_\alpha, \mu_a, \mu_b, \mu_{z_1}, \mu_{z_2})$ is provided by

$$
\begin{aligned}
b_i &\left(\mu_\alpha, \mu_a, \mu_b, \mu_{z_1}, \mu_{z_2}, \mu_{x_i}, \mu_{y_i} \right) = \\
&= \mu_b^2 \big[\cos^2 \mu_\alpha (\mu_{x_i} - \mu_{z_1})^2 + 2\cos\mu_\alpha \sin\mu_\alpha (\mu_{x_i} - \mu_{z_1})(\mu_{y_i} - \mu_{z_2}) + \\
&\qquad\qquad + \sin^2 \mu_\alpha (\mu_{y_i} - \mu_{z_2})^2 \big] + \\
&+ \mu_a^2 \big[\sin^2 \mu_\alpha (\mu_{x_i} - \mu_{z_1})^2 - 2\sin\mu_\alpha \cos\mu_\alpha (\mu_{x_i} - \mu_{z_1})(\mu_{y_i} - \mu_{z_2}) + \\
&\qquad\qquad + \cos^2 \mu_\alpha (\mu_{y_i} - \mu_{z_2})^2 \big] - \mu_a^2 \mu_b^2 = 0,
\end{aligned}
\tag{8.33}
$$

with $i \in \{1, \ldots, n/2\}$. Collecting the unknown parameters in the vector $\boldsymbol{\Xi}$, viz. $\boldsymbol{\Xi} = [\mu_\alpha, \mu_a, \mu_b, \mu_{z_1}, \mu_{z_2}]^T$, their values can then be estimated via a least-squares solution within the GHM as outlined in Section 8.6.1, where (8.29a) would be replaced by (8.33).

8.6.3 Example — 2D Similarity Transformation When Coordinates Have Been Observed in Both the Source and Target Systems

If $n/2$ coordinate pairs (X_i, Y_i) and (x_i, y_i) have been observed in both target ("new") and source ("old") coordinate systems, respectively, then the following
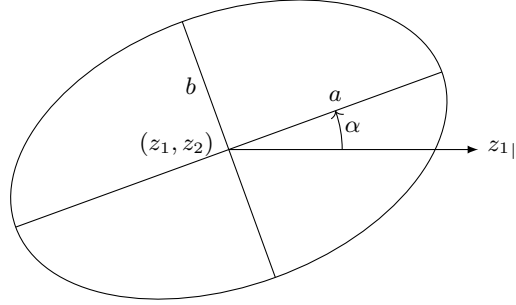
Figure 8.2: An ellipse with semi-major and semi-minor axes a and b, respectively, centered at (z_1, z_2) and rotated by angle α

GHM can be used to model a 2D similarity transformation:

$$\boldsymbol{b}(\boldsymbol{\mu}, \boldsymbol{\xi}) := \begin{bmatrix} \dots \\ X_i \\ Y_i \\ \dots \end{bmatrix} - \begin{bmatrix} \dots \\ e_{X_i} \\ eY_i \\ \dots \end{bmatrix} - \begin{bmatrix} \dots & \dots & \dots & \\ 1 & 0 & x_i - e_{x_i} & -(y_i - e_{y_i}) \\ 0 & 1 & y_i - e_{y_i} & x_i - e_{x_i} \\ \dots & & & \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \boldsymbol{0}, \quad (8.34a)$$

where

$\boldsymbol{y} := \begin{bmatrix} \dots, X_i, Y_i, \dots, x_i, y_i, \dots \end{bmatrix}^T$ is a $2n \times 1$ vector of observed coordinates,

$$(8.34b)$$

$\boldsymbol{e} := \begin{bmatrix} \dots, e_{X_i}, e_{Y_i}, \dots, e_{x_i}, e_{y_i}, \dots \end{bmatrix}^T$ is a $2n \times 1$ random error vector,   (8.34c)

$\boldsymbol{\mu} := \boldsymbol{y} - \boldsymbol{e}$ is a $2n \times 1$ vector of actual ("true") coordinates, and   (8.34d)

$\boldsymbol{\xi} := \begin{bmatrix} \xi_1, \xi_2, \xi_3, \xi_4 \end{bmatrix}^T$ is the 4×1 vector of unknown parameters, with   (8.34e)

$\xi_3 := \omega \cos\alpha$, and $\xi_4 := \omega \sin\alpha$.   (8.34f)

Here, ξ_1 and ξ_2 are translation parameters along the X- and Y-axis, respectively; ω is a scale factor, and α is a counter-clockwise rotation angle.

8.7   Some Published Examples

The following papers include numerical examples for the GHM that may be of interest to the reader:

1. Circle fitting: Schaffrin and Snow (2010).

2. Line fitting in 3D: Snow and Schaffrin (2016).

3. 2D similarity transformations: Neitzel and Petrovic (2008).

In addition to these sources, Pope's 1972 paper is highly recommended to understand how to best avoid potential pitfalls in adjusting data within iteratively linearized models.

8.8 Practice Problems

1. A circle is to be fitted to the set of measured coordinates shown in Table 8.1, which were presented in Schaffrin and Snow (2010). Both x- and y-coordinates were measured, and the associated random errors are considered to be iid.

Table 8.1: Measured coordinates for the fitting of a circle. Units are not given.

No.	x	y
1	0.7	4.0
2	3.3	4.7
3	5.6	4.0
4	7.5	1.3
5	6.4	−1.1
6	4.4	−3.0
7	0.3	−2.5
8	−1.1	1.3

(a) Setup an appropriate Gauss-Helmert Model with the coordinates of the center of the circle and its radius as the three unknown parameters. What is the redundancy of the model?

(b) Compute the least-squares estimates of the center of the circle and its radius. You may use the following initial approximations for the parameters: $\Xi_0 = [3, 1, 4]^T$ (in order of x and y coordinates of the center point followed by the radius).

(c) Compute the estimated variance component and the empirical rms of the estimated parameters (i.e., the square roots of the diagonal elements of the estimated dispersion matrix).

(d) What is the geometrical relationship between the estimated center of the circle and each respective pair of observed and adjusted coordinates?

2. An ellipse is to be fitted to the set of measured coordinates plotted in Figure 8.3 and listed in Table 8.2. Both z_1- and z_2-coordinates were measured, and the associated random errors are considered to be iid.
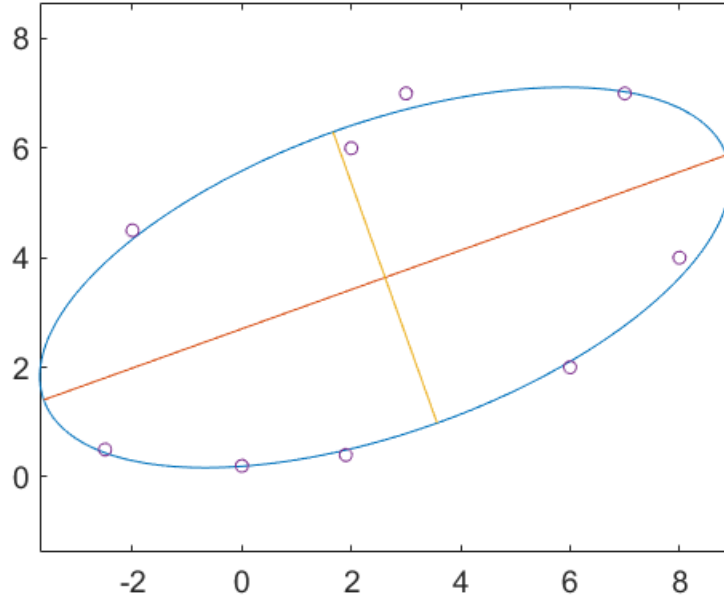
Figure 8.3: Fitted ellipse and measured coordinates in the 2-D plane (listed in Table 8.2)

(a) Setup an appropriate Gauss-Helmert Model with the coordinates of the center of the ellipse (z_1, z_2), its semi-major and semi-minor axes lengths

Table 8.2: Measured coordinates for the fitting of an ellipse. Units are not given.

No.	z_1	z_2
1	2.0	6.0
2	7.0	7.0
3	9.0	5.0
4	3.0	7.0
5	6.0	2.0
6	8.0	4.0
7	−2.0	4.5
8	−2.5	0.5
9	1.9	0.4
10	0.0	0.2

a and b, and the angle α between the z_1 axis and the semi-major axis as five unknown parameters (see Section 8.6.2). What is the redundancy of the model?

(b) Compute the least-squares estimates of the unknown parameters of the ellipse. You may use the following initial approximations for the parameters: $\Xi_0 = [0, 7, 3, 3, 4]^T$ (in order of μ_α^0, μ_a^0, μ_b^0, $\mu_{z_1}^0$, $\mu_{z_1}^0$).

(c) Compute the estimated variance component and the empirical rms of the estimated parameters (i.e., the square roots of the diagonal elements of the estimated dispersion matrix).

3. The data used for fitting the parabola shown in Figure 3.2 are listed in Table 8.3. Assume that the cofactor matrix for the x-coordinates is $Q_x = (0.010\,\mathrm{m})^2 \cdot I_n$ and that the cofactor matrix for the y-coordinates is $Q_y = (0.005\,\mathrm{m})^2 \cdot I_n$. Using the model presented in Section 8.6.1, compute the following:

(a) Estimates for the three unknown parameters of the parabola.

(b) The estimated variance component.

(c) The empirical rms of the estimated parameters (i.e., the square roots of the diagonal elements of the estimated dispersion matrix).

Table 8.3: Measured coordinates for the fitting of a parabola. The units are in meters.

No.	x	y
1	1.007	1.827
2	1.999	1.911
3	3.007	1.953
4	3.998	2.016
5	4.999	2.046
6	6.015	2.056
7	7.014	2.062
8	8.014	2.054
9	9.007	2.042
10	9.988	1.996
11	11.007	1.918
12	12.016	1.867

8.9 Summary Formulas for the Least-Squares Solution Within the Gauss-Helmert Model

See the last paragraph of Section 8.2 for comments about replacing the product BA used in earlier sections of this chapter with the matrix A itself, as was done in the following:

The linearized Gauss-Helmert Model (GHM) is given by

$$
\underset{(r+m)\times 1}{\boldsymbol{w}} = \underset{(r+m)\times m}{A}\ \boldsymbol{\xi} + \underset{(r+m)\times n}{B}\ \boldsymbol{e}, \quad \underset{n\times 1}{\boldsymbol{e}} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}).
$$

Table 8.4: Summary formulas for the LESS within the Gauss-Helmert Model

Quantity	Formula	Eq.
Model redundancy	$r = \operatorname{rk} B - \operatorname{rk} A$	(8.23)
Vector of estimated parameters	$\hat{\boldsymbol{\xi}} =$ $\left[A^T\left(BP^{-1}B^T\right)^{-1}A\right]^{-1}A^T\left(BP^{-1}B^T\right)^{-1}\boldsymbol{w}$	(8.11a)
Dispersion matrix for estimated parameters	$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 {\cdot} \left[A^T\left(BP^{-1}B^T\right)^{-1}A\right]^{-1}$	(8.12)
Vector of predicted residuals	$\tilde{\boldsymbol{e}} = P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}\left(\boldsymbol{w} - A\hat{\boldsymbol{\xi}}\right)$	(8.11b)
Dispersion matrix for residuals	$D\{\tilde{\boldsymbol{e}}\} =$ $P^{-1}B^T\left(BP^{-1}B^T\right)^{-1}\left[B{\cdot}D\{\boldsymbol{e}\}{\cdot}B^T - A{\cdot}D\{\hat{\boldsymbol{\xi}}\}{\cdot}A^T\right]\left(BP^{-1}B^T\right)^{-1}BP^{-1}$	(8.13)
Sum of squared residuals (SSR)	$\Omega = \tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}$	(8.21a)
Estimated variance component	$\hat{\sigma}_0^2 = \Omega/r$	(8.22)

Chapter 9

Statistical Analysis

It is assumed that the reader of these notes has had at least a first course in statistical methods or probability theory and thus has some familiarity with hypothesis testing in statistical analysis. Therefore, key terms and concepts will be described only briefly, and the main focus will be placed on the application of hypothesis testing to parameters estimated from least-squares adjustments as described in the preceding chapters. For a broader treatment of statistical methods, and an excellent refresher on hypothesis testing in particular, see Snedecor and Cochran (1980).

Consider a normally distributed random (scalar) variable y with the following first through fourth moments:

$$E\{y\} = \mu, \tag{9.1a}$$

$$E\{(y - \mu)^2\} = D\{y\} = \sigma^2, \tag{9.1b}$$

$$E\{(y - \mu)^3\} = 0, \tag{9.1c}$$

$$E\{(y - \mu)^4\} = 3(\sigma^2)^2. \tag{9.1d}$$

The third moment being zero in (9.1c) means there is no skewness in the distribution of the random variable. The right side of (9.1d) indicates that there is no kurtosis (peak) in the distribution.

If (9.1c) or (9.1d) are not satisfied, the variable is not normally distributed and can be characterized as follows:

$$E\{(y - \mu)^3\} > 0 \Leftrightarrow \text{the distribution is skewed to the positive side.} \tag{9.2a}$$

$$E\{(y - \mu)^3\} < 0 \Leftrightarrow \text{the distribution is skewed to the negative side.} \tag{9.2b}$$

$$E\{(y - \mu)^4\} - 3(\sigma^2)^2 > 0 \Leftrightarrow \text{the distribution has positive kurtosis.} \tag{9.2c}$$

$$E\{(y - \mu)^4\} - 3(\sigma^2)^2 < 0 \Leftrightarrow \text{the distribution has negative kurtosis.} \tag{9.2d}$$

Skewness appears in a graph of a sample of the random variable (e.g., a histogram) as a shift in the peak value from center. Positive kurtosis shows higher probability near the expected value μ, which results in a taller, narrower graph. Negative kurtosis shows higher probability in the tails of the graph; thus the graph appears flatter than that of a normally distributed variable.

The pdf (probability density function, or density function) of a normally distributed random (scalar) variable y is

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(y-\mu)^2/2\sigma^2}, \tag{9.3}$$

where μ is the expectation of the distribution (population mean), σ is standard deviation, σ^2 is variance, and e is Euler's number (i.e., the base of the natural logarithm, $e \approx 2.71828$). Note that the term $1/\sqrt{2\pi\sigma^2} \approx 0.4/\sigma$ denotes the amplitude of the graph of the curve, μ shows the offset of the peak from center, and σ is the distance from the center to the inflection points of the curve.

The cdf (cumulative distribution function, or distribution function) of a *normally distributed* random variable is expressed as

$$F(y) = \int_{-\infty}^{y} f(t)\, dt = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} e^{-(t-\mu)^2/2\sigma^2}\, dt. \tag{9.4}$$

Figure 9.1 shows pdf and cdf plots for the normal distribution using various values for μ and σ^2. Line colors and types match between the pdf and cdf plots. The solid, green line represents the respective standard normal pdf and cdf curves.

Note that, in geodetic-science applications, the random variable y might be an observation, an adjusted observation, a predicted residual, etc. We can standardize the random variable y with the following transformation, which subtracts out the mean and divides by the standard deviation:

$$z = \frac{y-\mu}{\sigma}. \tag{9.5}$$

The standardized random variable z has the following moments and probability functions:

$$E\{z\} = 0, \tag{9.6a}$$

$$D\{z\} = 1, \tag{9.6b}$$

$$\text{pdf}:\ f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}, \tag{9.6c}$$

$$\text{cdf}:\ F(z) = \int_{-\infty}^{z} f(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt. \tag{9.6d}$$

A plot of the pdf of z is shown in Figure 9.2, along with example Student's t-distribution curves (discussed below).

In the multivariate case, the random variable $\boldsymbol{y}$ is an $n \times 1$ vector, with an $n \times n$ dispersion (covariance) matrix $\Sigma = D\{\boldsymbol{y}\}$ and expectation vector $\boldsymbol{\mu} = E\{\boldsymbol{y}\}$, which is also size $n \times 1$. The pdf is then written as

$$f(\boldsymbol{y}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\Sigma}}\, e^{-(\boldsymbol{y}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{y}-\boldsymbol{\mu})/2}. \tag{9.7}$$
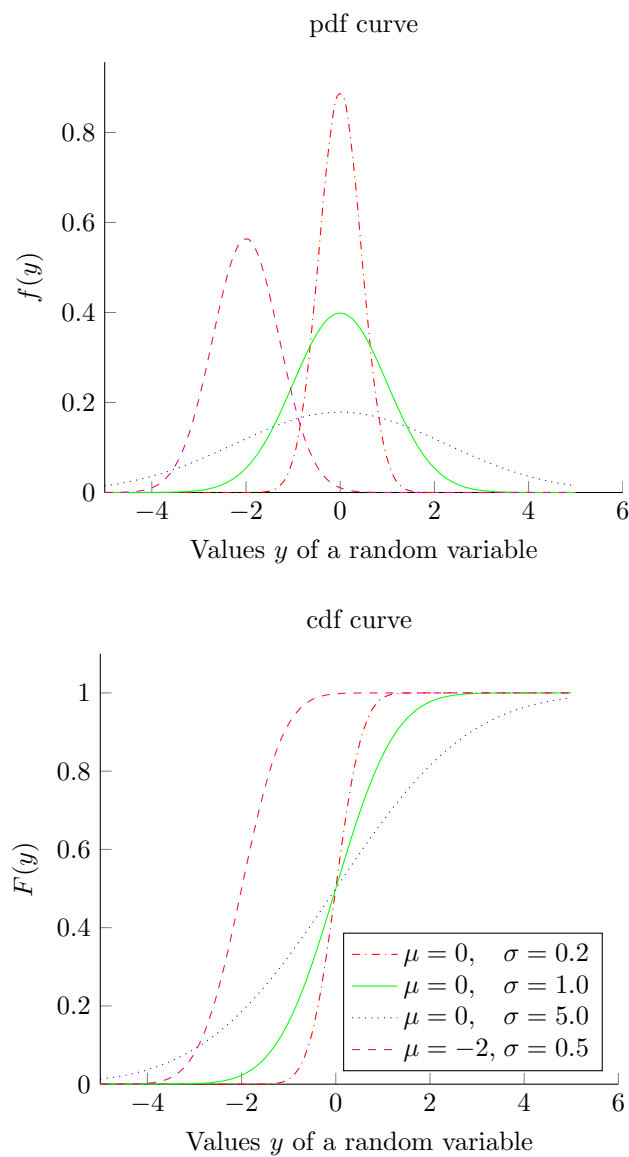
Figure 9.1: pdf curve (top) and cdf curve (bottom) for the normal distribution with matching line types and colors so that the legend pertains to both graphs

And the cdf is written as

$$F(y_1, \ldots, y_n) = \int\limits_{-\infty}^{y_n} \ldots \int\limits_{-\infty}^{y_1} f(t_1, \ldots, t_n) \, dt_1 \ldots dt_n \,. \qquad (9.8)$$
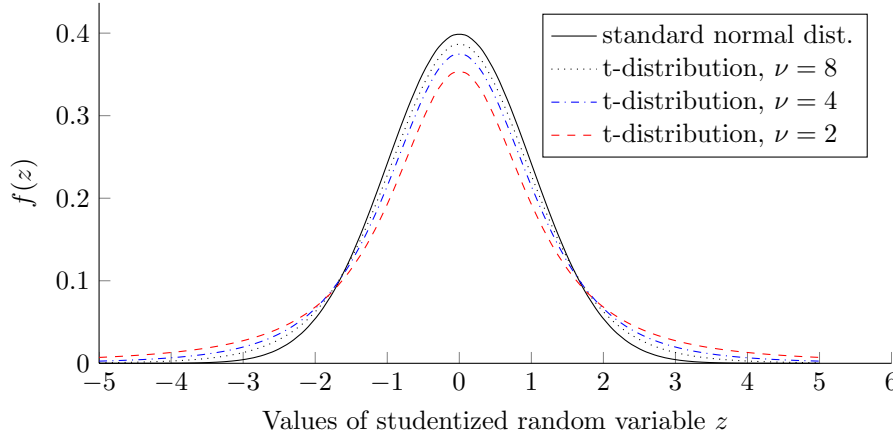
Figure 9.2: Curves of Student's t- and normal distributions for a standard-ized/studentized random variable

The elements of $\boldsymbol{y}$, i.e. $y_1, \ldots, y_n$, are statistically independent if, and only if,

$$f(t_1, \ldots, t_n) = f(t_1) \cdot f(t_2) \cdot \ldots f(t_n), \tag{9.9a}$$

which implies

$$C\{y_i, y_j\} = 0 \ \text{ for } i \neq j. \tag{9.9b}$$

Equation (9.9b) states that there is no covariance between the elements of random vector $\boldsymbol{y}$.

The third and fourth moments for the multivariate case are given in (9.10a) and (9.10b), respectively.

$$E\{(y_i - \mu_i)(y_j - \mu_j)(y_k - \mu_k)\} = 0 \ \text{ for } i, j, k = \{1, \ldots, n\} \tag{9.10a}$$

$$E\{(y_i - \mu_i)(y_j - \mu_j)(y_k - \mu_k)(y_l - \mu_l)\} = 3(\sigma_i^2)^2 \cdot \delta_{ijkl} \ \text{ for } i, j, k, l = \{1, \ldots, n\}, \tag{9.10b}$$

with δ_{ijkl} being the Kronecker-delta function satisfying $\delta_{ijkl} = 1$ if, and only if, $i = j = k = l$; otherwise $\delta_{ijkl} = 0$.

In the following, we discuss *studentized residuals*, which have a t-distribution (or Student's t-distribution). The pdf for a (scalar) variable having a t-distribution and $\nu = n - 1$ degrees of freedom is defined as follows:

$$f(t) = \frac{1}{\sqrt{(n-1)\pi}} \cdot \frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{\left(1 + \frac{t^2}{n-1}\right)^{n/2}}, \tag{9.11}$$

where the gamma function is defined by

$$\Gamma(n) := (n-1)\Gamma(n-1) = \int_0^\infty e^{-t} t^{n-1}\, dt = (n-1)! \ \text{ for } n \in \mathbb{N}. \tag{9.12}$$

As is known from introductory statistics, the pdf for the Student's t-distribution resembles the pdf of the standard normal distribution when ν is around 30. For $\nu = \infty$, the distributions are identical. A plot of the pdf for the Student's t-distribution, with $\nu = 2, 4, 8$, together with the pdf for the standard normal distribution, is shown in Figure 9.2.

9.1 Standardized and Studentized Residuals

We begin this section by restating the (full-rank) Gauss-Markov Model and writing the predicted vector of random errors within the model.

$$\boldsymbol{y} = A\boldsymbol{\xi} + \boldsymbol{e}, \ \ \boldsymbol{e} \sim \left(\boldsymbol{0}, \sigma_0^2 P^{-1}\right), \ \ \mathrm{rk}\,A = m \tag{9.13a}$$

$$\tilde{\boldsymbol{e}} = \left(I_n - AN^{-1}A^T P\right)\boldsymbol{y} = \left(I_n - AN^{-1}A^T P\right)\boldsymbol{e} \tag{9.13b}$$

As usual, the observation vector $\boldsymbol{y}$ is of size $n \times 1$, and the coefficient matrix A is of size $n \times m$, and normal equation variables are defined by $[N, \boldsymbol{c}] := A^T P[A, \boldsymbol{y}]$. Obviously, the far-right side of (9.13b) cannot be computed since $\boldsymbol{e}$ is an unknown variable. However, the expression is useful for analytical purposes.

In the following, we assume that the random error vector $\boldsymbol{e}$ has a normal distribution expressed by $\boldsymbol{e} \sim \mathcal{N}(\boldsymbol{0}, \sigma_0^2 P^{-1})$ (where the symbol $\mathcal{N}$ denotes normal distribution). This assumption is made for the sake of hypothesis testing in statistical analysis, which requires that test statistics[1] be computed as a function of a specified pdf. The justification of the assumption owes to the central limit theorem as stated by Bjerhammar (1973, p. 35) (see also the footnote on page 190). However, since $\boldsymbol{e}$ and σ_0^2 are unknown their respective prediction $\tilde{\boldsymbol{e}}$ and estimate $\hat{\sigma}_0^2$ are used instead; consequently, the Student's t-distribution is used in place of the normal distribution for formulating hypothesis tests.

9.1.1 Standardized Residuals

The so-called *standardized residual* vector is a function of the residual vector $\tilde{\boldsymbol{e}}$ and its dispersion matrix $D\{\tilde{\boldsymbol{e}}\}$ as shown in the following:

$$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2\left(P^{-1} - AN^{-1}A^T\right) =: \sigma_0^2 Q_{\tilde{e}}, \tag{9.14a}$$

$$\sigma_{\tilde{e}_j}^2 = \boldsymbol{\eta}_j^T D\{\tilde{\boldsymbol{e}}\}\boldsymbol{\eta}_j = E\{\tilde{e}_j^2\}, \tag{9.14b}$$

with

$$\boldsymbol{\eta}_j := \big[0, \ldots, 0, \underset{j\text{th}}{1}, 0, \ldots, 0\big]^T, \tag{9.14c}$$

as a unit vector that serves to extract the jth diagonal element from the dispersion matrix. Then, the jth standardized residual is defined as

$$\boxed{\tilde{z}_j := \tilde{e}_j / \sigma_{\tilde{e}_j}.} \tag{9.15}$$

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[1]The term *test statistic* is called *test criterion* by Snedecor and Cochran (1980, p. 65).

9.1.2 Studentized Residuals

Since the variance component σ_0^2 is considered unknown in the model (9.13a), we replace it with its estimate $\hat{\sigma}_0^2$, leading to the following analogous set of equations for the *studentized residual*:

$$\hat{\sigma}_0^2 = \frac{\tilde{e}^T P \tilde{e}}{n - \mathrm{rk}(A)} = \frac{y^T P y - c^T N^{-1} c}{n - m}, \tag{9.16a}$$

$$\hat{D}\{\tilde{e}\} = \hat{\sigma}_0^2 \left(P^{-1} - A N^{-1} A^T \right) =: \hat{\sigma}_0^2 Q_{\tilde{e}}, \tag{9.16b}$$

$$\hat{\sigma}_{\tilde{e}_j}^2 = \boldsymbol{\eta}_j^T \hat{D}\{\tilde{e}\} \boldsymbol{\eta}_j = \hat{E}\{\tilde{e}_j^2\}. \tag{9.16c}$$

Then the studentized residual is defined as

$$\boxed{\tilde{t}_j := \tilde{e}_j / \hat{\sigma}_{\tilde{e}_j}.} \tag{9.17}$$

Note that the denominator in (9.15) is constant (due to the unknown but constant variance component σ_0^2), whereas the denominator of (9.17) is random due to the introduction of the estimate $\hat{\sigma}_0^2$, which is random. Of course the numerator is random in both cases.

Using Q to represent cofactor matrices in general, we can rewrite the standardized and studentized residuals in the following alternative forms:

$$\text{Standardized residual:} \quad \tilde{z}_j := \tilde{e}_j / \sqrt{\sigma_0^2 \left(Q_{\tilde{e}} \right)_{jj}} \sim \mathcal{N}(0, 1). \tag{9.18a}$$

$$\text{Studentized residual:} \quad \tilde{t}_j := \tilde{e}_j / \sqrt{\hat{\sigma}_0^2 \left(Q_{\tilde{e}} \right)_{jj}} \sim t(n-1). \tag{9.18b}$$

Here $\sigma_0^2 Q_{\tilde{e}} = D\{\tilde{e}\}$, and $(Q_{\tilde{e}})_{jj}$ denotes the jth diagonal element of the residual cofactor matrix $Q_{\tilde{e}}$, and we have assumed that the standardized residuals are normally distributed, implying that the studentized residuals follow the Student's t-distribution. Again, it is noted that (9.18a) cannot be computed unless the variance component σ_0^2 is known.

9.1.2.1 Example of Studentized Residuals

This example treats studentized residuals within the model of direct observations of a single parameter μ with weight matrix $P = I_n$.

$$y = \boldsymbol{\tau}\mu + e, \;\; e \sim \mathcal{N}\left(\mathbf{0}, \sigma_0^2 I_n\right), \;\; \text{with } \boldsymbol{\tau} = [1, \ldots, 1]^T$$

$$\hat{\mu} = \frac{\boldsymbol{\tau}^T y}{\boldsymbol{\tau}^T \boldsymbol{\tau}} = \frac{1}{n}\left(y_1 + \ldots + y_n\right) \sim \mathcal{N}\left(\mu, \sigma_0^2 / n\right)$$

$$\tilde{e} = y - \boldsymbol{\tau}\hat{\mu} \sim \mathcal{N}\left(\mathbf{0}, \sigma_0^2 \left[I_n - n^{-1} \cdot \boldsymbol{\tau}\boldsymbol{\tau}^T\right]\right)$$

$$Q_{\tilde{e}} = I_n - n^{-1} \cdot \boldsymbol{\tau}\boldsymbol{\tau}^T$$

$$\hat{\sigma}_0^2 = \frac{\tilde{e}^T \tilde{e}}{(n-1)}$$

The formula for $Q_{\tilde{e}}$ in the above example means that $(Q_{\tilde{e}})_{jj} = (n-1)/n$, which shows that the more observations we have (i.e., the larger n is), the more the dispersion of the *predicted* random error $D\{\tilde{e}\}$ approaches the dispersion of the *true* random error $D\{e\}$. In this example the standardized and studentized residuals are written as follows:

$$\text{Standardized:} \qquad \tilde{z}_j = \frac{\tilde{e}_j}{\sqrt{\sigma_0^2 (Q_{\tilde{e}})_{jj}}} = \frac{\tilde{e}_j \sqrt{n}}{\sigma_0 \sqrt{n-1}} \sim \mathcal{N}(0,1). \qquad (9.19a)$$

$$\text{Or, alternatively:} \quad \tilde{z}_j = \frac{\tilde{e}_j}{\sqrt{(Q_{\tilde{e}})_{jj}}} \quad = \frac{\tilde{e}_j \sqrt{n}}{\sqrt{n-1}} \quad \sim \mathcal{N}(0,\sigma_0^2). \qquad (9.19b)$$

$$\text{Studentized:} \qquad \tilde{t}_j = \frac{\tilde{e}_j}{\sqrt{\hat{\sigma}_0^2 (Q_{\tilde{e}})_{jj}}} = \frac{\tilde{e}_j}{\sqrt{\tilde{e}^T \tilde{e}}} \sqrt{n} \sim t(n-1). \qquad (9.19c)$$

We extend the example by including a hypothesis test for the parameter estimate $\hat{\mu}$ against a specified value μ_0 at a significance level α.

$$\text{Hypothesis test:} \quad H_0 : E\{\hat{\mu}\} = \mu_0 \;\; \text{against}\; H_A : E\{\hat{\mu}\} \neq \mu_0.$$

$$\text{Test statistic:} \quad t = \frac{\hat{\mu} - \mu_0}{\sqrt{\hat{\sigma}_0^2}} \sqrt{n} \sim t(n-1).$$

We accept the null hypothesis H_0 if $t_{-\alpha/2} \leq t \leq t_{\alpha/2}$; otherwise we reject H_0. We may perform a similar test $H_0 : E\{\tilde{e}_j\} = 0$ for the jth residual. In this case the test statistic is the studentized residual computed by (9.19c).

9.2 Hypothesis Testing Within the Gauss-Markov Model

The hypothesis test introduced in Section 9.1 for direct observations of a single parameter is now extended to the Gauss-Markov Model (GMM). In introducing the GMM in Chapter 3, a probability density function was not given for the random observation errors; only the first and second moments of the random errors were specified. This is indeed all that is necessary to formulate and solve the least-squares estimation problem within the GMM. However, in order to perform classical hypothesis testing after the least-squares estimate has been computed, a probability distribution must be specified. Typically, we assume that the observation errors have a *normal distribution*. Then, the (full rank) GMM is written succinctly as

$$\underset{n \times 1}{y} = \underset{n \times m}{A} \, \xi + e, \;\; \text{rk}\, A = m, \;\; e \sim \mathcal{N}\big(0, \sigma_0^2 P^{-1}\big). \qquad (9.20)$$

where the symbol $\mathcal{N}$ denotes the normal distribution.

Minimization of the observation errors via a least-squares adjustment leads to the following vectors of parameter estimates and predicted random-errors, shown with their corresponding normal distributions (normal because the distribution of the observations were assumed to be normal for the sake of hypothesis testing):

$$\hat{\xi} = N^{-1} c \sim \mathcal{N}\big(\xi, \sigma_0^2 N^{-1}\big), \qquad (9.21a)$$

$$\tilde{e} = (I_n - AN^{-1}A^T P)y \sim \mathcal{N}(0, \sigma_0^2 [P^{-1} - AN^{-1}A^T]). \tag{9.21b}$$

Or equivalently, we could write for the predicted residual vector

$$\tilde{e} = (I_n - AN^{-1}A^T P)e = Q_{\tilde{e}} Py \sim \mathcal{N}(0, \sigma_0^2 Q_{\tilde{e}}), \tag{9.22a}$$

with its cofactor matrix provided by

$$Q_{\tilde{e}} := P^{-1} - AN^{-1}A^T. \tag{9.22b}$$

The jth standardized and studentized residuals are then written as

$$j\text{th standardized residual:} \quad \tilde{z}_j := \tilde{e}_j / \sqrt{\sigma_0^2 (Q_{\tilde{e}})_{jj}} \sim \mathcal{N}(0, 1), \tag{9.23}$$

$$j\text{th studentized residual:} \quad \tilde{t}_j := \tilde{e}_j / \sqrt{\hat{\sigma}_0^2 (Q_{\tilde{e}})_{jj}} \sim t(n - m). \tag{9.24}$$

As shown in Chapter 3, we compute the estimated reference variance within the GMM by

$$\hat{\sigma}_0^2 = \frac{\tilde{e}^T P \tilde{e}}{n - m}, \tag{9.25}$$

where $n - m$ is the redundancy of the model. The hypothesis test for the jth studentized residual then becomes

$$H_0 : E\{\tilde{e}_j\} = 0 \ \text{ versus } \ H_A : E\{\tilde{e}_j\} \neq 0. \tag{9.26}$$

Likewise, we may test individual elements of the estimated parameter vector $\hat{\boldsymbol{\xi}}$. For example, we may want to compare the jth element of the estimated parameter vector, $\hat{\xi}_j$, against some specified value $\xi_j^{(0)}$. In this case, the null hypothesis and computed test statistic are defined as follows:

$$H_0 : E\{\hat{\xi}_j\} = \xi_j^{(0)} \ \text{ versus } \ H_A : E\{\hat{\xi}_j\} \neq \xi_j^{(0)}, \tag{9.27a}$$

$$t_j = \frac{\hat{\xi}_j - \xi_j^{(0)}}{\sqrt{\hat{\sigma}_0^2 (N^{-1})_{jj}}} \sim t(n - m), \tag{9.27b}$$

or

$$t_j^2 = \frac{(\hat{\xi}_j - \xi_j^{(0)})[(N^{-1})_{jj}]^{-1}(\hat{\xi}_j - \xi_j^{(0)})/1}{(\tilde{e}^T P \tilde{e})/(n - m)} \sim F(1, n - m). \tag{9.27c}$$

From (9.27b) and (9.27c) we see that in this case the square of the test statistic having a Student's t-distribution has an F-distribution.

For a given significance level α, we accept H_0 if $t_{-\alpha/2} \leq t_j \leq t_{\alpha/2}$; otherwise we reject H_0. We can use a cdf table for the t-distribution to find the value of $t_{\alpha/2}(n - m)$. Note that α is the probability of making a Type I error (also called the significance level of the test), and $n - m$ is the degrees of freedom associated with $\hat{\sigma}_0^2$; for the F-distribution, 1 is the degrees of freedom associated with the numerator.

9.3 Confidence Intervals for Ellipses, Ellipsoids, and Hyperellipsoids

After we estimate the mean of a population, or the parameter of a data model, we might then like to make a statement about the accuracy of the estimated value. In statistics, a *probability statement* gives the probability that the estimated quantity falls within a certain interval centered on the true, but unknown mean (or model parameter). Such an interval is called a *confidence interval*, and its upper and lower bounds are called *confidence limits*. Confidence ellipses, ellipsoids, and hyperellipsoids are the respective 2-D, 3-D, and n-D analogues to confidence intervals.

9.3.1 Confidence Intervals — Univariate Case

By definition, the cdf (cumulative distribution function) of a random variable X is

$$F_X(x) = P(X \leq x), \quad -\infty < x < \infty, \tag{9.28}$$

which provides the probability that the unknown quantity X is less than or equal to the sampled value x. It follows, then, that the probability that X lies within the interval $(a, b]$ is

$$P(a < X \leq b) = F_X(b) - F_X(a). \tag{9.29}$$

Applying (9.29) to the standard normal random variable z of (9.5), we can write the following probabilities for confidence intervals bounded by $\pm 1\sigma$, $\pm 2\sigma$, $\pm 3\sigma$, respectively, from the mean, where $\sigma = 1$ since $z \sim \mathcal{N}(0, 1)$ according to (9.6a) and (9.6b):

$$P(-1 < z \leq 1) = P(\mu - \sigma < y \leq \mu + \sigma) = 68.3\% \tag{9.30a}$$
$$P(-2 < z \leq 2) = P(\mu - 2\sigma < y \leq \mu + 2\sigma) = 95.5\% \tag{9.30b}$$
$$P(-3 < z \leq 3) = P(\mu - 3\sigma < y \leq \mu + 3\sigma) = 99.7\% \tag{9.30c}$$

The intervals associated with these probability statements are commonly referred to as the "1-sigma," "2-sigma," and "3-sigma" confidence intervals, respectively. Other commonly used intervals are the so-called 90%, 95%, and 99% confidence intervals. For a normally distributed random variable z, their respective probability statements are

$$90\% = P(-1.645 < z \leq 1.645), \tag{9.31a}$$
$$95\% = P(-1.960 < z \leq 1.960), \tag{9.31b}$$
$$99\% = P(-2.576 < z \leq 2.576). \tag{9.31c}$$

Probability limits correspond to the area under the graph of the associated pdf. For example, the area between $\pm\sigma$ under the graph of the standard normal distribution shown in Figure 9.2 is 0.683, and it is 0.997 for $\pm 3\sigma$. The regions beyond these areas are called the *tails* of the graph. Figure 1.2 depicts a graphical representation of the areas comprising $\pm\sigma$, $\pm 2\sigma$, and $\pm 3\sigma$. It is shown again in Figure 9.3 for convenience.
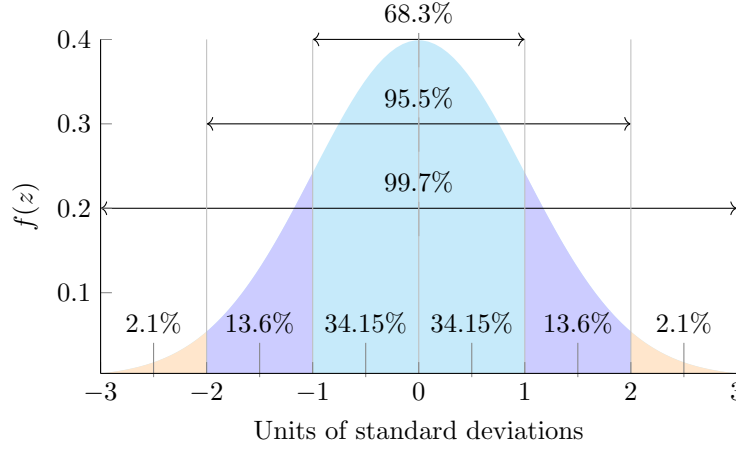
Figure 9.3: Normal distribution curve, with percent of areas under curve denoting probabilities. Image derived from TikZ code by John Canning, Senior Lecturer at the University of Brighton (http://johncanning.net/wp/?p=1202).

9.3.2   Confidence Ellipses — Bivariate Case

Now let us consider the bivariate (2-D) case where $\boldsymbol{y}$ is a random 2-D vector and $\boldsymbol{\mu}$ is its expected value; i.e., $\boldsymbol{\mu} = E\{\boldsymbol{y}\}$. Also, the dispersion of $\boldsymbol{y}$ is given by a 2×2 dispersion matrix Σ. In summary, we have

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = E\left\{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right\}, \quad \Sigma := D\{\boldsymbol{y}\} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \quad \sigma_{12} = \sigma_{21}. \tag{9.32}$$

When speaking of the elements of the vectors and matrix in (9.32), we say that μ_1 is the expected value of y_1; σ_1^2 is the variance of y_1 (with σ_1 called standard deviation), and σ_{12} is the covariance between y_1 and y_2.

The 2-D analogue to a confidence interval is a confidence ellipse, which can be generated from

$$(\boldsymbol{y} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{y} - \boldsymbol{\mu}) = \tag{9.33a}$$

$$= \frac{1}{(1 - \rho_{12}^2)} \left(\frac{(y_1 - \mu_1)^2}{\sigma_1^2} - 2\rho_{12} \frac{(y_1 - \mu_1)(y_2 - \mu_2)}{\sigma_1 \sigma_2} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2} \right) = k^2, \tag{9.33b}$$

where k is a constant, and ρ is the *correlation coefficient* defined by

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}. \tag{9.34}$$

By varying k, we generate a family of ellipses, each having an associated constant probability. Setting $k = 1$ results in the *standard confidence ellipse*. The ellipses actually originate by slicing the surface associated with a bivariate density function (pdf) with a plane parallel to the (y_1, y_2)-coordinate plane (see Figure 9.4) as described in the following.

Using the terms defined in (9.32), together with equation (9.7), we can write the *joint pdf* (or bivariate density function) of $\boldsymbol{y}$ explicitly as

$$f(\boldsymbol{y}) = f(y_1, y_2) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}} \cdot$$

$$\cdot \exp\left\{-\frac{\sigma_1^2\sigma_2^2}{2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}\left[\frac{(y_1 - \mu_1)^2}{\sigma_1^2} - 2\sigma_{12}\frac{(y_1 - \mu_1)}{\sigma_1^2}\frac{(y_2 - \mu_2)}{\sigma_2^2} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2}\right]\right\} = \tag{9.35a}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho_{12}^2}} \cdot$$

$$\cdot \exp\left\{-\frac{1}{2(1 - \rho_{12}^2)}\left[\left(\frac{y_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho_{12}\left(\frac{y_1 - \mu_1}{\sigma_1}\right)\left(\frac{y_2 - \mu_2}{\sigma_2}\right) + \left(\frac{y_2 - \mu_2}{\sigma_2}\right)^2\right]\right\}, \tag{9.35b}$$

where exp stands for the exponential function, e.g., $\exp\{x\} = e^x$. The density function has the form of a bell-shaped surface over the (y_1, y_2)-coordinate plane, centered at (μ_1, μ_2). By ignoring ρ, the respective *marginal pdf's* $f(y_1)$ and $f(y_2)$ can be written as

$$f(y_1) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(\frac{y_1 - \mu_1}{\sigma_1}\right)^2\right\}, \tag{9.36a}$$

and

$$f(y_2) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(\frac{y_2 - \mu_2}{\sigma_2}\right)^2\right\}. \tag{9.36b}$$

The bivariate density function $f(y_1, y_2)$ and the marginal density functions $f(y_1)$ and $f(y_2)$ are depicted in Figure 9.4 with ellipses traced out by slicing planes.

Each element of the vector $\boldsymbol{y}$ may be normalized according to (9.5), so that the jth element of the normalized vector $\boldsymbol{z}$ is expressed in terms of the corresponding jth element of $\boldsymbol{y}$; that is $z_j = (y_j - \mu_j)/\sigma_j$, $j = 1, 2$. Substituting z_j into (9.35b) we can write the following pdf for the normalized 2-D vector $\boldsymbol{z}$:

$$f(z_1, z_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho_{12}^2}} \cdot \exp\left\{-\frac{1}{2(1 - \rho_{12}^2)}\left(z_1^2 - 2\rho_{12}z_1z_2 + z_2^2\right)\right\}. \tag{9.37}$$

As noted above, a family of ellipses can be generated by slicing the bell-shaped surface generated by the density function (9.37) with planes parallel to the (y_1, y_2)-coordinate plane (see Figure 9.4). The formula for the ellipse can be defined by setting the density function to a constant value related to the height of the slicing plane, which after some simplification results in an equation of the form (9.33b). According to Mikhail and Gracie (1981, p. 221), the relationship between the height h of the slicing plane above the (y_1, y_2)-coordinate plane and the constant k in (9.33b) is given by $k^2 = \ln[4\pi^2h^2\sigma_1^2\sigma_2^2(1 - \rho_{12}^2)]^{-1}$. Setting $k = 1$ gives the equation for the standard confidence ellipse as follows:

$$z_1^2 - 2\rho_{12}z_1z_2 + z_2^2 = 1 - \rho_{12}^2. \tag{9.38}$$
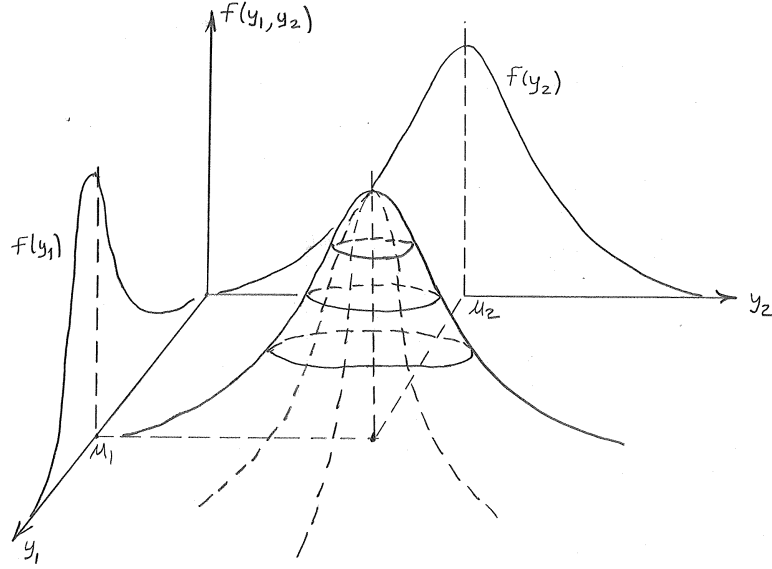
Figure 9.4: Bivariate and marginal density functions (pdf's) with ellipses traced from slicing planes, after Mikhail and Gracie (1981, p. 221)

The size, shape, and orientation of the confidence ellipse are determined by the eigenvalues and eigenvectors of the dispersion matrix Σ.

9.3.2.1   Eigenvector-eigenvalue decomposition of Σ

The eigenvector-eigenvalue decomposition of the 2×2 matrix Σ is described as follows: Denote the eigenvectors of Σ as $\boldsymbol{u}_j$ and the eigenvalues as λ_j, $j = 1, 2$. Then we have the relation

$$\Sigma \boldsymbol{u}_j = \lambda \boldsymbol{u}_j, \tag{9.39}$$

for which we write the following characteristic equation:

$$\det\left(\Sigma - \lambda I_2\right) = \left(\sigma_1^2 - \lambda\right)\left(\sigma_2^2 - \lambda\right) - \sigma_{12}^2 = \lambda^2 - \left(\sigma_1^2 + \sigma_2^2\right)\lambda + \left(\sigma_1^2 \sigma_2^2 - \sigma_{12}^2\right) = 0. \tag{9.40}$$

In (9.40), λ has been used in general to represent either eigenvalue λ_1 or λ_2. By convention, we require $\lambda_1 \geq \lambda_2 > 0$ and write the following solution for the roots of the characteristic equation (9.40):

$$\lambda_{1 \text{ or } 2} = \frac{\sigma_1^2 + \sigma_2^2}{2} \pm \sqrt{\left(\frac{\sigma_1^2 + \sigma_2^2}{2}\right)^2 - \frac{1}{4} 4\sigma_1^2 \sigma_2^2 + \frac{4\sigma_{12}^2}{4}} \Rightarrow \tag{9.41a}$$

$$\lambda_{1 \text{ or } 2} = \frac{\sigma_1^2 + \sigma_2^2}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 + 4\sigma_{12}^2} > 0, \tag{9.41b}$$

which shows that the eigenvalues must be greater than zero, since Σ is positive definite.

Now we must find the two corresponding eigenvectors. Let the matrix U be comprised of the two eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that $U := [\boldsymbol{u}_1, \ \boldsymbol{u}_2]$. Also define a diagonal matrix comprised of the corresponding eigenvalues $\Lambda := \text{diag}(\lambda_1, \ \lambda_2)$. Then according to (9.39) we have

$$\Sigma U = U\Lambda = \tag{9.42a}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \tag{9.42b}$$

$$= \begin{bmatrix} \sigma_1^2 u_{11} + \sigma_{12} u_{21} & \sigma_1^2 u_{12} + \sigma_{12} u_{22} \\ \sigma_{12} u_{11} + \sigma_2^2 u_{21} & \sigma_{12} u_{12} + \sigma_2^2 u_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1 \cdot u_{11} & \lambda_2 \cdot u_{12} \\ \lambda_1 \cdot u_{21} & \lambda_2 \cdot u_{22} \end{bmatrix}. \tag{9.42c}$$

Starting by equating the first columns on each side of (9.42c), and then the second columns, we can write the following four equations in the four unknowns u_{11}, u_{12}, u_{21}, and u_{22}:

$$u_{21} = \frac{(\lambda_1 - \sigma_1^2) u_{11}}{\sigma_{12}}, \quad u_{21} = \frac{\sigma_{12} u_{11}}{\lambda_1 - \sigma_2^2}, \quad u_{12} = \frac{\sigma_{12} u_{22}}{\lambda_2 - \sigma_1^2}, \quad u_{12} = \frac{(\lambda_2 - \sigma_2^2) u_{22}}{\sigma_{12}}. \tag{9.43}$$

The eigenvector $\boldsymbol{u}_1 = [u_{11}, u_{21}]^T$ defines the direction of the *semimajor axis* of the confidence ellipse, while the eigenvector $\boldsymbol{u}_2 = [u_{12}, u_{22}]^T$, orthogonal to $\boldsymbol{u}_1$, defines the *semiminor axis* direction. The square root of the eigenvalue λ_1 gives the semimajor-axis length, and the square root of the eigenvalue λ_2 gives the semiminor-axis length. Also, if θ is the angle measured counter clockwise from the positive z_1-axis to the semimajor axis of the confidence ellipse, then we can write the matrix U as

$$U = [\boldsymbol{u}_1, \ \boldsymbol{u}_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{9.44}$$

Using (9.43) and (9.44), the angle θ is derived as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{u_{21}}{u_{11}} = \frac{\lambda_1 - \sigma_1^2}{\sigma_{12}} = \frac{\sigma_{12}}{\lambda_1 - \sigma_2^2} = -\frac{u_{12}}{u_{22}} = \frac{\sigma_2^2 - \lambda_2}{\sigma_{12}} = \frac{\sigma_{12}}{\sigma_1^2 - \lambda_2} \text{ and} \tag{9.45a}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} = \left(\frac{2\sigma_{12}}{\lambda_1 - \sigma_2^2}\right) \frac{1}{1 - \frac{\sigma_{12}^2}{(\lambda_1 - \sigma_2^2)^2}} \left(\frac{\lambda_1 - \sigma_2^2}{\lambda_1 - \sigma_2^2}\right) \Rightarrow \tag{9.45b}$$

$$\tan(2\theta) = \frac{2\sigma_{12}(\lambda_1 - \sigma_2^2)}{(\lambda_1 - \sigma_2^2)^2 - \sigma_{12}^2} = \frac{2\sigma_{12}(\lambda_1 - \sigma_2^2)4}{[2(\lambda_1 - \sigma_2^2)]^2 - 4\sigma_{12}^2}. \tag{9.45c}$$

By manipulating (9.41b), we have

$$2(\lambda_1 - \sigma_2^2) = (\sigma_1^2 - \sigma_2^2) \pm \sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\sigma_{12}^2} \Rightarrow \tag{9.46a}$$

$$\left[2\left(\lambda_1 - \sigma_2^2\right)\right]^2 = 2\left(\sigma_1^2 - \sigma_2^2\right)^2 \pm 2\left(\sigma_1^2 - \sigma_2^2\right)\sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 + 4\sigma_{12}^2} + 4\sigma_{12}^2. \quad (9.46b)$$

Substituting (9.46a) and (9.46b) into (9.45c) gives

$$\tan(2\theta) = \frac{4\sigma_{12}\left[\left(\sigma_1^2 - \sigma_2^2\right) \pm \sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 + 4\sigma_{12}^2}\right]}{2\left(\sigma_1^2 - \sigma_2^2\right)\left[\left(\sigma_1^2 - \sigma_2^2\right) \pm \sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 + 4\sigma_{12}^2}\right]} \Rightarrow \quad (9.47a)$$

$$\boxed{\tan(2\theta) = \frac{2\sigma_{12}}{\sigma_1^2 - \sigma_2^2}.} \quad (9.47b)$$

As stated in Mikhail and Gracie (1981, p. 227), "the quadrant of 2θ is determined in the usual way from the signs of the numerator $2\sigma_{12}$ and the denominator $(\sigma_1^2 - \sigma_2^2)$."

Returning now to the notion of ellipses of constant probability represented by (9.33a), probabilities for various values of k are most easily determined by using a transformed system of equations centered on $\boldsymbol{\mu}$ and rotated so that the y_1 and y_2 axes coincide with the axes formed by the eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Then, instead of correlated coordinates y_1 and y_2, we end up with uncorrelated coordinates u_1 and u_2 with respective variances λ_1 and λ_2 from (9.41b). And so the probability statement for being on or within an ellipse having semimajor and semiminor axes $k\sqrt{\lambda_1}$ and $k\sqrt{\lambda_2}$, respectively, is

$$P\left\{\frac{u_1^2}{\lambda_1} + \frac{u_2^2}{\lambda_2} < k^2\right\} = P\{\chi^2 < k^2\} = 1 - \alpha, \quad (9.48)$$

where α is a specified level of significance. Because it is assumed that u_1 and u_2 are sampled from a normal distribution, the sum of their squares has a χ^2 distribution. See Section 9.4.1 for a description of the χ^2 distribution.

Given a value for $P = 1 - \alpha$, the value of k (or vice versa) can be determined from a table of values for the χ^2 density function. Users of MATLAB® can generate P given k^2 by using $P = \text{chi2cdf}(k^2, 2)$, and k^2 given P can be generated by $k^2 = \text{chi2inv}(P, 2)$. Commonly used values are shown in Table 9.1. Compare the probability of 39.4% associated with the 1-sigma confidence ellipse to the value of 68.3% shown in (9.30a) for the 1-sigma confidence interval in the univariate case.

Table 9.1: "k-sigma" probabilities for various confidence ellipses. $P = 1 - \alpha$.

P	0.394	0.500	0.900	0.950	0.990
k	1.000	1.177	2.146	2.448	3.035

9.3.3   Standard Empirical Error Ellipse

An empirical error ellipse differs from the confidence ellipse described above in that the covariance matrix Σ is replaced by the estimated matrix $\hat{\Sigma}$, such that

$\hat{\Sigma}^{-1} = \hat{\sigma}_0^{-2} Q^{-1}$, for example, where $\hat{\sigma}_0^2$ is an estimated variance component and Q is a given cofactor matrix. Then, for the standard empirical error ellipse ($k := 1$), rather than (9.33a), we have

$$\frac{(\boldsymbol{y} - \hat{\boldsymbol{\mu}})^T Q^{-1} (\boldsymbol{y} - \hat{\boldsymbol{\mu}})}{\hat{\sigma}_0^2} = k^2 = 1. \tag{9.49}$$

If we are evaluating $n/2$ number of 2-D points, so that Q^{-1} is of size $n \times n$, we may simply work with each of the ($n/2$ number of) 2×2 block diagonal matrices of $\hat{\sigma}_0^{-2} Q^{-1}$ independently to form the empirical error ellipse of each point. However, we must bear in mind that these block diagonal matrices do not tell the whole story since the off-block-diagonal elements have been ignored. In any case, it may be prudent to verify that the associated correlation-coefficients of the off-block-diagonal elements are relatively small in magnitude.

Note that error ellipses and confidence ellipses have the same shape, but are centered differently; namely, error ellipses in the estimated point (e.g. $(\hat{\mu}_1, \hat{\mu}_2)$) and confidence ellipses in the "true" point (e.g. (μ_1, μ_2)). Accordingly, the interpretation is that the 1-sigma error ellipse shows about 40% likelihood for the "true" point to fall inside, whereas the 1-sigma confidence ellipse shows us the area where, with about 40% likelihood, the estimated point can be found. In case that an estimated variance component is involved (e.g. $\hat{\sigma}_0^2$), the term "empirical error ellipse" is favored.

9.3.3.1   2-D examples within the Gauss-Markov Model

The following two examples apply to the Gauss-Markov Model (GMM):

1. Consider the GMM (9.20), with an associated least-squares solution and dispersion given in (9.21a). Assume that the parameter vector $\boldsymbol{\xi}$ is comprised of successive 2-D point coordinates such that $(\hat{\xi}_{2i-1}, \hat{\xi}_{2i})$ represents the coordinate estimates of the ith point. Now, also assume that we wish to compare the estimates with given (fixed) values $(\xi_{2i-1}^0, \xi_{2i}^0)$, perhaps from published results of a previous adjustment. Then we may write the following equations for the null hypothesis and the standard empirical error ellipse, where, for convenience, $k := 2i$ and $j := k - 1$ are used for indices (obviously, the index k is not the same as the constant k used in (9.33b) and (9.48)):

$$H_0 : E\{ [\hat{\xi}_j, \hat{\xi}_k]^T \} = [\xi_j^0, \xi_k^0]^T, \tag{9.50a}$$

$$\frac{1}{\hat{\sigma}_0^2} \begin{bmatrix} \hat{\xi}_j - \xi_j^0 \\ \hat{\xi}_k - \xi_k^0 \end{bmatrix}^T \begin{bmatrix} N_{j,j} & N_{j,k} \\ N_{k,j} & N_{k,k} \end{bmatrix} \begin{bmatrix} \hat{\xi}_j - \xi_j^0 \\ \hat{\xi}_k - \xi_k^0 \end{bmatrix} = 1. \tag{9.50b}$$

2. Suppose that instead of comparing the solution to given, fixed values we want to compare the results (2-D coordinate estimates) of two adjustments. Using the previously defined indices, let the estimates of the ith point of the second adjustment be represented by $(\hat{\hat{\xi}}_j, \hat{\hat{\xi}}_k)$. We ask the question: is the outcome

of the second adjustment statistically equivalent to the first? Unless there is statistically significant overlap of the respective error ellipses, the answer is no. The null hypothesis H_0 and the test statistic f are defined as follows:

$$H_0 : E\{[\hat{\xi}_j, \ \hat{\xi}_k]^T\} = E\{[\hat{\hat{\xi}}_j, \ \hat{\hat{\xi}}_k]^T\}, \tag{9.51a}$$

$$f := \frac{1/2}{\hat{\sigma}_0^2/\sigma_0^2} \begin{bmatrix} \hat{\xi}_j - \hat{\hat{\xi}}_j \\ \hat{\xi}_k - \hat{\hat{\xi}}_k \end{bmatrix}^T D\left\{ \begin{bmatrix} \hat{\xi}_j - \hat{\hat{\xi}}_j \\ \hat{\xi}_k - \hat{\hat{\xi}}_k \end{bmatrix} \right\}^{-1} \begin{bmatrix} \hat{\xi}_j - \hat{\hat{\xi}}_j \\ \hat{\xi}_k - \hat{\hat{\xi}}_k \end{bmatrix} \sim F(2, n - \mathrm{rk}\, A). \tag{9.51b}$$

Here, $1/2$ in the numerator reflects the first degrees of freedom, 2, owing to two elements of the parameter vector being tested. Also note that the unknown variance component σ_0^2 shown in the denominator cancels with the same term occurring in the dispersion matrix. Moreover, in computing the test statistic f, it is assumed that the estimated variance component $\hat{\sigma}_0^2$ is common to both adjustments. This assumption can be verified by a homogeneity test $H_0 : E\{\hat{\sigma}_0^2\} = E\{\hat{\hat{\sigma}}_0^2\}$, which is discussed in Section 9.4. Here, we also assume that the rank of matrix A is equivalent in both adjustments, which is equal to the number of unknown parameters m according to the model definition (9.20). Note that in the case that the two adjustments are uncorrelated, we could replace the inverted dispersion matrix of parameter differences with the inverse of the sum of the two respective dispersion matrices.

3. From a least-squares adjustment of a 3D network, the following variances and correlation coefficient were obtained for the estimates of the horizontal coordinates of one of the points:

$$\sigma_x^2 = (0.035)^2 \, \mathrm{m}^2, \ \ \sigma_y^2 = (0.022)^2 \, \mathrm{m}^2, \ \ \rho_{xy} = 0.31$$

Draw the empirical error ellipse for the point (assumed centered at its estimated coordinate values). What is the probability that the "true" coordinates lie within the error ellipse centered at the estimated coordinates? How does the ellipse change if, instead of the standard error ellipse, we seek the ellipse for which there is a 95% probability that the true point lies within the ellipse centered on the estimated point?

Table 9.2: Solution for standard empirical error ellipse

semimajor axis length	$a = 0.035\,989$ m
semiminor axis length	$b = 0.020\,341$ m
rotation angle	$\theta = 16.396123°$
probability	39.4%
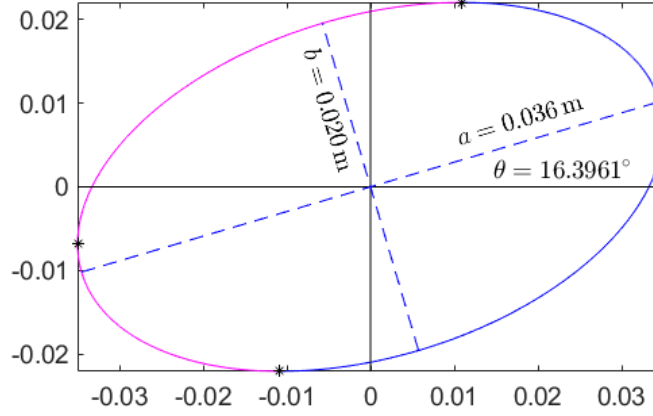
Figure 9.5: Standard empirical error ellipse

According to Table 9.1, the semimajor and semiminor axes lengths would increase by a factor of 2.447 for a 95% error ellipse. However, the orientation of the ellipse would not change.

9.3.4 Confidence Ellipsoids and Hyperellipsoids — Multivariate Case

In the 3-D case, confidence ellipses are extended to confidence ellipsoids. But, in our general formulation of the GMM we may be working with a higher-dimensional space, and thus we speak of confidence hyperellipsoids. Since 3-D and higher dimensions are natural extensions of the 2-D case, no further discussion is necessary. However, we do list probabilities associated with confidence ellipsoids for the 3-D case in Table 9.3. The table entries can be generated using the same MATLAB® commands shown in the previous section, except that the second argument must be 3 (degrees of freedom) instead of 2.

Table 9.3: "k-sigma" probabilities for various confidence ellipsoids. $P = 1 - \alpha$.

P	0.199	0.500	0.900	0.950	0.990
k	1.000	1.538	2.500	2.796	3.365

9.4 χ^2-distribution, Variance Testing, and F-distribution

This section includes the statistical topics of χ^2- and F-distributions as well as the topic of variance testing.

9.4.1 χ^2-Distribution

The χ^2-distribution is attributed to the German geodesist F.R. Helmert from 1876. If we claim that the (unknown) random error vector e from the GMM is normally distributed as $e \sim \mathcal{N}(\mathbf{0}, \sigma_0^2 P^{-1})$, then the quadratic product $e^T Pe$ has a χ^2-distribution with $\nu := \operatorname{rk} P = n$ *degrees of freedom*, expressed by

$$\frac{e^T Pe}{\sigma_0^2} \sim \chi^2(\nu). \tag{9.52}$$

Now, define $x := e^T Pe/\sigma_0^2$ (which cannot actually be computed since both e and σ_0^2 are unknown). Therefore, the pdf of x is written as

$$f(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu-2)/2} e^{-x/2} & \text{for } x > 0 \\ 0 & \text{for } x \le 0, \end{cases} \tag{9.53}$$

where e is Euler's number $2.71828\ldots$ The gamma function $\Gamma(\cdot)$ was defined in (9.12). Figure 9.6 shows plots of the χ^2-distribution for $\nu = \{1, 3, 5, 8, 10, 30\}$ with respective colors: black, magenta, cyan, red, green, blue. Note that the peaks of the curves move to the right as ν increases and that the curves appear to approximate the normal-distribution curve as ν grows to 10 and larger. This agrees with our expectation that the χ^2-distribution is *asymptotically normal*, due to the *central limit theorem*.[2]

From the variance component derivations in Section 3.3, we have $E\{e^T Pe\} = n \cdot \sigma_0^2$, and, furthermore, we can write

$$E\{e^T Pe/\sigma_0^2\} = \operatorname{tr}\big(P \cdot E\{ee^T\}/\sigma_0^2\big) = \operatorname{tr} I_n = n, \tag{9.54a}$$

$$E\{\tilde{e}^T P\tilde{e}/\sigma_0^2\} = \operatorname{tr}\big(P \cdot E\{\tilde{e}\tilde{e}^T\}/\sigma_0^2\big) = \operatorname{tr}\big(I_n - AN^{-1}A^T P\big) = n - \operatorname{rk} A = n - m. \tag{9.54b}$$

Then, with the help of (9.25) and (9.54b), the relation

$$\tilde{e}^T P\tilde{e}/\sigma_0^2 = \nu\hat{\sigma}_0^2/\sigma_0^2 \sim \chi^2(\nu), \tag{9.55a}$$

is established, with

$$\nu := n - m \tag{9.55b}$$

as the degrees of freedom (usually denoted r for redundancy elsewhere in these notes).

Note that though we have been discussing the random error vector e and the predicted residual $\tilde{e}$, the relations expressed in (9.55a) apply to all quadratic forms in normally distributed variables. Thus, when we have a vector of normally distributed variables, the corresponding quadratic form will have a χ^2-distribution.

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[2]According to Bjerhammar (1973, Section 2.15), the central limit theorem says that "the sum of n independent stochastic variables having equal expectation and variance will have a distribution that converges towards the normal distribution for $n \to \infty$."
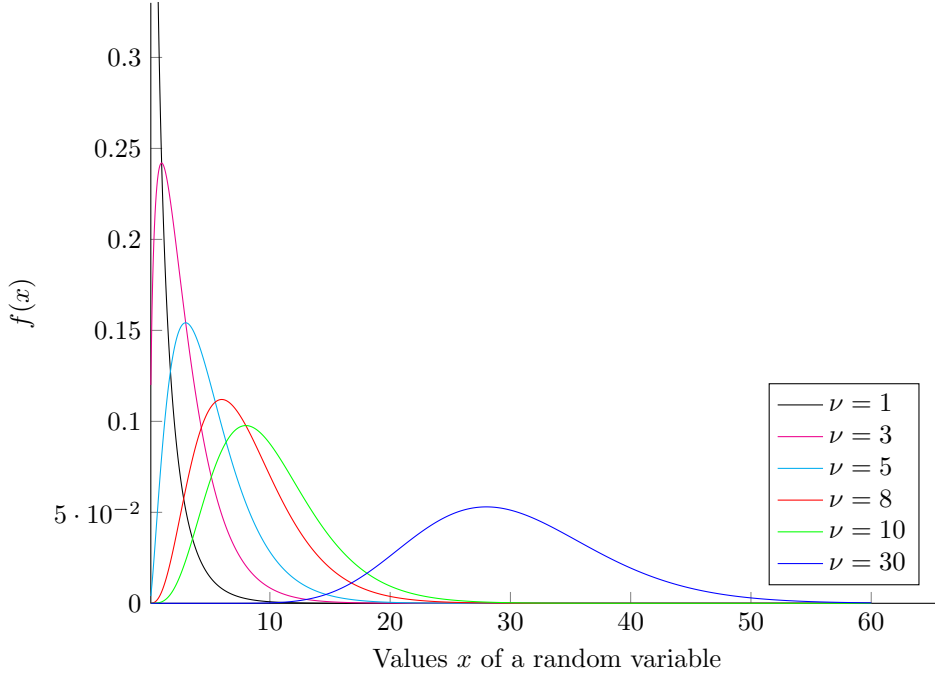
Figure 9.6: Curve of χ^2-distribution with various degrees of freedom ν

9.4.2   Variance Testing

Suppose we want to compare the estimated variance component $\hat{\sigma}_0^2$ to a given quantity σ^2 (in the latter, the 0-subscript is not used so as not to confuse the given value with the unknown "true value"). We do so by performing the following hypothesis test at a chosen significance level α (e.g., $\alpha = 0.05$):

$$H_0 : E\{\hat{\sigma}_0^2\} \leq \sigma^2 \ \text{ vs. } \ H_A : E\{\hat{\sigma}_0^2\} > \sigma^2 \tag{9.56a}$$

$$t := (n-m) \cdot \left(\hat{\sigma}_0^2/\sigma^2\right) \sim \chi^2(n-m) \tag{9.56b}$$

$$\text{If } t \leq \chi^2_{\alpha,n-m} \ \text{ accept } H_0; \text{ else reject } H_0. \tag{9.56c}$$

The test as shown is referred to as a one-tailed test, because the null hypothesis only states that the expectation of the estimated quantity is less than or equal to a given value (the use of $\geq$ in H_0 would also constitute a one-tailed test). In contrast, a two-tailed test would require an equals sign in the null hypothesis. The jargon one- and two-tailed comes from the fact that $1-\alpha$ represents the area under the pdf curve left of the right tail in the one-tailed case, and it represents the area between both the left and right tails (each of which have area $\alpha/2$) in the two-tailed case.

Under the assumption that the data model is correct, if the estimate $\hat{\sigma}_0^2$ turns out statistically to be less than the given value σ^2, we deem our measurements to be more precise than that reflected in the weight matrix P. On the other hand, if $\hat{\sigma}_0^2$

proves statistically to be greater than the given value, we deem our measurements to be less precise. Usually our main concern is that $\hat{\sigma}_0^2$ reflects that our measurements are at least as precise as what is reflected by the elements of the weight matrix P, thus the use of a single-tailed hypothesis may be more commonly used in practice.

On the other hand, if we need to test for equality between the estimated variance component $\hat{\sigma}_0^2$ and a chosen value σ^2, the above hypothesis test should be modified to depend on $\alpha/2$ as follows:

$$H_0 : E\{\hat{\sigma}_0^2\} = \sigma^2 \ \ \text{vs.} \ \ H_A : E\{\hat{\sigma}_0^2\} \neq \sigma^2 \tag{9.57a}$$

$$t := (n - m) \cdot \left(\hat{\sigma}_0^2/\sigma^2\right) \sim \chi^2(n - m) \tag{9.57b}$$

$$\text{If } \chi_{1-\alpha/2,n-m}^2 < t < \chi_{\alpha/2,n-m}^2 \ \ \text{accept } H_0; \text{ else reject } H_0. \tag{9.57c}$$

Note: Some tables of the χ^2 distribution list *percentiles* that equal the area under the curve less than $\chi_{p,\text{df}}^2$ rather than the area under the curve right of $\chi_{\alpha,\text{df}}^2$ shown in other tables (where df stands for degrees of freedom, sometimes denoted as ν). Either type of table can be used as long as the relationship $p = 1 - \alpha$ is considered.

In the case where we need to compare two estimated reference variances $\hat{\sigma}_{0,1}^2$ and $\hat{\sigma}_{0,2}^2$ from two independent adjustments, we must compute a ratio of test statistics, which has an F-distribution (assuming both the numerator and denominator have χ^2-distributions). Let t_1 and t_2 be the test statistics from the respective adjustments; then we can write

$$\frac{t_1/(n_1 - m_1)}{t_2/(n_2 - m_2)} = \hat{\sigma}_{0,1}^2/\hat{\sigma}_{0,2}^2 \sim F(n_1 - m_1, n_2 - m_2), \tag{9.58}$$

where $n_i - m_i$, $i = 1, 2$, are the respective degrees of freedom of the two independent adjustments.

9.4.3 F-Distribution

The F-distribution was named for its discover R.A. Fisher (1925) by G.W. Snedacor (1935). It is a distribution for the ratio of two mutually independent random variables that have χ^2-distributions with degrees of freedom $v_1 := m$ and $v_2 := n - m$, respectively. The pdf of such a variable is given by

$$f(w) = \frac{\Gamma\left(\frac{m}{2} + \frac{n-m}{2}\right)m^{m/2}(n - m)^{(n-m)/2}w^{(m/2)-1}}{\Gamma(\frac{m}{2})\Gamma\left(\frac{n-m}{2}\right)(n - m + mw)^{(m/2+(n-m)/2)}} = \tag{9.59a}$$

$$= \frac{(v_1/v_2)^{v_1/2}\Gamma\left((v_1 + v_2)/2\right)w^{(v_1/2)-1}}{\Gamma(v_1/2)\Gamma(v_2/2)\left(1 + v_1w/v_2\right)^{(v_1+v_2)/2}}. \tag{9.59b}$$

As n becomes large compared to m, the curve of the F-distribution approaches the curve of the normal distribution.

9.5 Hypothesis Testing on the Estimated Parameters

In the GMM, we may wish to perform a global model-check by comparing a specified parameter vector $\boldsymbol{\xi}^0$ to the estimated vector $\hat{\boldsymbol{\xi}}$. In such a case, we may use as the test statistic the ratio of weighted norms of the difference vector $\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0$ and the predicted residual vector $\tilde{\boldsymbol{e}}$ as follows:

$$w := \frac{(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)^T A^T P A (\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)}{\sigma_0^2 m} \cdot \frac{\sigma_0^2 (n-m)}{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}} \sim F(m, n-m). \tag{9.60}$$

Here we have assumed that matrix A has full rank, i.e., $\operatorname{rk} A = m$. Since the numerator and denominator are statistically independent of one another, the test statistic w has an F-distribution with m and $n-m$ degrees of freedom, as shown in (9.60). Therefore, our global model-check is made by the following hypothesis test:

$$H_0 : E\{\hat{\boldsymbol{\xi}}\} = \boldsymbol{\xi}^0 \quad \text{vs.} \quad H_A : E\{\hat{\boldsymbol{\xi}}\} \neq \boldsymbol{\xi}^0 \tag{9.61a}$$

$$\text{If } w \leq F_{\alpha, m, n-m} \text{ accept } H_0; \text{ else reject } H_0. \tag{9.61b}$$

We now show that the numerator and denominator of w are indeed independent, as required for use of the F-distribution. To do so, we only need to show that

$$C\{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}, (\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})^T (A^T P A)(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})\} = 0. \tag{9.62}$$

Note that, without loss of generality, we have replaced $\boldsymbol{\xi}^0$ with $\boldsymbol{\xi}$. From (4.5e) we have $\tilde{\boldsymbol{e}} = [I_n - AN^{-1}A^T P]\boldsymbol{e}$. Therefore,

$$\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}} = \boldsymbol{e}^T [I_n - PAN^{-1}A^T] P [I_n - AN^{-1}A^T P] \boldsymbol{e} =$$
$$= \boldsymbol{e}^T [P - PAN^{-1}A^T P] \boldsymbol{e} =: \boldsymbol{e}^T M_1 \boldsymbol{e}. \tag{9.63a}$$

Also

$$A(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}) = \boldsymbol{e} - \tilde{\boldsymbol{e}} = \boldsymbol{e} - (I_n - AN^{-1}A^T P)(A\boldsymbol{\xi} + \boldsymbol{e}) = (AN^{-1}A^T P)\boldsymbol{e} \Rightarrow \tag{9.63b}$$
$$(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})^T (A^T P A)(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}) = \boldsymbol{e}^T (PAN^{-1}A^T) P (AN^{-1}A^T P)\boldsymbol{e} = \tag{9.63c}$$
$$= \boldsymbol{e}^T (PAN^{-1}A^T P)\boldsymbol{e} =: \boldsymbol{e}^T M_2 \boldsymbol{e}. \tag{9.63d}$$

By substitution of (9.63a) and (9.63d), the condition (9.62) is equivalent to the condition that $\boldsymbol{e}^T M_1 \boldsymbol{e}$ and $\boldsymbol{e}^T M_2 \boldsymbol{e}$ are independent, which holds if, and only if,

$$M_1 D\{\boldsymbol{e}\} M_2 = 0 \tag{9.63e}$$

(cf. Searle and Khuri (2017, Theorem 10.3)), which is true since

$$(P - PAN^{-1}A^T P)(\sigma_0^2 P^{-1})(PAN^{-1}A^T P) = 0. \tag{9.63f}$$

9.6 Checking an Individual Element (or 2-D or 3-D Point) of the Parameter Vector

We may use an $l \times m$ matrix K to select a subset of size l from the $m \times 1$ vector of estimated parameters $\hat{\boldsymbol{\xi}}$ for hypothesis testing as follows:

$$H_0 : E\{K\hat{\boldsymbol{\xi}}\} = K\boldsymbol{\xi}^0 = \boldsymbol{\kappa}_0, \tag{9.64a}$$

$$H_A : E\{K\hat{\boldsymbol{\xi}}\} = K\boldsymbol{\xi}^0 \neq \boldsymbol{\kappa}_0. \tag{9.64b}$$

If $l = 1$, K is a unit row vector that extracts the relevant element from the parameter vector, in which case κ_0 is simply a scalar quantity. The following examples show the matrix K used for extracting a single element, a 2-D point, and a 3-D point, respectively:

$$K := \begin{bmatrix} 0, \ldots, 0, 1, 0, \ldots, 0 \end{bmatrix}, \quad \text{where 1 appears at the } j\text{th element;} \tag{9.65a}$$

$$K := \begin{bmatrix} 0_2, \ldots, 0_2, I_2, 0_2, \ldots, 0_2 \end{bmatrix}, \quad \text{where } K \text{ is size } 2 \times m; \tag{9.65b}$$

$$K := \begin{bmatrix} 0_3, \ldots, 0_3, I_3, 0_3, \ldots, 0_3 \end{bmatrix}, \quad \text{where } K \text{ is size } 3 \times m. \tag{9.65c}$$

For 2-D and 3-D points, the subscripts denote the dimension of the square sub-matrices (zero matrix or identity matrix), and I_n ($n \in \{2, 3\}$) is the jth sub-matrix of K, which means it "selects" the jth point from $\hat{\boldsymbol{\xi}}$.

The test statistic is then defined as

$$w := \frac{\left[K(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)\right]^T D\{K(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)\}^{-1} \left[K(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)\right]/\operatorname{rk} K}{\hat{\sigma}_0^2/\sigma_0^2} = \tag{9.66a}$$

$$= \frac{\left[K\hat{\boldsymbol{\xi}} - \boldsymbol{\kappa}_0\right]^T \left[KN^{-1}K^T\right]^{-1} \left[K\hat{\boldsymbol{\xi}} - \boldsymbol{\kappa}_0\right]/l}{\hat{\sigma}_0^2} =: \frac{R/l}{(\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}})/(n - m)}. \tag{9.66b}$$

Note that σ_0^2 appears in the denominator of (9.66a) in order to cancel out the same term hidden inside the dispersion matrix in the numerator. Also note that since $\boldsymbol{\xi}^0$ is a specified (and therefore non-random) quantity to test against, the dispersion is not affected by it, i.e.,

$$D\{K(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)\} = D\{K\hat{\boldsymbol{\xi}}\} = \sigma_0^2 K N^{-1} K^T. \tag{9.67}$$

The symbols R and Ω are used for convenience and are analogous to the symbols introduced in Sections 5.5 and 6.4, respectively. They are statistically independent of one another and have the following distributions:

$$R \sim \chi^2(l), \ \ \Omega \sim \chi^2(n - m). \tag{9.68}$$

Statistical independence between the random variables R and Ω means that their joint pdf is equivalent to the product of their individual pdf's: $f(R, \Omega) = f(R) \cdot f(\Omega)$. Independence can be shown by following the same line of thought as that used at the end of the previous section, where M_1 remains unchanged and M_2 is now

$PAN^{-1}K^T[KN^{-1}K^T]^{-1}KN^{-1}A^TP$. Therefore, the test statistic (9.66b) has an F-distribution represented by

$$w \sim F(l, n - m). \tag{9.69}$$

An alternative, more compact, form for w when $l = 1$ is given by

$$w = \frac{(\hat{\xi}_j - (\kappa_0)_j)^2}{\hat{\sigma}_0^2 (N^{-1})_{jj}} \sim F(1, n - m). \tag{9.70}$$

The decision to accept or reject the null hypothesis is made analogous to (9.61b).

9.6.1 Non-central F-Distribution

If the null hypothesis H_0 is false, the test statistic w is said to have a non-central F-distribution (denoted here as F'), which requires a non-centrality parameter θ so that $w \sim F'(v_1, v_2, \theta)$ under H_A, where v_1 and v_2 have been used to denote the degrees of freedom, in general. The qualification "under H_A" implies that we must pose a specific alternative hypothesis H_A in this case, rather than just the negation of H_0. For a one-tailed test, the area under the non-central F-distribution curve and to the right of F_α (from the F-distribution table) is denoted as β. The value of β is also the probability of making an *error of the second kind*, namely to accept the null hypothesis H_0 when the specified alternative hypothesis H_A is actually true. The quantity $1 - \beta$ is known as the *power of the test*. As the value of θ increases, so does the value $1 - \beta$. Below we have rewritten (9.69) for the non-central case, with the theoretical formula for 2θ following.

$$w \sim F'(l, n - m, \theta) \tag{9.71a}$$

$$2\theta = (K\boldsymbol{\xi} - \boldsymbol{\kappa}_0)^T (KN^{-1}K^T)^{-1} (K\hat{\boldsymbol{\xi}} - \boldsymbol{\kappa}_0) \tag{9.71b}$$

Note that the non-centrality property is reflected in (9.71b) by including both the true (unknown) vector of parameters $\boldsymbol{\xi}$ and its estimate $\hat{\boldsymbol{\xi}}$ in bilinear form.

9.7 Detection of a Single Outlier in the Gauss-Markov Model

A model that expresses the jth observation as a potential outlier can be written as

$$y_j = \boldsymbol{a}_j^T \boldsymbol{\xi}^{(j)} + \xi_0^{(j)} + e_j. \tag{9.72}$$

The terms of the model are described as follows:

y_j is the jth element of the $n \times 1$ observation vector $\boldsymbol{y}$.

$\boldsymbol{a}_j$ is an $m \times 1$ column vector that is comprised of the m elements of the jth row of matrix A so that $[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n]^T := A$.

$\boldsymbol{\xi}^{(j)}$ denotes the $m \times 1$ vector of unknown parameters associated with that set of observations whose jth element is considered an outlier, as opposed to $\boldsymbol{\xi}$, which is associated with the same set of observations except that the jth one is not considered as an outlier.

e_j is the jth element of the unknown random error vector $\boldsymbol{e}$.

$\xi_0^{(j)}$ is an unknown (scalar) parameter that accounts for an outlier. In other words, it accounts for a non-random error in the observation. The formula for its estimate is developed below.

The following example may be illustrative: Suppose the observation y_j should have been $100\,\text{m}$ but only a value of $10\,\text{m}$ was recorded, then $\xi_0^{(j)}$ accounts for a $90\,\text{m}$ blunder.

A modified GMM whose jth observation might be deemed an outlier is expressed as

$$\underset{n \times 1}{\boldsymbol{y}} = \underset{n \times m}{A}\, \boldsymbol{\xi}^{(j)} + \underset{n \times 1}{\boldsymbol{\eta}_j}\, \xi_0^{(j)} + \boldsymbol{e}, \ \ \boldsymbol{\eta}_j := \big[0,\ldots,0,1,0,\ldots,0\big]^T, \tag{9.73a}$$

$$\boldsymbol{e} \sim \mathcal{N}(\boldsymbol{0}, \sigma_0^2 P^{-1}). \tag{9.73b}$$

Note that the number 1 in $\boldsymbol{\eta}_j$ appears at the jth element; all other elements are 0. We must compare the model in (9.73) with the original GMM (3.1), which is assumed to not include an outlier. Since the model (9.73) assumes only one outlier in the data set, n comparisons of the two models are necessary in order to test all y_i $(i = 1, \ldots, n)$ observations independently. For each comparison we introduce the constraint equation

$$\xi_0^{(j)} = K \begin{bmatrix} \boldsymbol{\xi}^{(j)} \\ \xi_0^{(j)} \end{bmatrix} = \kappa_0 = 0. \tag{9.74}$$

Here $K := [0, 0, \ldots, 1]$ is of size $1 \times (m+1)$. When we impose the constraint (9.74) upon the model (9.73), we obtain a model equivalent to the original GMM (3.1) that does not include an additional parameter to model an outlier.

Diagonal weight matrix P assumed   *Note: For the remainder of this section, we will assume that the weight matrix P is diagonal: $P = \text{diag}(p_1, \ldots, p_n)$, where p_i is the weight of the ith observation.* See Schaffrin (1997b) for a treatment of outlier detection with correlated observations.

Now, we begin with the following Lagrange target function to derive a least-squares estimator in the unconstrained model (9.73):

$$\Phi\big(\boldsymbol{\xi}^{(j)}, \xi_0^{(j)}\big) = \big(\boldsymbol{y} - A\boldsymbol{\xi}^{(j)} - \boldsymbol{\eta}_j\xi_0^{(j)}\big)^T P\big(\boldsymbol{y} - A\boldsymbol{\xi}^{(j)} - \boldsymbol{\eta}_j\xi_0^{(j)}\big), \tag{9.75}$$

which is made stationary with respect to $\boldsymbol{\xi}^{(j)}$ and $\xi_0^{(j)}$ by setting the first partial derivatives of (9.75) to zero, resulting in the following Euler-Lagrange necessary

conditions:

$$\frac{1}{2}\left[\frac{\partial\Phi}{\partial\boldsymbol{\xi}^{(j)}}\right]^T = -A^T P\boldsymbol{y} + A^T P\boldsymbol{\eta}_j\hat{\xi}_0^{(j)} + A^T PA\hat{\boldsymbol{\xi}}^{(j)} \doteq \boldsymbol{0}, \qquad (9.76a)$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\xi_0^{(j)}} = -\boldsymbol{\eta}_j^T P\boldsymbol{y} + \boldsymbol{\eta}_j^T PA\hat{\boldsymbol{\xi}}^{(j)} + \boldsymbol{\eta}_j^T P\boldsymbol{\eta}_j\hat{\xi}_0^{(j)} \doteq 0. \qquad (9.76b)$$

Of course the second partial derivatives are functions of P, which is positive definite by definition, thereby satisfying the sufficient condition required for obtaining the minimum of (9.75). In matrix form we have

$$\begin{bmatrix} N & A^T P\boldsymbol{\eta}_j \\ \boldsymbol{\eta}_j^T PA & \boldsymbol{\eta}_j^T P\boldsymbol{\eta}_j \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{\xi}}^{(j)} \\ \hat{\xi}_0^{(j)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{\eta}_j^T P\boldsymbol{y} \end{bmatrix}, \qquad (9.77a)$$

or, because P was assumed to be diagonal,

$$\begin{bmatrix} N & \boldsymbol{a}_j p_j \\ p_j\boldsymbol{a}_j^T & p_j \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{\xi}}^{(j)} \\ \hat{\xi}_0^{(j)} \end{bmatrix} =: N_1\begin{bmatrix} \hat{\boldsymbol{\xi}}^{(j)} \\ \hat{\xi}_0^{(j)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ p_j y_j \end{bmatrix}. \qquad (9.77b)$$

Here, as in previous chapters, we have used the definition $[N, \boldsymbol{c}] := A^T P[A, \boldsymbol{y}]$. Using (A.15) for the inverse of a partitioned matrix, and decomposing the resulting inverse into a sum of two matrices, results in

$$\begin{bmatrix} \hat{\boldsymbol{\xi}}^{(j)} \\ \hat{\xi}_0^{(j)} \end{bmatrix} = \begin{bmatrix} N^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{c} \\ p_j y_j \end{bmatrix} + \begin{bmatrix} N^{-1}\boldsymbol{a}_j p_j \\ -1 \end{bmatrix}\left(p_j - p_j\boldsymbol{a}_j^T N^{-1}\boldsymbol{a}_j p_j\right)^{-1}\cdot$$
$$\cdot\begin{bmatrix} p_j\boldsymbol{a}_j^T N^{-1} & -1 \end{bmatrix}\begin{bmatrix} \boldsymbol{c} \\ p_j y_j \end{bmatrix}, \qquad (9.78a)$$

or

$$\begin{bmatrix} \hat{\boldsymbol{\xi}}^{(j)} \\ \hat{\xi}_0^{(j)} \end{bmatrix} = \begin{bmatrix} N^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{c} \\ p_j y_j \end{bmatrix} - \begin{bmatrix} N^{-1}\boldsymbol{a}_j p_j \\ -1 \end{bmatrix}\left(p_j - p_j^2\boldsymbol{a}_j^T N^{-1}\boldsymbol{a}_j\right)^{-1}p_j\left(y_j - \boldsymbol{a}_j^T N^{-1}\boldsymbol{c}\right). \qquad (9.78b)$$

From (9.78b), and recalling that $\hat{\boldsymbol{\xi}} = N^{-1}\boldsymbol{c}$ is associated with a data model that assumes no outliers, we can write the following difference between estimations:

$$\hat{\boldsymbol{\xi}}^{(j)} - \hat{\boldsymbol{\xi}} = -N^{-1}\boldsymbol{a}_j\left(\frac{y_j - \boldsymbol{a}_j^T\hat{\boldsymbol{\xi}}}{p_j^{-1} - \boldsymbol{a}_j^T N^{-1}\boldsymbol{a}_j}\right) = -N^{-1}\boldsymbol{a}_j\frac{\tilde{e}_j}{(Q_{\tilde{\boldsymbol{e}}})_{jj}}, \qquad (9.79)$$

where $(Q_{\tilde{\boldsymbol{e}}})_{jj}$ is the jth diagonal element of the cofactor matrix for the residual vector $\tilde{\boldsymbol{e}}$. For the estimated non-random error in the observation y_j we have

$$\hat{\xi}_0^{(j)} = \frac{y_j - \boldsymbol{a}_j^T\hat{\boldsymbol{\xi}}}{1 - p_j\boldsymbol{a}_j^T N^{-1}\boldsymbol{a}_j} = \frac{\tilde{e}_j}{(Q_{\tilde{\boldsymbol{e}}}P)_{jj}} = \frac{\tilde{e}_j/p_j}{(Q_{\tilde{\boldsymbol{e}}})_{jj}}, \qquad (9.80)$$

where the last equality holds since P was said to be diagonal.

The hypothesis test for the jth observation being an outlier is then written as

$$H_0 : E\{\hat{\xi}_0^{(j)}\} = 0 \ \text{ versus } \ H_A : E\{\hat{\xi}_0^{(j)}\} \neq 0. \tag{9.81}$$

The test statistic has an F-distribution and is computed by

$$T_j = \frac{R_j/1}{(\Omega - R_j)/(n - m - 1)} \sim F(1, n - m - 1). \tag{9.82}$$

The definition of R_j, in terms of $\hat{\xi}_0^{(j)}$, is

$$R_j := \frac{\left(\hat{\xi}_0^{(j)} - 0\right)^2}{KN_1^{-1}K^T} = \frac{\left(\hat{\xi}_0^{(j)}\right)^2}{\left(p_j - p_j^2 \boldsymbol{a}_j^T N^{-1} \boldsymbol{a}_j\right)^{-1}} = \frac{\tilde{e}_j^2}{(Q_{\tilde{e}}P)_{jj}^2} p_j (Q_{\tilde{e}}P)_{jj} = \frac{\tilde{e}_j^2}{(Q_{\tilde{e}})_{jj}}. \tag{9.83}$$

The matrix N_1 was defined as

$$N_1 := \begin{bmatrix} N & \boldsymbol{a}_j p_j \\ p_j \boldsymbol{a}_j^T & p_j \end{bmatrix}, \tag{9.84}$$

in (9.77b). Pre- and post-multiplying N_1^{-1} by K extracts only its last diagonal element, which, upon applying the formula for the inverse of a partitioned matrix, turns out to be the scalar quantity $(p_j - p_j^2 \boldsymbol{a}_j^T N^{-1} \boldsymbol{a}_j)^{-1}$.

It is important to note that the symbols $\tilde{e}$ and $Q_{\tilde{e}}$ appearing in (9.83) represent the residual vector and its cofactor matrix, respectively, as predicted within the GMM (3.1) — see (3.9) and (3.14c). As was already mentioned, when we impose the constraint (9.74) on model (9.73b) we reach a solution identical to the LESS within model (3.1). It is also important to understand the terms in the denominator of (9.82). As stated previously, the symbol R is used to account for that portion of the P-weighted residual norm due to the constraints. The first parenthetical term in the denominator, $(\Omega - R_j)$, accounts for that part of the norm coming from the unconstrained solution. Here we have used $\Omega := \tilde{e}^T P \tilde{e}$, with $\tilde{e}$ belonging to the constrained solution (determined within the model (3.1)). Therefore, we must subtract R from Ω, as it is defined here, to arrive at the portion of the norm coming from the unconstrained LESS computed within model (9.73).

We note again that the equations from (9.77b) *to* (9.83) *hold only in the case of a diagonal weight matrix* P. Regardless of whether or not P is diagonal, the quantity

$$r_j := (Q_{\tilde{e}}P)_{jj} \tag{9.85a}$$

is the jth so-called *redundancy number*, for the unconstrained solution in this case. The following properties hold for r_j:

$$0 < r_j \leq 1 \ \text{ for } j = \{1, \ldots, n\} \ \text{ and } \ \sum_j r_j = n - \text{rk}\, A. \tag{9.85b}$$

Note that $(Q_{\tilde{e}}P)_{jj} = p_j \cdot (Q_{\tilde{e}})_{jj}$ for the case that matrix P is diagonal, as reflected in (9.80).

We comment that outlier detection at the 2-D and 3-D level can also be performed, for example, in testing whether observed 2-D and 3-D points are outliers. The 3-D case is also appropriate for GPS baseline adjustments. Its development is shown by Snow (2002); see also Snow and Schaffrin (2003).

A strategy for outlier detection   Since the model (9.73) only accounts for an outlier at the jth observation. A strategy is needed to check for outliers at all observations including the case where more than one outlier might be present in the observation vector $\boldsymbol{y}$. The usual way of approaching this problem is to perform n independent outlier tests, allowing j to run from 1 to n. If the null hypothesis cannot be accepted for one or more of the tests, the observation associated with the largest value for the test statistic T_j is flagged as a potential outlier and removed from the observation vector $\boldsymbol{y}$. The entire process is repeated until the null hypothesis can be accepted for all remaining observations, with n being reduced by 1 for each successive set of tests.

To be more conservative, after each set of tests that results in an observation being flagged as a potential outlier and removed, the previously removed observations are added back in one at a time (in the opposite order they were removed) to see if they can remain in the observation vector or if they once again must be removed. Eventually, one would hope to reach a point where all outliers have been detected and removed, implying that finally the null hypothesis can be accepted for all remaining residuals.

The reason for this conservative step is that an outlier at the jth element of the observation vector may result in a larger test statistic for some residuals other than $\tilde{e}_j$. To see how this could be, we repeat the formula for the vector of residuals shown in (3.9):

$$\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}} = \left(I_n - AN^{-1}A^T P\right)\boldsymbol{y} = Q_{\tilde{e}}P\boldsymbol{y} =: R\boldsymbol{y}, \tag{9.86a}$$

where the symbol R has been used to denote the matrix whose diagonal elements are the so-called redundancy numbers as shown in (9.85a). If R is expressed as matrix of column vectors, viz. $R = [\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n]$, then it is easy to see that

$$\tilde{\boldsymbol{e}} = \boldsymbol{r}_1 \cdot y_1 + \boldsymbol{r}_2 \cdot y_2 + \cdots + \boldsymbol{r}_n \cdot y_n, \tag{9.86b}$$

revealing that each element of $\tilde{\boldsymbol{e}}$ is potentially a linear combination of all the elements of $\boldsymbol{y}$ (since R is not expected to be a diagonal matrix, in general). This means that an outlier at the jth element of $\boldsymbol{y}$ could "bleed into" residuals other than $\tilde{e}_j$, perhaps giving the impression that some different observation is an outlier, when in fact it is not. This linear relationship between the residuals and the observations attests to the challenge of successful outlier detection after an adjustment. It may or may not succeed in identifying all outliers and in avoiding wrongly misidentifying some observations as outliers when they in fact are not. The challenge of successful outlier detection underscores the importance of avoiding making errors in observations and in finding strategies to find all blunders in the data before an adjustment is performed.

Chapter 10

Answers to Practice Problems

The following list contains partial answers to selected practice problems.

Chapter 2 TODO

Chapter 3

3.a; 3.c $\hat{\xi} = 110.1176\,\text{m}$; $\hat{\sigma}_0^2 = 2.205883$.

4.a; 4.b $\hat{a} = 0.00252$, $\hat{b} = 0.00288$, $\hat{c} = 9.98620$, $\hat{\sigma}_0^2 = (1.987)^2$; $\hat{a} = -6.1 \times 10^{-5}$, $\hat{b} = -5.6 \times 10^{-5}$, $\hat{c} = 9.9 \times 10^{-6}$, $\hat{d} = 2.52 \times 10^{-3}$, $\hat{e} = 2.88 \times 10^{-3}$, $\hat{f} = 10.010$, $\hat{\sigma}_0^2 = 1.407^2$.

5.a; 5.d $\hat{P}_x = 72.997\,\text{m}$, $\hat{P}_y = 92.009\,\text{m}$; $\hat{\sigma}_0^2 = (0.690)^2$.

6.b; 6.b $\hat{P}_x = 1065.201\,\text{m}$, $\hat{P}_y = 825.198\,\text{m}$; $\hat{\sigma}_0^2 = (1.758)^2$.

7.a; 7.b $\hat{\xi}_1 = -0.5396$ (slope), $\hat{\xi}_2 = 5.7612$ (y-intercept); $\hat{\sigma}_0^2 = (0.316)^2$.

8.a $\hat{\xi}_1 = 168.149\,\text{mm}$, $\hat{\xi}_2 = 160.300\,\text{mm}$, $\hat{\omega}_1 = 1.000011$, $\hat{\omega}_2 = 1.000021$, $\hat{\beta} = 0°12'22.0''$, $\widehat{\beta + \epsilon} = 00°13'08.5''$.

9.b; 9.c $\hat{H}_A = 1679.509\,\text{ft}$, $\hat{H}_B = 1804.043\,\text{ft}$, $\hat{H}_C = 2021.064\,\text{ft}$, $\hat{H}_E = 1507.075\,\text{ft}$, $\hat{H}_F = 1668.148\,\text{ft}$, $H_D^0 = 1928.277\,\text{ft}$; $\hat{\sigma}_0^2 = (0.081)^2$.

9.e $\hat{H}_B = 1803.966\,\text{ft}$, $\hat{H}_C = 2020.986\,\text{ft}$, $\hat{H}_D = 1928.200\,\text{ft}$, $\hat{H}_E = 1506.998\,\text{ft}$, $\hat{H}_F = 1668.071\,\text{ft}$, $H_A^0 = 1679.432\,\text{ft}$, $\hat{\sigma}_0^2 = (0.081)^2$.

Chapter 4

3. $\tilde{e} = [8.1, 8.8, -5.3, 3.4, -8.8, -9.4]^T \text{arcsec}$, $\hat{\sigma}_0^2 = (0.879453)^2$,

$$Q = \begin{bmatrix} 200 & 0 & 0 & 0 & 0 & 0 \\ 0 & 200 & -100 & 0 & 0 & 0 \\ 0 & -100 & 200 & 0 & 0 & 0 \\ 0 & 0 & 0 & 200 & -100 & 0 \\ 0 & 0 & 0 & -100 & 200 & 0 \\ 0 & 0 & 0 & 0 & 0 & 200 \end{bmatrix} \text{arcsec}^2 \text{ (to be converted)}.$$

4. $\hat{\mu}_{y_4} = 500.214 \,\text{m} \pm 5\,\text{mm}$.

5. $\hat{\sigma}_0^2 = (1.1321)^2$, $Q = \begin{bmatrix} 200 & -100 & 0 & 0 & 0 \\ -100 & 200 & 0 & 0 & 0 \\ 0 & 0 & 200 & 0 & 0 \\ 0 & 0 & 0 & 200 & 0 \\ 0 & 0 & 0 & 0 & 200 \end{bmatrix} \text{arcsec}^2 \text{ (to be converted)}.$

Chapter 5

5.a; 5.b $r = 3$, $\hat{\sigma}_0^2 = (0.015)^2$; $r = 4$, $\hat{\sigma}_0^2 = (0.013)^2$.

6.a; 5.b $r = 3$, $\hat{P}_1 = (589.979, 374.998)\,\text{m}$.

7. $\hat{a} = -0.00735466$, $\Omega = 7.57541$, $R = 0.162439$.

Chapter 6

3.a; 3.b $r = 4$, $\hat{\sigma}_0^2 = (0.08063)^2$; $r = 5$, $\hat{\sigma}_0^2 = (0.07305)^2$, $T = 0.104487$.

4. $r = 3$, $\hat{\sigma}_0^2 = (4.599140)^2$, $T = 33.07538$.

5. $\hat{a} = -0.00729396$, $\Omega = 7.57541$, $R = 0.0234899$.

6.a $\hat{\xi}^T = \begin{bmatrix} 68.8534 & 66.9512 & 68.1542 & 66.0026 & 67.9917 & 68.5199 & 67.6955 \end{bmatrix}^T \text{m}$,
$\hat{\sigma}_0^2 = (1.00036)^2$.

Chapter 7

1.a See answers to Problems 9.b and 9.c of Chapter 3.

1.b $\hat{\hat{\xi}} = [1679.497, 1804.053, 2021.126, 1507.062, 1668.156, 1928.277]^T \text{ft}$,
$\hat{\sigma}_0^2 = (0.08197)^2 = 0.006719$.

2. Estimated height in feet: $\hat{H}_A = 1679.493$, $\hat{H}_B = 1804.072$, $\hat{H}_C = 2021.150$,
$\hat{H}_E = 1507.068$, $\hat{H}_F = 1668.159$, $\hat{H}_G = 1858.255$.

Chapter 8

1.b; 1.c $\hat{\xi} = [3.04324, 0.74568, 4.10586]^T$; $\hat{\sigma}_0^2 = (0.243289)^2 = 0.059190$.

2.b; 2.c $\hat{\xi} = [19.700\,975°, 6.6284, 2.8227, 2.6177, 3.6400]^T$; $\hat{\sigma}_0^2 = (0.263559)^2 = 0.069463$.

3.a; 3.b $\hat{\xi} = [1.73586328, 0.098057768, -0.0072771964]^T$; $\hat{\sigma}_0^2 = (1.830478)^2 = 3.350650$.

Part II

Advanced Adjustment Computations

Chapter 11

Review of the Gauss-Markov Model

The nonlinear form of the Gauss-Markov Model (GMM) is written as

$$\boldsymbol{Y} = a(\boldsymbol{\Xi}) + \boldsymbol{e}, \tag{11.1a}$$

$$\boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}). \tag{11.1b}$$

The symbols in equation (11.1) are defined as follows:

$\boldsymbol{Y}$ is a given $n \times 1$ vector of observations.

a is a (known) nonlinear function such that $a : \mathbb{R}^m \to \mathbb{R}^n$, $m < n$.

$\boldsymbol{\Xi}$ is an $m \times 1$ vector of unknown, non-random parameters.

$\boldsymbol{e}$ is an $n \times 1$ vector of unknown, random errors.

σ_0^2 is an unknown variance component.

P is a given $n \times n$ matrix of observation weights.

We linearize (11.1a) by Taylor-series expansion with respect to a fixed (i.e., non-random) approximation $\boldsymbol{\Xi}_0$ (called expansion point), which leads to the following *linearized GMM*:

$$\boldsymbol{y} := \boldsymbol{Y} - a(\boldsymbol{\Xi}_0) = \left.\frac{\partial a(\boldsymbol{\Xi})}{\partial \boldsymbol{\Xi}^T}\right|_{\boldsymbol{\Xi}=\boldsymbol{\Xi}_0} (\boldsymbol{\Xi} - \boldsymbol{\Xi}_0) + (\text{higher order terms}) + \boldsymbol{e}. \tag{11.2}$$

Neglecting the higher order terms and defining a coefficient matrix $A := \partial a / \partial \boldsymbol{\Xi}^T :$ $\mathbb{R}^m \to \mathbb{R}^n$, $m < n$, and an incremental parameter vector $\boldsymbol{\xi} := \boldsymbol{\Xi} - \boldsymbol{\Xi}_0$, we can then write the linearized GMM as

$$\boldsymbol{y} = A\boldsymbol{\xi} + \boldsymbol{e}, \tag{11.3a}$$

$$\boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}). \tag{11.3b}$$

Note that $\operatorname{rk} A = \dim \mathcal{R}(A) = m$ if, and only if, there is no rank deficiency in the model. For the linearized case this relation holds everywhere in the m-dimensional space, but it may not hold in the nonlinear case.

We review two approaches to estimating the parameter vector $\boldsymbol{\xi}$:

I. Algebraic-geometric approach (e.g., least-squares adjustment).

II. Statistical approach (e.g., minimum variance, unbiased estimators).

I. Algebraic-geometric approach: This approach uses a weighted least-squares adjustment, which is derived from the minimization of the weighted L_2-norm

$$\|\boldsymbol{y} - A\boldsymbol{\xi}\|_P^2 = \min_{\boldsymbol{\xi}} \tag{11.4}$$

and leads to an estimate $\hat{\boldsymbol{\xi}}$ for the unknown parameters. Or, being more general, we can express the problem as a minimization of the random error vector $\boldsymbol{e}$, which leads to both the parameter estimate $\hat{\boldsymbol{\xi}}$ and the predicted random error (or residual) vector $\tilde{\boldsymbol{e}}$.

$$\|\boldsymbol{e}\|_P^2 = \min_{\boldsymbol{e},\boldsymbol{\xi}}\{\boldsymbol{e} = \boldsymbol{y} - A\boldsymbol{\xi}\} \tag{11.5}$$

The Lagrange target function (or Lagrangian function) to minimize is a scalar-valued function that is quadratic in the unknown random error vector $\boldsymbol{e}$. It is written as

$$\Phi(\boldsymbol{e},\boldsymbol{\xi},\boldsymbol{\lambda}) = \boldsymbol{e}^T P \boldsymbol{e} + 2\boldsymbol{\lambda}^T(\boldsymbol{y} - A\boldsymbol{\xi} - \boldsymbol{e}), \tag{11.6}$$

which must be made stationary with respect to the unknown variables $\boldsymbol{e}$, $\boldsymbol{\xi}$, and $\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ is an $n \times 1$ vector of Lagrange multipliers. Accordingly, the Euler-Lagrange necessary conditions (or first-order conditions) lead to a minimization of (11.6). In forming the Euler-Lagrange necessary conditions, we take the first partial derivatives of the target function (11.6), set them to be zero, and use hat and tilde symbols to denote the particular solutions to these condition equations.

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}} = P\tilde{\boldsymbol{e}} - \hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0} \tag{11.7a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\xi}} = -A^T\hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0} \tag{11.7b}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}} - \tilde{\boldsymbol{e}} \doteq \boldsymbol{0} \tag{11.7c}$$

For convenience we define *normal equation* variables N and $\boldsymbol{c}$ as

$$[N,\, \boldsymbol{c}] := A^T P [A,\, \boldsymbol{y}]. \tag{11.8}$$

In the following, we often refer to matrix N as the *normal equations matrix*.

Solving the three equations (11.7) in the three unknowns $\tilde{\boldsymbol{e}}$, $\hat{\boldsymbol{\xi}}$, and $\hat{\boldsymbol{\lambda}}$ leads to

$$\hat{\boldsymbol{\lambda}} = P\tilde{\boldsymbol{e}}, \tag{11.9a}$$

$$A^T\hat{\boldsymbol{\lambda}} = A^T P\tilde{\boldsymbol{e}} = \boldsymbol{0}, \tag{11.9b}$$

implying that

$$A^T\hat{\boldsymbol{\lambda}} = A^T P(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}) = \boldsymbol{0}, \tag{11.9c}$$

which leads to the LEast-Squares Solution (LESS) for $\boldsymbol{\xi}$ as

$$\boxed{\hat{\boldsymbol{\xi}} = (A^T P A)^{-1}A^T P\boldsymbol{y} = N^{-1}\boldsymbol{c}} \tag{11.9d}$$

and the predicted residual vector

$$\boxed{\tilde{\boldsymbol{e}} = (I - AN^{-1}A^T P)\boldsymbol{y}.} \tag{11.9e}$$

Note the following relation between the predicted residual vector $\tilde{\boldsymbol{e}}$ and the unknown random error vector $\boldsymbol{e}$:

$$\tilde{\boldsymbol{e}} = (I - AN^{-1}A^T P)\boldsymbol{y} = (I - AN^{-1}A^T P)(\boldsymbol{y} - A\boldsymbol{\xi}) = (I - AN^{-1}A^T P)\boldsymbol{e}. \tag{11.10}$$

Equation (11.10) shows that, in general, the predicted random error vector $\tilde{\boldsymbol{e}}$ is not the same as the true (unknown) random error vector $\boldsymbol{e}$. They would only be the same if $\boldsymbol{e}$ belonged to the nullspace of $A^T P$, which is hardly possible since random measurement errors are involved.

The expectations of the estimated parameter vector $\hat{\boldsymbol{\xi}}$ and the predicted random error vector $\tilde{\boldsymbol{e}}$ are given as follows:

$$E\{\hat{\boldsymbol{\xi}}\} = N^{-1}A^T P \cdot E\{\boldsymbol{y}\} = N^{-1}A^T P A\boldsymbol{\xi} = \boldsymbol{\xi}. \tag{11.11}$$

Equation (11.11) holds for all $\boldsymbol{\xi} \in \mathbb{R}^m$. Therefore, $\hat{\boldsymbol{\xi}}$ is said to be a *uniformly unbiased estimate* of $\boldsymbol{\xi}$.

$$E\{\tilde{\boldsymbol{e}}\} = (I - AN^{-1}A^T P) \cdot E\{\boldsymbol{y}\} = (I - AN^{-1}A^T P)A\boldsymbol{\xi} = A\boldsymbol{\xi} - A\boldsymbol{\xi} = \boldsymbol{0} = E\{\boldsymbol{e}\} \tag{11.12}$$

Because the $n \times 1$ vector $\boldsymbol{0}$ is only one element of $\mathbb{R}^n$, $\tilde{\boldsymbol{e}}$ is considered to be a *weakly unbiased prediction* of $\boldsymbol{e}$.

The associated dispersion and covariance matrices are derived as follows:

$$D\{\hat{\boldsymbol{\xi}}\} = N^{-1}A^T P \cdot D\{\boldsymbol{y}\} \cdot PAN^{-1} = N^{-1}A^T P(\sigma_0^2 P^{-1})PAN^{-1} \Rightarrow$$

$$\boxed{D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 N^{-1},} \tag{11.13}$$

$$D\{\tilde{\boldsymbol{e}}\} = (I - AN^{-1}A^T P) \cdot D\{\boldsymbol{y}\} \cdot (I - PAN^{-1}A^T) =$$
$$= (I - AN^{-1}A^T P)(\sigma_0^2 P^{-1})(I - PAN^{-1}A^T) =$$
$$= \sigma_0^2(P^{-1} - AN^{-1}A^T)(I - PAN^{-1}A^T) =$$
$$= \sigma_0^2(P^{-1} - AN^{-1}A^T) - P^{-1}PAN^{-1}A^T + AN^{-1}A^T PAN^{-1}A^T \Rightarrow$$

$$\boxed{D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2(P^{-1} - AN^{-1}A^T),} \tag{11.14}$$

$$C\{\hat{\boldsymbol{\xi}}, \tilde{\boldsymbol{e}}\} = N^{-1}A^T P \cdot D\{\boldsymbol{y}\} \cdot (I - PAN^{-1}A^T) =$$
$$= N^{-1}A^T P(\sigma_0^2 P^{-1})(I - PAN^{-1}A^T) =$$
$$= \sigma_0^2 N^{-1}A^T - \sigma_0^2 N^{-1}A^T PAN^{-1}A^T = \sigma_0^2(N^{-1}A^T - N^{-1}A^T) \Rightarrow$$

$$\boxed{C\{\hat{\boldsymbol{\xi}}, \tilde{\boldsymbol{e}}\} = 0.} \tag{11.15}$$

Equation (11.15) shows the "covariance orthogonality." This is opposed to the algebraic orthogonality depicted in Figure 11.1, where it is shown that the residual vector $\tilde{\boldsymbol{e}}$ is added to $\boldsymbol{y}$ to make equation (11.3) consistent. Through the least-squares principle, we have found a particular residual vector $\tilde{\boldsymbol{e}}$ that is closest to (geometrically orthogonal to) the column space of matrix A.
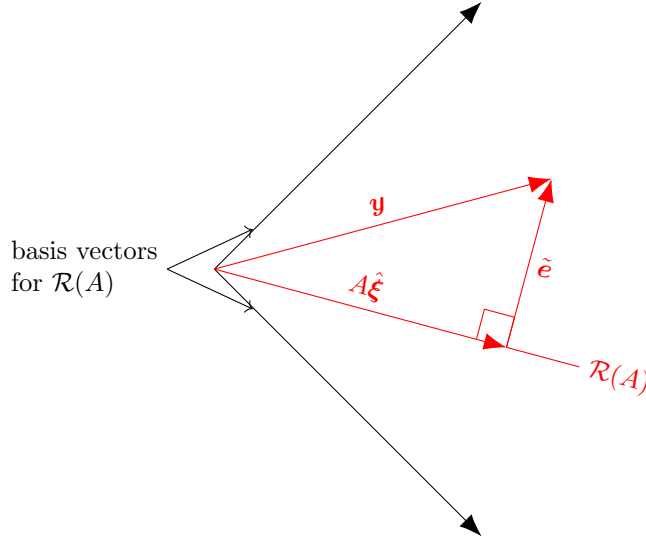
Figure 11.1: Orthogonality between $\mathcal{R}(A)$ and $\tilde{\boldsymbol{e}}$

Let us consider the P-weighted L_2-norm of the residual vector $\tilde{\boldsymbol{e}}$, which can be expressed in the following forms:

$$\|\tilde{\boldsymbol{e}}\|_P^2 = \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}} = (\boldsymbol{y} - A\hat{\boldsymbol{\xi}})^T P(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}) = \tag{11.16a}$$

$$= \boldsymbol{y}^T P(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}) - \hat{\boldsymbol{\xi}}^T A^T P\tilde{\boldsymbol{e}} = \tag{11.16b}$$

$$= \boldsymbol{y}^T P(\boldsymbol{y} - A\hat{\boldsymbol{\xi}}) = \boldsymbol{y}^T P\boldsymbol{y} - \boldsymbol{c}^T \hat{\boldsymbol{\xi}} = \boldsymbol{y}^T P\boldsymbol{y} - \hat{\boldsymbol{\xi}}^T N\hat{\boldsymbol{\xi}}. \tag{11.16c}$$

In (11.16) we have used the orthogonality property $A^T P\tilde{\boldsymbol{e}} = \boldsymbol{0}$ shown in (11.9b). However, we have not yet shown that equations (11.16) are connected to the "best" estimate for the variance component σ_0^2. That is, the connection between $\tilde{\boldsymbol{e}}$ and $\hat{\sigma}_0^2$ is unknown at this point. Furthermore, there is no algebraic principle that allows us to make this connection; we must use statistics.

II) Statistical approach: This approach gives a more indirect estimate of the unknown parameters $\boldsymbol{\xi}$. Our principal unknown is an $m \times n$ matrix L that the parameter estimate $\hat{\boldsymbol{\xi}}$ can be derived from. We seek a linear estimator. That is, the parameter estimates $\hat{\boldsymbol{\xi}}$ must depend linearly on the observation vector $\boldsymbol{y}$. Furthermore, we require the estimate to be *uniformly unbiased* and to be *best* in terms of minimum variance. Together these requirements comprise the *Best Linear Uniformly Unbiased Estimate*, or BLUUE, of $\boldsymbol{\xi}$. The components of BLUUE are outlined in the following:

(i) Linear requirement:

$$\hat{\boldsymbol{\xi}} = L\boldsymbol{y} + \boldsymbol{\gamma}, \text{ where } L \in \mathbb{R}^{m \times n} \text{ and } \boldsymbol{\gamma} \in \mathbb{R}^m. \tag{11.17a}$$

Equation (11.17a) is an inhomogeneous linear form due the $m \times 1$ vector $\boldsymbol{\gamma}$. It requires that both L and $\boldsymbol{\gamma}$ be determined. Therefore, there are $m(n+1)$ unknowns, a relatively large number.

(ii) Uniformly unbiased requirement:

$$\boldsymbol{\xi} = E\{\hat{\boldsymbol{\xi}}\} \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^m \Rightarrow$$
$$E\{L\boldsymbol{y} + \boldsymbol{\gamma}\} = LE\{\boldsymbol{y}\} + \boldsymbol{\gamma} = LA\boldsymbol{\xi} + \boldsymbol{\gamma}, \tag{11.17b}$$

leading to the two requirements

$$LA = I_m \text{ and } \boldsymbol{\gamma} = \boldsymbol{0}. \tag{11.17c}$$

Equation (11.17c) specifies $m \times m$ constraints in the first equation and m constraints in the second equation. Thus the number of unknowns minus the number of constraints is $m(n+1) - m(m+1) = m(n-m)$. Therefore, we have reduced our search space somewhat from $m(n+1)$.

(iii) Best requirement: By "best" we mean minimum average-variance. An average variance can be computed by dividing the trace of the $m \times m$ parameter dispersion matrix by m. However, division by m only scales the quantity to be minimized, so we can just as well minimize the trace itself.

$$\operatorname{tr} D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 \operatorname{tr}(LP^{-1}L^T) = \min_L\{LA = I_m\} \tag{11.17d}$$

The quadratic form $LP^{-1}L^T$ in (11.17d) is the term to minimize. The term $LA = I_m$ imposes $m \times m$ constraints.

Because unbiasedness is required, the dispersion of $\hat{\boldsymbol{\xi}}$ is the same as the MSE of $\hat{\boldsymbol{\xi}}$. Thus we can write

$$\operatorname{tr} \operatorname{MSE}\{\hat{\boldsymbol{\xi}}\} = \operatorname{tr} E\{(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})^T\} = E\{(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})^T(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})\} = E\{\|\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\|^2\}. \tag{11.18}$$

The result is an expectation of a vector norm, which is a scalar. Note that the property of the trace being invariant with respect to a cyclic transformation was used in (11.18).

The Lagrange target function associated with (11.17d) is

$$\Phi(L, \boldsymbol{\lambda}) = \operatorname{tr} L P^{-1} L^T + 2 \operatorname{tr} \Lambda (LA - I_m), \tag{11.19}$$

which must be made stationary with respect to L and Λ. Here, Λ is an $m \times m$ symmetric matrix of Lagrange multipliers. Accordingly, the Euler-Lagrange necessary conditions are formed by

$$\frac{1}{2} \frac{\partial \Phi}{\partial L} = \hat{L} P^{-1} + \hat{\Lambda} A^T \doteq 0 \tag{11.20a}$$

$$\frac{1}{2} \frac{\partial \Phi}{\partial \Lambda} = \hat{L} A - I_m \doteq 0. \tag{11.20b}$$

In equation (11.20a) we have used rules (12) and (4) from section 10.3.2 of Lütkepohl (1996) for derivatives of the trace. Likewise, in equation (11.20b) we have used rule (5) from the same section of Lütkepohl (see equations (A.47) herein). The two condition equations are solved simultaneously as follows:

$$\hat{L} = -\hat{\Lambda}^T A^T P$$

implies that

$$-\hat{\Lambda}^T A^T P A = I_m \Rightarrow \hat{\Lambda} = -(A^T P A)^{-1},$$

finally leading to

$$\hat{L} = (A^T P A)^{-1} A^T P. \tag{11.21}$$

Substituting the solution for $\hat{L}$ into (11.17a) and using the condition $\boldsymbol{\gamma} = \mathbf{0}$ in (11.17c) leads to the BLUUE for the parameters $\boldsymbol{\xi}$ as

$$\hat{\boldsymbol{\xi}} = \hat{L} \boldsymbol{y} + \boldsymbol{\gamma} = (A^T P A)^{-1} A^T P \boldsymbol{y}. \tag{11.22}$$

Comparing (11.22) to (11.9d), reveals that the BLUUE of $\boldsymbol{\xi}$ is equivalent to the LESS of $\boldsymbol{\xi}$ within the (full-rank) Gauss-Markov model.

Chapter 12

Introducing the vec Operator and the Kronecker-Zehfuss Product

The vec operator forms a column vector from the columns of the matrix that it operates on by stacking one column on top of the next, from first to last.

$$\underset{p \times q}{G} := \left[\boldsymbol{g}_1, \ldots, \boldsymbol{g}_q \right] \Rightarrow \underset{pq \times 1}{\operatorname{vec} G} := \left[\boldsymbol{g}_1^T, \ldots, \boldsymbol{g}_q^T \right]^T \tag{12.1}$$

Here $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_q$ are the $p \times 1$ column vectors of G. Note that the reverse operation *is not unique.*

Given two $p \times q$ matrices A, B such that $\underset{p \times q}{A} := [a_{ij}]$ and $\underset{p \times q}{B} := [b_{ij}]$, the following relationship between the trace and the vec operator holds:

$$\operatorname{tr}(A^T B) = \operatorname{tr}(BA^T) = \quad \text{(trace invariant with respect to a cyclic transf.)} \tag{12.2}$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} b_{ij} = \quad \text{(first sum for trace, second for matrix product)}$$

$$= (\operatorname{vec} A)^T \operatorname{vec} B =$$

$$= \sum_{ij} a_{ij} b_{ij}, \quad \text{(multiplies corresponding elements)}$$

which finally allows us to write

$$\operatorname{tr}(A^T B) = (\operatorname{vec} A)^T \operatorname{vec} B. \tag{12.3}$$

Given matrices A of size $p \times q$, B of size $q \times r$, and C of size $s \times r$, we have the following important relationship, which connects the vec operator and the Kronecker-Zehfuss product (also called direct product, Kronecker product, and

Zehfuss product—see Searle and Khuri (2017, p. 78)).

$$\underset{ps \times 1}{\text{vec}(ABC^T)} = \underset{ps \times qr}{(C \otimes A)} \underset{qr \times 1}{\text{vec}\, B}, \tag{12.4}$$

where the Kronecker product is defined by

$$(C \otimes A) := [c_{ij}A] = \begin{bmatrix} c_{11}A & c_{12}A & \dots & c_{1r}A \\ c_{21}A & \ddots & & \\ \vdots & & & \\ c_{s1}A & c_{s2}A & \dots & c_{sr}A \end{bmatrix}. \tag{12.5}$$

The definition of the Kronecker product, as well as many of its properties, is given in the appendix and is used in several of the following sections.

Now we generalize formula (12.4) using a quadruple product of matrices, which is commonly found in the variance-component estimation problem. Because the trace is invariant with respect to a cyclic transformation, we can write

$$\text{tr}(ABC^T D^T) = \text{tr}(D^T ABC^T) = \tag{12.6a}$$

$$= (\text{vec}\, D)^T \text{vec}(ABC^T) = \tag{12.6b}$$

$$= (\text{vec}\, D)^T (C \otimes A) \text{vec}\, B, \tag{12.6c}$$

where (12.3) and (12.4) where used in the last two lines, respectively.

It is also required at times to apply the vec operator to a vector outer-product. Given a vector $\boldsymbol{a}$, substitute into equation (12.4) $A = \boldsymbol{a}$ and $C^T = \boldsymbol{a}^T$; also let $B = I_1 = 1$. Then we have

$$\text{vec}(\boldsymbol{a}\boldsymbol{a}^T) = \boldsymbol{a} \otimes \boldsymbol{a}. \tag{12.7}$$

Commutation matrices appear in the rules for the Kronecker product in the appendix. Here we comment that a commutation matrix K is square and has only ones and zeros for its elements. Each row has exactly a single one, and likewise for each column. Thus the identity matrix is one example of a commutation matrix. The commutation matrix is not symmetric (except for the identity matrix), but it is orthogonal, meaning that $K^{-1} = K^T$. A commutation matrix is also a vec-permutation matrix. We illustrate this property by the following example:

$$K := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \Rightarrow \text{vec}\, A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \end{bmatrix},$$

$$\text{vec } A^T = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \end{bmatrix} = K \text{ vec } A.$$

To demonstrate the usefulness of the Kronecker product, we now show certain applications of it to the Gauss-Markov Model (GMM) and the associated Best Linear Uniformly Unbiased Estimate (BLUUE). We begin by deriving an alternative form for the target function (11.19) in order to exploit the Kronecker product.

$$\Phi(L, \Lambda) = \text{tr } LP^{-1}L^T + 2\,\text{tr } \Lambda^T(LA - I_m) = \tag{12.8a}$$
$$= \text{tr } LP^{-1}L^T I_m + 2\,\text{tr}(LA - I_m)\Lambda I_m = \quad \text{(noting the symmetry of } \Lambda)$$
$$= \text{tr } LP^{-1}L^T I_m + 2\,\text{tr } LA\Lambda I_m - 2\,\text{tr } I_m\Lambda I_m =$$
$$= (\text{vec } L^T)^T(I_m \times P^{-1})\,\text{vec } L^T +$$
$$\quad + 2(\text{vec } L^T)^T(I_m \times A)\,\text{vec } \Lambda - 2(\text{vec } I_m)^T\,\text{vec } \Lambda \Rightarrow$$
$$\Phi(\boldsymbol{l}, \boldsymbol{\lambda}) = \boldsymbol{l}^T(I_m \times P^{-1})\boldsymbol{l} + 2\big[\boldsymbol{l}^T(I_m \times A) - (\text{vec } I_m)^T\big]\boldsymbol{\lambda} \tag{12.8b}$$

Here, $\boldsymbol{l}$ and $\boldsymbol{\lambda}$ are variables for the vectorized forms of the unknown matrices L and Λ and are define as follows:

$$\boldsymbol{l} := \text{vec}(L^T) \text{ is an } nm \times 1 \text{ vector containing the rows of } L \text{ in vector form.} \tag{12.8c}$$

$$\boldsymbol{\lambda} := \text{vec } \Lambda \text{ is an } m^2 \times 1 \text{ vector comprised of the columns of } \Lambda. \tag{12.8d}$$

Using vectors $\boldsymbol{l}$ and $\boldsymbol{\lambda}$, the following Lagrange target function can be written as an alternative to (11.19):

$$\Phi(\boldsymbol{l}, \boldsymbol{\lambda}) = \boldsymbol{l}^T(I_m \otimes P^{-1})\boldsymbol{l} - 2\big[\boldsymbol{l}^T(I_m \otimes A) - (\text{vec } I_m)^T\big]\boldsymbol{\lambda}, \tag{12.9}$$

which must be made stationary with respect to $\boldsymbol{l}$ and $\boldsymbol{\lambda}$. Accordingly, the Euler-Lagrange necessary conditions are written as

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{l}} = (I_m \otimes P^{-1})\hat{\boldsymbol{l}} - (I_m \otimes A)\hat{\boldsymbol{\lambda}} \doteq 0, \tag{12.10a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = -(I_m \otimes A^T)\hat{\boldsymbol{l}} + \text{vec } I_m \doteq 0. \tag{12.10b}$$

The normal equations are then solved for $\hat{\boldsymbol{l}}$ and $\hat{\boldsymbol{\lambda}}$ as follows: Equation (12.10a) implies that

$$\hat{\boldsymbol{l}} = (I_m \otimes P^{-1})^{-1}(I_m \otimes A)\hat{\boldsymbol{\lambda}} = (I_m \otimes PA)\hat{\boldsymbol{\lambda}}, \tag{12.11a}$$

which, together with (12.10b), further implies

$$(I_m \otimes A^T PA)\hat{\boldsymbol{\lambda}} = \text{vec } I_m, \tag{12.11b}$$

leading to the estimates

$$\hat{\boldsymbol{\lambda}} = (I_m \otimes A^T P A)^{-1} \operatorname{vec} I_m = \operatorname{vec}(A^T P A)^{-1}, \tag{12.11c}$$

$$\hat{\boldsymbol{l}} = \operatorname{vec}(\hat{L}^T) = (I_m \otimes P A) \operatorname{vec}(A^T P A)^{-1} = \operatorname{vec}\big[P A (A^T P A)^{-1}\big], \tag{12.11d}$$

and finally to

$$\hat{L} = (A^T P A)^{-1} A^T P. \tag{12.11e}$$

The sufficient condition for minimization is satisfied by

$$\frac{1}{2} \frac{\partial^2 \Phi}{\partial \boldsymbol{l} \partial \boldsymbol{l}^T} = \frac{\partial (I_m \otimes P^{-1}) \hat{\boldsymbol{l}}}{\partial \boldsymbol{l}^T} = I_m \otimes P^{-1}, \tag{12.12}$$

which is positive definite.

The Best Linear Uniformly Unbiased Estimation (BLUUE) of $\boldsymbol{\xi}$ and its dispersion are, respectively,

$$\hat{\boldsymbol{\xi}} = \hat{L} \boldsymbol{y} = \hat{\Lambda}^T A^T P \boldsymbol{y} = (A^T P A)^{-1} A^T P \boldsymbol{y}, \tag{12.13a}$$

and

$$D\{\hat{\boldsymbol{\xi}}\} = \hat{\Lambda}^T A^T P (\sigma_0^2 P^{-1}) P A \hat{\Lambda} = \sigma_0^2 (A^T P A)^{-1} = \sigma_0^2 N^{-1}. \tag{12.13b}$$

Likewise, the predicted residual vector and its dispersion are, respectively,

$$\tilde{\boldsymbol{e}} := \boldsymbol{y} - A \hat{\boldsymbol{\xi}} = \big[I_n - A(A^T P A)^{-1} A^T P\big] \boldsymbol{y}, \tag{12.14a}$$

and

$$D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2 \big[P^{-1} - A(A^T P A)^{-1} A^T\big]. \tag{12.14b}$$

Corollary: In the Gauss-Markov Model (GMM) with full-rank matrices A and P, the BLUUE of $\boldsymbol{\xi}$ is automatically generated by the LESS $\hat{\boldsymbol{\xi}}$ with the associated dispersion matrix $D\{\hat{\boldsymbol{\xi}}\}$ and residual vector $\tilde{\boldsymbol{e}}$. This fact is called "*Gauss' second argument in favor of the least-squares adjustment.*"

Chapter 13

Variance Component Estimation

In this chapter we develop estimators for the unknown variance component σ_0^2 from the Gauss-Markov Model (GMM) (11.3b). We begin by restating the full-rank GMM from chapter Chapter 11.

$$\boldsymbol{y} = A\boldsymbol{\xi} + \boldsymbol{e}, \ \ \operatorname{rk} A = m < n, \ \ \boldsymbol{e} \sim (\boldsymbol{0}, \sigma_0^2 P^{-1}) \tag{13.1}$$

Our goal is to determine the estimated variance component $\hat{\sigma}_0^2$ in such a way that it is independent of the estimated parameter vector $\hat{\boldsymbol{\xi}}$.

First note that

$$E\{\boldsymbol{e}\boldsymbol{e}^T\} = \sigma_0^2 P^{-1} = D\{\boldsymbol{e}\}, \tag{13.2a}$$

and

$$E\{\boldsymbol{e}^T\boldsymbol{e}\} = E\{\operatorname{tr}\boldsymbol{e}^T\boldsymbol{e}\} = E\{\operatorname{tr}\boldsymbol{e}\boldsymbol{e}^T\} = \operatorname{tr} E\{\boldsymbol{e}\boldsymbol{e}^T\} = \sigma_0^2 \operatorname{tr} P^{-1} =$$
$$= E\{(\boldsymbol{y} - A\boldsymbol{\xi})^T(\boldsymbol{y} - A\boldsymbol{\xi})\} = E\{\boldsymbol{y}^T\boldsymbol{y}\} - E\{\boldsymbol{y}\}^T E\{\boldsymbol{y}\} = E\{\boldsymbol{y}^T\boldsymbol{y}\} - \boldsymbol{\xi}^T A^T A\boldsymbol{\xi}. \tag{13.2b}$$

We see from (13.2b) that σ_0^2 and $\boldsymbol{\xi}$ are not decoupled. But we want the estimates $\hat{\sigma}_0^2$ and $\hat{\boldsymbol{\xi}}$ to be decoupled so that estimating the variance component σ_0^2 has nothing to do with estimating the parameter vector $\boldsymbol{\xi}$. To this end we seek the *Best Invariant Quadratic Uniformly Unbiased Estimate*, or BIQUUE. Each term in the acronym BIQUUE is explained below.

BIQUUE

(i) Quadratic requirement: $\hat{\sigma}_0^2$ is quadratic in the observation vector $\boldsymbol{y}$, such that

$$\hat{\sigma}_0^2 = \boldsymbol{y}^T M \boldsymbol{y}, \tag{13.3a}$$

where M is an unknown $n \times n$ matrix to be determined. Since $\hat{\sigma}_0^2$ is a scalar, it is equal to its transpose; therefore

$$\hat{\sigma}_0^2 = (\boldsymbol{y}^T M \boldsymbol{y})^T = \boldsymbol{y}^T M^T \boldsymbol{y} = \boldsymbol{y}^T \left(\frac{M + M^T}{2} \right) \boldsymbol{y}. \qquad (13.3\text{b})$$

Thus we can use M or M^T; it does not matter. So, without loss of generality, we require the matrix M to be symmetric. This reduces the number of unknowns from n^2 to $n(n+1)/2$.

(ii) Invariant requirement: We require the estimate to be invariant with respect to translation, i.e., invariant with respect to a shift of $\boldsymbol{y}$ along the range space of A. The motivation for this requirement is to ensure that $\hat{\sigma}_0^2$ is independent of the estimated parameter vector $\hat{\boldsymbol{\xi}}$, an objective already stated above.

$$\hat{\sigma}_0^2 = (\boldsymbol{y} - A\boldsymbol{\xi})^T M (\boldsymbol{y} - A\boldsymbol{\xi}) \ \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^m. \qquad (13.4\text{a})$$

Obviously, (13.4a) includes the estimate $\hat{\boldsymbol{\xi}}$ since it also belongs to $\mathbb{R}^m$. Due to invariance we can write

$$\begin{aligned} \hat{\sigma}_0^2 &= \boldsymbol{y}^T M \boldsymbol{y} = (\boldsymbol{y} - A\boldsymbol{\xi})^T M (\boldsymbol{y} - A\boldsymbol{\xi}) = \\ &= \boldsymbol{y}^T M \boldsymbol{y} - \boldsymbol{y}^T M A \boldsymbol{\xi} - \boldsymbol{\xi}^T A^T M \boldsymbol{y} + \boldsymbol{\xi}^T A^T M A \boldsymbol{\xi}, \end{aligned} \qquad (13.4\text{b})$$

implying that

$$\boldsymbol{\xi}^T A^T M A \boldsymbol{\xi} = 2\boldsymbol{y}^T M A \boldsymbol{\xi} \ \text{ for any } \boldsymbol{y} \text{ and any } \boldsymbol{\xi}. \qquad (13.4\text{c})$$

"For any $\boldsymbol{\xi}$" means that $\boldsymbol{\xi}$ could be positive or negative. The left side of (13.4c) would not change if $\boldsymbol{\xi}$ is replaced by $-\boldsymbol{\xi}$, but the right side would change in sign. The only quantity that remains equal when we change the sign of $\boldsymbol{\xi}$ in (13.4c) is zero. Therefore, the condition becomes

$$\boldsymbol{y}^T M A \boldsymbol{\xi} = \boldsymbol{0} \ \text{ for any } \boldsymbol{y} \text{ and any } \boldsymbol{\xi}, \text{ which is true if, and only if, } MA = 0. \tag{13.4d}$$

This matrix constraint satisfies the invariant condition; that is, the "decoupling" between $\hat{\boldsymbol{\xi}}$ and $\hat{\sigma}_0^2$ guarantees invariance.

(iii) Uniformly Unbiased requirement: The equality

$$\sigma_0^2 = E\{\hat{\sigma}_0^2\} = E\{\boldsymbol{y}^T M \boldsymbol{y}\} = E\{\boldsymbol{e}^T M \boldsymbol{e}\} \qquad (13.5\text{a})$$

holds since $\hat{\sigma}_0^2 = \boldsymbol{y}^T M \boldsymbol{y}$ according to (13.3a). Also, due to the invariant principle, $\boldsymbol{e} = \boldsymbol{y} - A\boldsymbol{\xi}$ holds for all $\boldsymbol{\xi}$. Thus, we may continue with

$$\sigma_0^2 = \operatorname{tr} E\{M \boldsymbol{e}\boldsymbol{e}^T\} = \operatorname{tr}(M D\{\boldsymbol{e}\}) = \sigma_0^2 \operatorname{tr}(M P^{-1}) \ \text{ for all } \sigma_0^2 \in \mathbb{R}^+ \Leftrightarrow \tag{13.5b}$$

$$1 = \operatorname{tr}(M P^{-1}). \qquad (13.5\text{c})$$

Here, $\mathbb{R}^+$ denotes the field of positive real numbers. Equation (13.5c) provides the uniformly-unbiased condition; it holds due to the invariance principle.

(iv) Best requirement: "Best" means that the dispersion of the estimated variance component $\hat{\sigma}_0^2$ must be minimized.

$$D\{\hat{\sigma}_0^2\} = \min_{M} \begin{cases} MA = 0 \\ \operatorname{tr}(MP^{-1}) = 1 \end{cases} \tag{13.6}$$

We are dealing with the dispersion of a quadratic form, which is a fourth moment. Therefore we need to make an additional assumption. We must assume quasi-normality, which says that the third and fourth moments behave as if the random errors $\boldsymbol{e}$ are normally distributed.

Aside: For the normal distribution, all moments can be written as a function of the first and second moments, $E\{e_i\} = 0$ and $E\{e_i^2\} = \sigma_i^2$, respectively. Moreover, under normality, the following relations hold for the fourth and all odd-order moments of the ith random error e_i:

$$E\{e_i^4\} = 3(\sigma_i^2)^2 \ \text{ and } \ E\{e_i^{2k+1}\} = 0, \ k = 0, 2, 4, \ldots \tag{13.7}$$

The left side of (13.6) can also be expressed as follows:

$$D\{\hat{\sigma}_0^2\} = D\{\boldsymbol{e}^T M \boldsymbol{e}\} = E\{\boldsymbol{e}^T M \boldsymbol{e} \boldsymbol{e}^T M \boldsymbol{e}\} - E\{\boldsymbol{e}^T M \boldsymbol{e}\}^2 = \tag{13.8a}$$

$$= E\{\boldsymbol{e}^T M \boldsymbol{e} \boldsymbol{e}^T M \boldsymbol{e}\} - \left[\sigma_0^2 \operatorname{tr}(MP^{-1})\right]^2 = E\{\boldsymbol{e}^T M \boldsymbol{e} \boldsymbol{e}^T M \boldsymbol{e}\} - (\sigma_0^2)^2. \tag{13.8b}$$

Note that in numerical computations we would replace $\boldsymbol{e}$ with $\boldsymbol{y}$ in (13.8) since $\boldsymbol{e}$ is unknown. However, analytically the results are the same due to the invariance property.

Now, the expectation term in (13.8b) consists of products in the random variable $\boldsymbol{e}$; therefore, it can be expressed as a sum of the expectations of all combinations of the products. We illustrate this with symbols under the respective occurrences of $\boldsymbol{e}$ as follows (obviously each of these accented vectors $\boldsymbol{e}$ are actually equivalent to one another):

$$E\{\underline{\boldsymbol{e}}^T M \underline{\boldsymbol{e}} \, \underline{\underline{\boldsymbol{e}}}^T M \underline{\underline{\boldsymbol{e}}}\} = E\{\underline{\boldsymbol{e}}^T M \underline{\boldsymbol{e}}\}E\{\underline{\underline{\boldsymbol{e}}}^T M \underline{\underline{\boldsymbol{e}}}\} + E\{M \underline{\boldsymbol{e}} \, \underline{\underline{\boldsymbol{e}}}^T\}E\{M \underline{\boldsymbol{e}} \, \underline{\underline{\boldsymbol{e}}}^T\} +$$

$$+ E\{\underline{\boldsymbol{e}} \, \underline{\boldsymbol{e}}^T\}E\{M \underline{\underline{\boldsymbol{e}}} \, \underline{\underline{\boldsymbol{e}}}^T M^T\} =$$

$$= (\sigma_0^2)^2 + M E\{\boldsymbol{e}\boldsymbol{e}^T\}M E\{\boldsymbol{e}\boldsymbol{e}^T\} + E\{\boldsymbol{e}\boldsymbol{e}^T\}M E\{\boldsymbol{e}\boldsymbol{e}^T\}M^T =$$

$$= (\sigma_0^2)^2 + (\sigma_0^2)^2 M P^{-1} M P^{-1} + (\sigma_0^2)^2 P^{-1} M P^{-1} M^T. \tag{13.9}$$

Noting that M is symmetrical and substituting (13.9) into (13.8b) and then applying the trace operator yields

$$D\{\hat{\sigma}_0^2\} = 2(\sigma_0^2)^2 \operatorname{tr}(P^{-1} M P^{-1} M^T) = 2(\sigma_0^2)^2 (\operatorname{vec} M)^T (P^{-1} \otimes P^{-1}) \operatorname{vec} M, \tag{13.10}$$

leading to the following expression that must be minimized:

$$D\{\hat{\sigma}_0^2\} = \min_{M}\{MA = 0, \operatorname{tr}(MP^{-1}) = 1\}. \tag{13.11}$$

Introducing an $m \times n$ matrix of Lagrange multipliers Λ and a scalar Lagrange multiplier λ_0 leads to the following Lagrange target function:

$$\Phi(M, \Lambda, \lambda_0) = \operatorname{tr}(MP^{-1}MP^{-1}) - 2\operatorname{tr}(M^T A\Lambda) - 2\big[\operatorname{tr}(MP^{-1}) - 1\big]\lambda_0, \quad (13.12)$$

which must be made stationary with respect to the unknown variables M, Λ, and λ_0.

An alternative form of the Lagrangian function can be written as

$$\Phi(\operatorname{vec} M, \boldsymbol{\lambda}, \lambda_0) = 2(\operatorname{vec} M)^T (P^{-1} \otimes P^{-1}) \operatorname{vec} M +$$
$$+ 4(\operatorname{vec} M)^T (I_n \otimes A)\boldsymbol{\lambda} + 4\big[(\operatorname{vec} M)^T \operatorname{vec} P^{-1} - 1\big]\lambda_0, \qquad (13.13)$$

which must be made stationary with respect to $\operatorname{vec} M$, $\boldsymbol{\lambda}$, and λ_0.

Note that $\operatorname{vec} M$ is an $n^2 \times 1$ vector; $\boldsymbol{\lambda}$ is an $nm \times 1$ vector, and λ_0 is a scalar. The equivalence between (13.12) and (13.13) is seen by noting that $\operatorname{vec} \Lambda = \boldsymbol{\lambda}$ and by use of (12.2) and (12.5).

The Euler-Lagrange necessary conditions result in

$$\frac{1}{4}\frac{\partial \Phi}{\partial \operatorname{vec} M} = (P^{-1} \otimes P^{-1}) \operatorname{vec} M + (I_n \otimes A)\hat{\boldsymbol{\lambda}} + \operatorname{vec} P^{-1}\hat{\lambda}_0 \doteq \mathbf{0}, \qquad (13.14\text{a})$$

$$\frac{1}{4}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = (I_n \otimes A^T) \operatorname{vec} M \doteq \mathbf{0}, \qquad (13.14\text{b})$$

$$\frac{1}{4}\frac{\partial \Phi}{\partial \lambda_0} = (\operatorname{vec} P^{-1})^T \operatorname{vec} M - 1 \doteq 0. \qquad (13.14\text{c})$$

For convenience, a hat symbol is not used over the particular vector $\operatorname{vec} M$ that we solve for in the minimization, as has been done for $\hat{\boldsymbol{\lambda}}$ and $\hat{\lambda}_0$. From (13.14a) we get the following expression for $\operatorname{vec} M$:

$$\operatorname{vec} M = -(P^{-1} \otimes P^{-1})^{-1}(I_n \otimes A)\hat{\boldsymbol{\lambda}} - (P^{-1} \otimes P^{-1})^{-1} \operatorname{vec}(P^{-1})\hat{\lambda}_0 =$$
$$= -(P \otimes P)(I_n \otimes A)\hat{\boldsymbol{\lambda}} - (P \otimes P) \operatorname{vec}(P^{-1})\hat{\lambda}_0 =$$
$$= -(PI_n \otimes PA)\hat{\boldsymbol{\lambda}} - \operatorname{vec}(PP^{-1}P)\hat{\lambda}_0 \Rightarrow$$
$$\operatorname{vec} M = -(P \otimes PA)\hat{\boldsymbol{\lambda}} - \operatorname{vec}(P)\hat{\lambda}_0. \qquad (13.15\text{a})$$

Now, substituting (13.15a) into (13.14b) and dropping the negative sign results in

$$(I_n \otimes A^T)\big[(P \otimes PA)\hat{\boldsymbol{\lambda}} + \operatorname{vec}(P)\hat{\lambda}_0\big] = 0$$
$$(I_n \otimes A^T)(P \otimes PA)\hat{\boldsymbol{\lambda}} + (I_n \otimes A^T) \operatorname{vec}(P)\hat{\lambda}_0 = 0$$
$$(I_n P \otimes A^T PA)\hat{\boldsymbol{\lambda}} + \operatorname{vec}(A^T PI_n^T)\hat{\lambda}_0 = 0$$
$$(P \otimes N)\hat{\boldsymbol{\lambda}} + \operatorname{vec}(A^T P)\hat{\lambda}_0 = 0. \qquad (13.15\text{b})$$

Finally, we can write the estimated vector of Lagrange multipliers as

$$\hat{\boldsymbol{\lambda}} = -(P^{-1} \otimes N^{-1}) \operatorname{vec}(A^T P)\hat{\lambda}_0 = -\operatorname{vec}(N^{-1}A^T PP^{-1})\hat{\lambda}_0 = -\operatorname{vec}(N^{-1}A^T)\hat{\lambda}_0. \qquad (13.15\text{c})$$

Substituting (13.15c) into (13.15a) yields

$$\text{vec}\,M = (P \otimes PA)\,\text{vec}\,(N^{-1}A^T)\hat{\lambda}_0 - \text{vec}\,(P)\hat{\lambda}_0 =$$
$$= \text{vec}\,(PAN^{-1}A^TP)\hat{\lambda}_0 - \text{vec}\,(P)\hat{\lambda}_0 \Rightarrow$$
$$\text{vec}\,M = -\big[\text{vec}\,(P) - \text{vec}\,(PAN^{-1}A^TP)\big]\hat{\lambda}_0. \tag{13.15d}$$

Then we substitute (13.15d) into (13.14c) to obtain

$$-(\text{vec}\,P^{-1})^T\big[\text{vec}\,(P) - \text{vec}\,(PAN^{-1}A^TP)\big]\hat{\lambda}_0 = 1. \tag{13.15e}$$

Using (12.2) allows us to rewrite (13.15e) and solve for $\hat{\lambda}_0$ as follows:

$$-\text{tr}\big(P^{-1}\big[P - PAN^{-1}A^TP\big]\big)\hat{\lambda}_0 = 1 \Rightarrow$$
$$-\text{tr}(I_n - AN^{-1}A^TP)\hat{\lambda}_0 = 1 \Rightarrow$$
$$-\big[\text{tr}\,I_n - \text{tr}(N^{-1}A^TPA)\big]\hat{\lambda}_0 = 1 \Rightarrow$$
$$(n - m)\hat{\lambda}_0 = -1 \Rightarrow$$
$$\hat{\lambda}_0 = -1/(n - m). \tag{13.15f}$$

Now we substitute (13.15f) into (13.15d) and obtain

$$\text{vec}\,M = \text{vec}\,\big\{(n - m)^{-1}\big[P - PAN^{-1}A^TP\big]\big\}. \tag{13.15g}$$

Because the matrices within the vec operator in (13.15g) are of the same size, we can write

$$M = (n - m)^{-1}[P - PAN^{-1}A^TP]. \tag{13.15h}$$

Now we substitute (13.15h) into (13.3a) to obtain an expression for the estimated variance component as

$$\hat{\sigma}_0^2 = \boldsymbol{y}^T M\boldsymbol{y} = \boldsymbol{y}^T\big[(n - m)^{-1}(P - PAN^{-1}A^TP)\big]\boldsymbol{y} = \tag{13.16a}$$
$$= (n - m)^{-1}\big(\boldsymbol{y}^T P\boldsymbol{y} - \boldsymbol{y}^T PAN^{-1}A^TP\boldsymbol{y}\big). \tag{13.16b}$$

Equation (13.16b) is the BIQUUE for the unknown variance component σ_0^2. The estimated variance component $\hat{\sigma}_0^2$ has been determined independently from the parameter estimate $\hat{\boldsymbol{\xi}}$, which was our objective.

Let us verify that the two conditions stated in (13.6) are satisfied for matrix M.

First condition: $MA = 0$

$$MA = \big\{(n - m)^{-1}[P - PAN^{-1}A^TP]\big\}A = (n - m)^{-1}[PA - PAN^{-1}A^TPA] \Rightarrow$$
$$MA = (n - m)^{-1}[PA - PA] = 0 \tag{13.17a}$$

Second condition: $\operatorname{tr}(MP^{-1}) = 1$

$$\operatorname{tr}(MP^{-1}) = \operatorname{tr}\big(\{(n-m)^{-1}[P - PAN^{-1}A^T P]\}\big)P^{-1} \Rightarrow$$

$$\operatorname{tr}(MP^{-1}) = (n-m)^{-1}[\operatorname{tr} I_n - \operatorname{tr} PAN^{-1}A^T] = (n-m)^{-1}(\operatorname{tr} I_n - \operatorname{tr} I_m) = 1$$
$$\tag{13.17b}$$

Now using the symbols N and $\boldsymbol{c}$ introduced in (11.8), we can rewrite BIQUUE (13.16b) as follows:

$$\boxed{\hat{\sigma}_0^2 = \frac{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T \hat{\boldsymbol{\xi}}}{n-m} = \frac{\boldsymbol{y}^T P \boldsymbol{y} - \hat{\boldsymbol{\xi}}^T N \hat{\boldsymbol{\xi}}}{n-m} = \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{n-m}.} \tag{13.18}$$

The vector $\tilde{\boldsymbol{e}}$ in (13.18) is the same as the predicted residual vector associated with the BLUUE of $\boldsymbol{\xi}$. Thus the BIQUUE variance component $\hat{\sigma}_0^2$ agrees with that associated with BLUUE for $\boldsymbol{\xi}$. Also note that the use of the symbol $\hat{\boldsymbol{\xi}}$ in (13.18) is only done for convenience and does not mean that BIQUUE depends on the estimate for the parameter vector.

Incidentally, if we omit the uniformly-unbiased condition of (13.6), we arrive at BIQE, which differs from BIQUUE by an addition of 2 in the denominator.

$$\hat{\hat{\sigma}}_0^2 = \frac{\tilde{\boldsymbol{e}}^T P \tilde{\boldsymbol{e}}}{n-m+2} = \mathrm{BIQE}\{\sigma_0^2\} \tag{13.19}$$

The BIQUUE variance component $\hat{\sigma}_0^2$ is a random variable, so we want to find its expectation and dispersion.

First it is useful to compute the expectation $E\{\boldsymbol{y}\boldsymbol{y}^T\}$.

$$E\{\boldsymbol{y}\boldsymbol{y}^T\} = E\{(A\boldsymbol{\xi} + \boldsymbol{e})(A\boldsymbol{\xi} + \boldsymbol{e})^T\} = E\{A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T + A\boldsymbol{\xi}\boldsymbol{e}^T + \boldsymbol{e}\boldsymbol{\xi}^T A^T + \boldsymbol{e}\boldsymbol{e}^T\} =$$
$$= E\{A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T\} + E\{A\boldsymbol{\xi}\boldsymbol{e}^T\} + E\{\boldsymbol{e}\boldsymbol{\xi}^T A^T\} + E\{\boldsymbol{e}\boldsymbol{e}^T\} =$$
$$= A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T + A\boldsymbol{\xi}E\{\boldsymbol{e}^T\} + E\{\boldsymbol{e}\}\boldsymbol{\xi}^T A^T + D\{\boldsymbol{e}\} \Rightarrow$$
$$E\{\boldsymbol{y}\boldsymbol{y}^T\} = \sigma_0^2 P^{-1} + A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T \tag{13.20a}$$

Next we compute the expectation of $\hat{\sigma}_0^2$.

$$(n-m)E\{\hat{\sigma}_0^2\} = E\{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{y}^T PAN^{-1}A^T P \boldsymbol{y}\} =$$
$$= \operatorname{tr} E\{\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{y}^T PAN^{-1}A^T P \boldsymbol{y}\} = \operatorname{tr}(PE\{\boldsymbol{y}\boldsymbol{y}^T\}) - \operatorname{tr}(PAN^{-1}A^T PE\{\boldsymbol{y}\boldsymbol{y}^T\}) =$$
$$= \operatorname{tr}\big[P(\sigma_0^2 P^{-1} + A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T)\big] - \operatorname{tr}\big[PAN^{-1}A^T P(\sigma_0^2 P^{-1} + A\boldsymbol{\xi}\boldsymbol{\xi}^T A^T)\big] =$$
$$= \operatorname{tr}(\sigma_0^2 I_n + \boldsymbol{\xi}^T N \boldsymbol{\xi}) - \operatorname{tr}(PAN^{-1}A^T \sigma_0^2 + PAN^{-1}A^T PA\boldsymbol{\xi}\boldsymbol{\xi}^T A^T) =$$
$$= \operatorname{tr}(\sigma_0^2 I_n + \boldsymbol{\xi}^T N \boldsymbol{\xi}) - \operatorname{tr}(I_m \sigma_0^2 + \boldsymbol{\xi}^T N \boldsymbol{\xi}) \Rightarrow$$
$$(n-m)E\{\hat{\sigma}_0^2\} = \sigma_0^2(n-m) \tag{13.20b}$$

Finally, we can write the expectation of the BIQUUE variance component as

$$E\{\hat{\sigma}_0^2\} = \sigma_0^2. \tag{13.21}$$

Equation (13.21) shows that BIQUUE $\hat{\sigma}_0^2$ is indeed an unbiased estimate of σ_0^2.

The dispersion of BIQUUE $\hat{\sigma}_0^2$ is computed as follows: Considering (13.10) and that matrix M is symmetric, we write

$$D\{\hat{\sigma}_0^2\} = 2(\sigma_0^2)^2 \operatorname{tr}(MP^{-1}MP^{-1}).$$

Then, considering (13.15h)

$$D\{\hat{\sigma}_0^2\} = 2(\sigma_0^2)^2 \operatorname{tr}([P - PAN^{-1}A^T P]P^{-1}[P - PAN^{-1}A^T P]P^{-1})(n-m)^{-2} =$$
$$= 2(\sigma_0^2)^2 \operatorname{tr}([I_n - PAN^{-1}A^T]^2)(n-m)^{-2} =$$

(because $[I_n - PAN^{-1}A^T]$ is idempotent)

$$= 2(\sigma_0^2)^2 [\operatorname{tr} I_n - \operatorname{tr}(A^T PAN^{-1})](n-m)^{-2} =$$
$$= 2(\sigma_0^2)^2 (n-m)(n-m)^{-2},$$

finally resulting in

$$D\{\hat{\sigma}_0^2\} = 2(\sigma_0^2)^2/(n-m). \tag{13.22}$$

Equation (13.22) shows the true dispersion of the BIQUUE variance component $D\{\hat{\sigma}_0^2\}$ in terms of the true variance component σ_0^2. Equation (13.22) also implies that the estimated dispersion is provided by

$$\boxed{\hat{D}\{\hat{\sigma}_0^2\} = 2(\hat{\sigma}_0^2)^2/(n-m).} \tag{13.23}$$

From equation (13.23) we see that the estimated dispersion of the BIQUUE variance component will turn out to be relatively large unless the model redundancy $n-m$ is large.

We gave the solution to BIQE above; it can be shown that its dispersion is given by

$$D\{\hat{\hat{\sigma}}_0^2\} = 2(\sigma_0^2)^2/(n-m+2). \tag{13.24}$$

Chapter 14

Expectation-Dispersion Correspondence

An alternative approach to estimating the variance component σ_0^2 exploits the vec operator to a larger degree by changing the quadratic estimate to a linear estimate. Mathematically, this change is expressed by

$$\boldsymbol{y}^T M \boldsymbol{y} \to \text{vec}(\boldsymbol{y}^T M \boldsymbol{y}) = (\text{vec}\, M)^T (\boldsymbol{y} \otimes \boldsymbol{y}). \tag{14.1}$$

The first term in (14.1) is quadratic in $\boldsymbol{y}$. The rightmost term is linear in $(\boldsymbol{y} \otimes \boldsymbol{y})$. Note that the equation in (14.1) holds since (using (A.32))

$$\boldsymbol{y}^T M \boldsymbol{y} = \text{tr}(\boldsymbol{y}^T M \boldsymbol{y}) = \text{tr}(M^T \boldsymbol{y} I_1 \boldsymbol{y}^T) = (\text{vec}\, M)^T (\boldsymbol{y} \otimes \boldsymbol{y}), \tag{14.2}$$

where M is symmetric by definition.

The key idea is to change our original (quadratic) model so that the Best Linear Uniformly Unbiased Estimate (BLUUE) of the variance component in the revised (linear) model is the same as the BIQUUE for the variance component in the original model. We call this equivalence Expectation-Dispersion (E-D) Correspondence, so named because we rephrase the dispersion $D\{\hat{\sigma}_0^2\}$ as an expectation.

We begin by computing the expectation of the Kronecker product $\boldsymbol{y} \otimes \boldsymbol{y}$ in (14.1). Using (12.7) and (13.20a), we can write

$$E\{\boldsymbol{y} \otimes \boldsymbol{y}\} = \text{vec}\, E\{\boldsymbol{y}\boldsymbol{y}^T\} = \text{vec}(\sigma_0^2 P^{-1} + A \boldsymbol{\xi} \boldsymbol{\xi}^T A^T) = \tag{14.3a}$$

(applying (12.7) to $A\boldsymbol{\xi}$)

$$= (\text{vec}\, P^{-1})\sigma_0^2 + (A\boldsymbol{\xi} \otimes A\boldsymbol{\xi}) \Rightarrow \tag{14.3b}$$

(using (A.38))

$$E\{\boldsymbol{y} \otimes \boldsymbol{y}\} = (\text{vec}\, P^{-1})\sigma_0^2 + (A \otimes A)(\boldsymbol{\xi} \otimes \boldsymbol{\xi}). \tag{14.3c}$$

Both unknown quantities σ_0^2 and $\boldsymbol{\xi}$ appear in equation (14.3c). Note that we could estimate the term $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$ appearing (14.3c); however, $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$ tells us nothing about $\boldsymbol{\xi}$ itself, and therefore we consider $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$ to be a nuisance parameter. Note that the

size of $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$ is $m^2 \times 1$; however the product contains only $(m+1)/2$ independent elements. We need a matrix B that, when multiplied on the left of (14.3c), will eliminate the nuisance parameters $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$, i.e., $B(A \otimes A) = 0$. For example, we could choose B as

$$B := (I_n - AN^{-1}A^T P) \otimes (I_n - AN^{-1}A^T P). \tag{14.4}$$

It is apparent from (14.4) that $B(A \otimes A) = 0$. Also, using (A.43), we have $\operatorname{tr} B = (n-m)^2$, which is the number of independent equations left in the model. The rank of matrix B is easily computed by noting that the matrix within the parenthetical terms in (14.4) is idempotent and that the rank of an idempotent matrix equals its trace, and by using (A.43).

After multiplication by matrix B, the resulting model (now linear in $\boldsymbol{y} \otimes \boldsymbol{y}$) is not equivalent to the original model (which is linear in $\boldsymbol{y}$), but we choose to proceed with this matrix B anyway. Our next step is to find the expectation $E\{\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}\}$, and to do so we begin with $B(\boldsymbol{y} \otimes \boldsymbol{y})$ since

$$B(\boldsymbol{y} \otimes \boldsymbol{y}) = \big[(I_n - AN^{-1}A^T P) \otimes (I_n - AN^{-1}A^T P)\big](\boldsymbol{y} \otimes \boldsymbol{y}) = \tag{14.5a}$$

$$= (I_n - AN^{-1}A^T P)\boldsymbol{y} \otimes (I_n - AN^{-1}A^T P)\boldsymbol{y} = \tag{14.5b}$$

$$= (\boldsymbol{y} - A\hat{\boldsymbol{\xi}}) \otimes (\boldsymbol{y} - A\hat{\boldsymbol{\xi}}) = \tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}. \tag{14.5c}$$

Here, $\hat{\boldsymbol{\xi}}$ is the BLUUE for the parameter vector in the Gauss-Markov Model (GMM). Continuing, with the help of (14.3c), we find

$$E\{\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}\} = B \cdot E\{\boldsymbol{y} \otimes \boldsymbol{y}\} = B\big[(\operatorname{vec} P^{-1})\sigma_0^2 + (A \otimes A)(\boldsymbol{\xi} \otimes \boldsymbol{\xi})\big] = \tag{14.6a}$$

$$= B(\operatorname{vec} P^{-1})\sigma_0^2 + B(A \otimes A)(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \Rightarrow \tag{14.6b}$$

$$\boxed{E\{\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}\} = B(\operatorname{vec} P^{-1})\sigma_0^2.} \tag{14.6c}$$

With equation (14.6c) we have a linear model in σ_0^2. An alternative expression for $E\{\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}\}$ is derived as follows:

$$E(\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}) = B(\operatorname{vec} P^{-1})\sigma_0^2 =$$
$$= \big[(I_n - AN^{-1}A^T P) \otimes (I_n - AN^{-1}A^T P)\big](\operatorname{vec} P^{-1}) \cdot \sigma_0^2 = \tag{14.7a}$$

(applying (A.31))

$$= \operatorname{vec}\big[(I_n - AN^{-1}A^T P)P^{-1}(I_n - PAN^{-1}A^T) \cdot \sigma_0^2\big] = \tag{14.7b}$$

(transposing the symmetrical part)

$$= \operatorname{vec}\big[(I_n - AN^{-1}A^T P)(I_n - AN^{-1}A^T P)P^{-1} \cdot \sigma_0^2\big] \Rightarrow \tag{14.7c}$$

(exploiting the idempotent property)

$$\boxed{E(\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}) = \operatorname{vec}(P^{-1} - AN^{-1}A^T) \cdot \sigma_0^2.} \tag{14.7d}$$

Note that the matrix expression $(P^{-1} - AN^{-1}A^T)$ might be singular, but this poses no problem due to use of the vec operator. We now derive the dispersion of $\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}$.

First, note that

$$\tilde{e} = y - A\hat{\xi} = (A\xi + e) - A\hat{\xi} = e + A(\xi - \hat{\xi}). \tag{14.8}$$

This relation, along with the invariance principle, is exploited in the following:

$$D\{\tilde{e} \otimes \tilde{e}\} = D\{B(y \otimes y)\} = D\{B[(y - A\xi) \otimes (y - A\xi)]\} = \tag{14.9a}$$

$$= D\{B(e \otimes e)\} = BD\{e \otimes e\}B^T = \tag{14.9b}$$

$$= B\big(E\{(e \otimes e)(e \otimes e)^T\} - E\{e \otimes e\}E\{e \otimes e\}^T\big)B^T. \tag{14.9c}$$

Now, considering the first expectation term in (14.9c), and temporarily using various symbols beneath the variables to illustrate how the combinations are formed, we find

$$E\{(\underline{e} \otimes \underline{e})(\underline{e} \otimes \underline{e})^T\} = E\{\underline{e} \otimes \underline{e}\}E\{(\underline{e} \otimes \underline{e})^T\}+$$
$$+ E\{\underline{e}\,\underline{e}^T\} \otimes E\{\underline{e}\,\underline{e}^T\} + K(E\{\underline{e}\,\underline{e}^T\} \otimes E\{\underline{e}\,\underline{e}^T\}), \tag{14.10a}$$

or, more simply

$$E\{(e \otimes e)(e \otimes e)^T\} = E\{(e \otimes e)\}E\{(e \otimes e)^T\}+$$
$$+ E\{ee^T\} \otimes E\{ee^T\} + K(E\{ee^T\} \otimes E\{ee^T\}). \tag{14.10b}$$

In equation (14.10b) a commutation matrix K has been introduced by way of (A.40). Inserting (14.10b) into (14.9c) and making use of (13.2a) leads to

$$D\{\tilde{e} \otimes \tilde{e}\} = B(I + K)(P^{-1} \otimes P^{-1}) \cdot (\sigma_0^2)^2 \cdot B^T = \tag{14.11a}$$

$$= (\sigma_0^2)^2 \cdot (I + K)[B(P^{-1} \otimes P^{-1})B^T] \Rightarrow \tag{14.11b}$$

$$D\{\tilde{e} \otimes \tilde{e}\} = (\sigma_0^2)^2 \cdot (I + K)[(P^{-1} - AN^{-1}A^T) \otimes (P^{-1} - AN^{-1}A^T)]. \tag{14.11c}$$

In (14.11b) we have used the fact that B is a Kronecker product of the same matrix (see (14.4)), so that $BK = KB$. We may now combine equations (14.7d) and (14.11c) into one succinct expression describing the distribution of $\tilde{e} \otimes \tilde{e}$ as follows:

$$\boxed{\begin{aligned} \tilde{e} \otimes \tilde{e} \sim (\sigma_0^2 \cdot \text{vec}(P^{-1} - AN^{-1}A^T), \\ (\sigma_0^2)^2 \cdot (I + K)[(P^{-1} - AN^{-1}A^T) \otimes (P^{-1} - AN^{-1}A^T)]). \end{aligned}} \tag{14.12}$$

Note that both the expectation and dispersion in (14.12) contain the parameter σ_0^2. Also note that the matrix comprised of the Kronecker product is singular. This equation has some similarities to the GMM, enough to try the least-squares solution (LESS) approach to estimate σ_0^2. We call this approach *E-D Correspondence*, the concept of which is summarized in the diagram below.

$$\boxed{y \sim (A\xi, \sigma_0^2 P^{-1}) \underset{\text{Correspondence}}{\overset{\text{E-D}}{\rightarrow}} \tilde{e} \otimes \tilde{e} \sim (E\{\tilde{e} \otimes \tilde{e}\}, D\{\tilde{e} \otimes \tilde{e}\})} \tag{14.13}$$

In order to proceed with the estimation of σ_0^2 using LESS, we need to handle the singular dispersion matrix in (14.12), which requires a generalized inverse (g-inverse). The g-inverse of a $p \times q$ matrix G is defined as the $q \times p$ matrix G^- such that

$$GG^-G = G. \tag{14.14}$$

We seek a g-inverse for the matrix $(P^{-1} - AN^{-1}A^T)$, which is provided by $(P - PAN^{-1}A^TP)$, since

$$(P^{-1} - AN^{-1}A^T)(P - PAN^{-1}A^TP)(P^{-1} - AN^{-1}A^T) = (P^{-1} - AN^{-1}A^T). \tag{14.15}$$

Furthermore, we define the (singular) cofactor matrix from (14.12) as

$$Q := (I + K)\big[(I_n - AN^{-1}A^TP)P^{-1} \otimes (I_n - AN^{-1}A^TP)P^{-1}\big], \tag{14.16}$$

where the term $(I+K)$ is essentially a factor of 2 in Q. Let the g-inverse of matrix Q be called W, so that $QWQ = Q$. The following matrix satisfies this equation:

$$\boxed{W := \frac{1}{4}(I + K)\big[P(I_n - AN^{-1}A^TP) \otimes P(I_n - AN^{-1}A^TP)\big].} \tag{14.17}$$

Note that multiplication of matrix W on both the right and left by Q yields

$$QWQ = \frac{1}{4}(I + K)^3\big[(I_n - AN^{-1}A^TP)P^{-1} \otimes (I_n - AN^{-1}A^TP)P^{-1}\big], \tag{14.18a}$$

but

$$\frac{1}{4}(I + K)^3 = \frac{1}{4}(I + 3K + 3K^2 + K^3) = \frac{1}{4}(I + 3K + 3I + IK) = I + K. \tag{14.18b}$$

So, indeed, W is a g-inverse of Q.

In the GMM we reach LESS by minimization of the target function $\Phi = (\boldsymbol{y} - A\boldsymbol{\xi})^T P(\boldsymbol{y} - A\boldsymbol{\xi})$. Now we are able to write an analogous LESS target function for the estimated variance component σ_0^2 using the g-inverse W derived above.

LESS target function:

$$\begin{aligned}
\big[(\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}) - E\{\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}\}\big]^T W\big[(\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}) - E\{\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}\}\big] = \\
= \{(\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}) - \mathrm{vec}\big[(I_n - AN^{-1}A^TP)P^{-1}\big]\sigma_0^2\}^T \cdot \\
\cdot W \cdot \{(\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}) - \mathrm{vec}\big[(I_n - AN^{-1}A^TP)P^{-1}\big]\sigma_0^2\} = \min_{\sigma_0^2}.
\end{aligned} \tag{14.19}$$

Following the LESS approach, we write a system of normal equations directly, based on the target function (14.19).

LESS normal equations:

$$\begin{aligned}
\big\{\mathrm{vec}\big[(I_n - AN^{-1}A^TP)P^{-1}\big]\big\}^T W\, \mathrm{vec}\big[(I_n - AN^{-1}A^TP)P^{-1}\big] \cdot \hat{\sigma}_0^2 = \\
= \big\{\mathrm{vec}\big[(I_n - AN^{-1}A^TP)P^{-1}\big]\big\}^T W(\tilde{\boldsymbol{e}} \otimes \tilde{\boldsymbol{e}}).
\end{aligned} \tag{14.20}$$

To derive a solution for the estimated variance component σ_0^2, we first simplify

$$\{\mathrm{vec}[(I_n - AN^{-1}A^TP)P^{-1}]\}^TW,$$

since it appears in both sides of (14.20). Here we use (A.31) and the fact that $(I+K)$ is equivalent to a factor of 2 when multiplied in W.

$$\{\mathrm{vec}[(I_n - AN^{-1}A^TP)P^{-1}]\}^TW =$$
$$= \frac{1}{2}\{\mathrm{vec}[(I_n - PAN^{-1}A^T)P(I_n - AN^{-1}A^TP)P^{-1}P(I_n - AN^{-1}A^TP)]\}^T =$$
$$= \frac{1}{2}[\mathrm{vec}(P - PAN^{-1}A^TP)]^T = \{\mathrm{vec}[(I_n - AN^{-1}A^TP)P^{-1}]\})^TW \qquad (14.21\mathrm{a})$$

Now we substitute (14.21a) into the right side of (14.20) to arrive at

$$\{\mathrm{vec}[(I_n - AN^{-1}A^TP)P^{-1}]\}^TW(\tilde{e}\otimes\tilde{e}) = \frac{1}{2}[\mathrm{vec}(P - PAN^{-1}A^TP)]^T(\tilde{e}\otimes\tilde{e}) =$$
$$= \frac{1}{2}\tilde{e}^T(P - PAN^{-1}A^TP)\tilde{e} = \frac{1}{2}\tilde{e}^TP\tilde{e}, \qquad (14.21\mathrm{b})$$

since $A^TP\tilde{e} = \mathbf{0}$ according to (11.9b). Next we substitute (14.21a) into the left side of (14.20).

$$\{\mathrm{vec}[(I_n - AN^{-1}A^TP)P^{-1}]\}^TW\,\mathrm{vec}[(I_n - AN^{-1}A^TP)P^{-1}]\hat{\sigma}_0^2 =$$
$$= \frac{1}{2}[\mathrm{vec}(P - PAN^{-1}A^TP)]^T\,\mathrm{vec}[(I_n - AN^{-1}A^TP)P^{-1}]\hat{\sigma}_0^2 =$$

(Continuing with help of (A.32), where matrices A and C are identity in that equation)

$$= \frac{1}{2}\mathrm{tr}[(I_n - AN^{-1}A^TP)P^{-1}(P - PAN^{-1}A^TP)]\hat{\sigma}_0^2 =$$
$$= \frac{1}{2}\mathrm{tr}[(I_n - AN^{-1}A^TP)(I_n - AN^{-1}A^TP)]\hat{\sigma}_0^2 =$$
$$= \frac{1}{2}\mathrm{tr}[(I_n - AN^{-1}A^TP)]\hat{\sigma}_0^2 = \frac{1}{2}\mathrm{tr}\{(I_n - N^{-1}A^TPA)\}\hat{\sigma}_0^2 =$$
$$= \frac{1}{2}(n - m)\hat{\sigma}_0^2. \qquad (14.21\mathrm{c})$$

Finally, we equate the left side (14.21b) and right side (14.21c) to obtain

$$\frac{1}{2}\tilde{e}^TP\tilde{e} = \frac{1}{2}(n-m)\hat{\sigma}_0^2, \qquad (14.21\mathrm{d})$$

resulting in the following estimate for the variance component σ_0^2:

$$\boxed{\hat{\sigma}_0^2 = \frac{\tilde{e}^TP\tilde{e}}{(n-m)}.} \qquad (14.22)$$

We conclude that LESS for the model (14.12) is equivalent to BIQUUE in the GMM (13.1), which is evident from the respective formulas for the estimated variance component (14.22) and (13.18).

We can also use E-D correspondence to derive the *Best Linear Estimate*, BLE, of σ_0^2. This is done by expressing the estimate $\hat{\hat{\sigma}}_0^2$ as a linear function of $\tilde{e} \otimes \tilde{e}$ and minimizing its MSE. The solution is equivalent to the BIQE mentioned in Chapter 13. The problem is setup below.

- Linear requirement:

$$\hat{\hat{\sigma}}_0^2 = L^T (\tilde{e} \otimes \tilde{e}). \tag{14.23}$$

- Best requirement:

$$\mathrm{MSE}\{\hat{\hat{\sigma}}_0^2\} = D\{\hat{\hat{\sigma}}_0^2\} + (E\{\hat{\hat{\sigma}}_0^2\} - \sigma_0^2)^2 =$$
$$= L^T D\{\tilde{e} \otimes \tilde{e}\} + L^T E\{\tilde{e} \otimes \tilde{e}\} E\{\tilde{e} \otimes \tilde{e}\}^T L - 2\sigma_0^2 L^T E\{\tilde{e} \otimes \tilde{e}\} + (\sigma_0^2)^2 = \min_{L}. \tag{14.24}$$

- Solution:

$$\hat{\hat{\sigma}}_0^2 = \frac{\tilde{e}^T P \tilde{e}}{n - m + 2} = \mathrm{BLE}\{\sigma_0^2\} = \mathrm{BIQE}\{\sigma_0^2\}. \tag{14.25}$$

Chapter 15

The Rank-Deficient
Gauss-Markov Model

The rank-deficient Gauss-Markov Model (GMM) describes the case where the co-efficient matrix A (also called design matrix or information matrix) does not have full column rank. As usual, we are dealing with n observations and m parameters so that the (linearized) observation vector $\boldsymbol{y}$ is of size $n \times 1$, while matrix A is of size $n \times m$. The model is stated as follows:

$$\boldsymbol{y} = A\boldsymbol{\xi} + \boldsymbol{e}, \;\; \boldsymbol{e} \sim (\boldsymbol{0}, \Sigma = \sigma_0^2 P^{-1}), \;\; \mathrm{rk}\, A =: q < m < n. \tag{15.1}$$

As in earlier chapters, the least-squares normal equations are written compactly as

$$N\hat{\boldsymbol{\xi}} = \boldsymbol{c}, \tag{15.2a}$$

where

$$\begin{bmatrix} N, \, \boldsymbol{c} \end{bmatrix} := A^T P \begin{bmatrix} A, \, \boldsymbol{y} \end{bmatrix}. \tag{15.2b}$$

The ranks of the $m \times m$ normal-equations matrix N and the $n \times m$ coefficient matrix A are related by

$$\mathrm{rk}\, N = \dim \mathcal{R}(N) \leq \dim \mathcal{R}(A^T) = \mathrm{rk}\, A^T = \mathrm{rk}\, A = q < m, \tag{15.3}$$

where the symbol $\mathcal{R}$ stands for range space (also called column space or kernel).

Here we have assumed that $m < n$, meaning that it is not a lack of observations that gives rise to the rank deficiency but that the system of redundant observation equations does not carry enough information about the parameters to estimate all of them. In terms of the columns of matrix A, it can be said that only q of them are linearly independent and that each of the remaining $m - q$ of them can be expressed as a linear combination of the q independent ones.

The less-than-or-equals sign in (15.3) denotes a more general relationship than what is needed here. We may change it to the equality sign since the weight matrix P

is positive definite, which means $\dim \mathcal{R}(N) = \dim \mathcal{R}(A^T)$. Therefore,

$$\operatorname{rk} N = \operatorname{rk} A = q < m. \tag{15.4}$$

Likewise, we can make a statement about the range spaces of matrices N and A^T as follows:

$$\mathcal{R}(N) \subset \mathcal{R}(A^T) \ \text{ and } \ \dim \mathcal{R}(N) = \operatorname{rk} N = \operatorname{rk} A^T = \dim \mathcal{R}(A^T) \Leftrightarrow \tag{15.5a}$$
$$\mathcal{R}(N) = \mathcal{R}(A^T). \tag{15.5b}$$

Question: Do solutions for $\hat{\boldsymbol{\xi}}$ always exist? Yes, because

$$\boldsymbol{c} := A^T P \boldsymbol{y} \in \mathcal{R}(A^T) = \mathcal{R}(N). \tag{15.6}$$

In other words, the vector $\boldsymbol{c}$ is in the range (column) space of N, which guarantees that we have solutions for $N\hat{\boldsymbol{\xi}} = \boldsymbol{c}$.

Question: How many solutions for $\hat{\boldsymbol{\xi}}$ are there and how do we represent them? The general solution $\hat{\boldsymbol{\xi}}$ belongs to a solution hyperspace that is shifted out of the origin by a particular solution $\hat{\boldsymbol{\xi}}_{\text{part}}$, where $\hat{\boldsymbol{\xi}}_{\text{part}}$ is a solution to an inhomogeneous system of equations. Running parallel to the set of all particular solutions, and through the origin of the solution hyperspace, is the nullspace of N, which is comprised of all the solutions to the homogeneous system $N\hat{\boldsymbol{\xi}} = \boldsymbol{0}$. Therefore we can write the general solution, as the sum of the particular solutions and the nullspace of N, denoted $\mathcal{N}(N)$, as in

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}_{\text{part}} + \mathcal{N}(N). \tag{15.7}$$

Figure 15.1 shows a graphical representation of equation (15.7). Obviously there are infinite choices for the particular solution $\hat{\boldsymbol{\xi}}_{\text{part}}$, and thus there are infinitely many solutions for the unknown parameters within the rank deficient GMM.

Using set notation, we may characterize the nullspaces of N and A by

$$\mathcal{N}(N) := \{\boldsymbol{\alpha} \mid N\boldsymbol{\alpha} = \boldsymbol{0}\} \ \text{ and } \ \mathcal{N}(A) := \{\boldsymbol{\alpha} \mid A\boldsymbol{\alpha} = \boldsymbol{0}\}, \tag{15.8}$$

respectively. When $A\boldsymbol{\alpha} = \boldsymbol{0}$ so does $N\boldsymbol{\alpha}$; therefore

$$\mathcal{N}(N) \subset \mathcal{N}(A). \tag{15.9a}$$

Also

$$\dim \mathcal{N}(N) = \dim \mathcal{N}(A) = m - q, \tag{15.9b}$$

since

$$\mathcal{N}(A) \overset{\perp}{\oplus} \mathcal{R}(A^T) = \mathbb{R}^m, \tag{15.9c}$$

which says that the nullspace of matrix A and the range space of A^T are both complimentary and orthogonal subspaces of one another. Because of (15.9a) and (15.9b), we can state that

$$\mathcal{N}(N) = \mathcal{N}(A), \tag{15.9d}$$
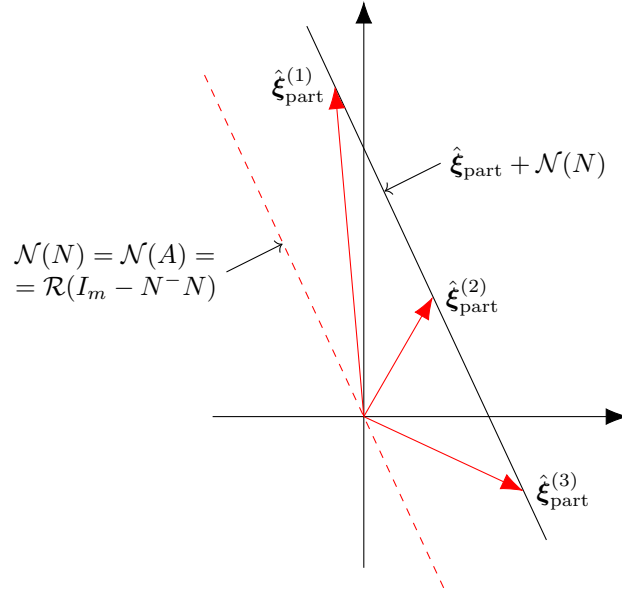
Figure 15.1: A schematic representation of the solution space for $\hat{\boldsymbol{\xi}}$, showing three example particular solutions out of an infinite number of them

allowing us to extend (15.7) to

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}_{\text{part}} + \mathcal{N}(N) = \hat{\boldsymbol{\xi}}_{\text{part}} + \mathcal{N}(A). \tag{15.10}$$

Thus we can generate all solutions $\hat{\boldsymbol{\xi}}$ if we know how to find a particular solution $\hat{\boldsymbol{\xi}}_{\text{part}}$ and if we know how to generate the nullspace of matrix A (or the nullspace of N). To find the nullspace, we must turn to the topic of generalized inverses.

15.1 Generalized Inverses

Generalized inverses (called g-inverses by Rao (1965)) are important for solving systems of equations that have singular coefficient matrices. Let G be the g-inverse of matrix N (with both G and N of size $m \times m$), then

$$NGN = N, \tag{15.11a}$$

implying that

$$N(I_m - GN) = 0, \tag{15.11b}$$

which further implies that

$$\mathcal{R}(I_m - GN) \subset \mathcal{N}(N). \tag{15.11c}$$

Question: are the two spaces shown in (15.11c) equivalent? The answer is yes, as shown below.

The matrix $(I_m - GN)$ is idempotent since

$$(I_m - GN)(I_m - GN) = (I_m - GN) - GN(I_m - GN) = (I_m - GN). \quad (15.12a)$$

Therefore $\mathrm{rk}(I_m - GN) = \mathrm{tr}(I_m - GN)$, and the dimension of the range space is

$$\dim \mathcal{R}(I_m - GN) = \mathrm{rk}(I_m - GN) = \mathrm{tr}(I_m - GN) = m - \mathrm{tr}(GN). \quad (15.12b)$$

But, GN itself is also idempotent; therefore:

$$\dim \mathcal{R}(I_m - GN) = m - \mathrm{tr}(GN) = m - \dim \mathcal{R}(GN) = m - q =$$
$$= m - \dim \mathcal{R}(N) = \dim \mathcal{N}(N). \quad (15.12c)$$

Because of (15.11c) and (15.12c), we can indeed say that the spaces in (15.11c) are equivalent, i.e.,

$$\mathcal{R}(I_m - GN) = \mathcal{N}(N). \quad (15.12d)$$

We have relied on the fact that, if one space is contained in another and the dimension of both spaces are the same, then the spaces must be equivalent.

Let the g-inverse matrix G be represented by the symbol N^-, then

$$\mathcal{R}(I_m - N^- N) = \mathcal{N}(N) \text{ for any } N^- \text{ of } N. \quad (15.13)$$

Using (15.13) together with (15.10), we are now ready to write the complete solution space of $\hat{\boldsymbol{\xi}}$ as

$$\boxed{\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}_{\mathrm{part}} + (I_m - N^- N)\boldsymbol{z} \text{ for any } \boldsymbol{z} \in \mathbb{R}^m \text{ and any chosen g-inverse } N^-.}$$

$$(15.14)$$

As an aside, we show the dimension of the range space of the idempotent matrix GN used in (15.12c). The rank of a product of matrices must be less than or equal to the rank of any factor. Therefore,

$$\mathrm{rk}(NGN) \leq \mathrm{rk}(GN) \leq \mathrm{rk}\, N = \mathrm{rk}(NGN), \quad (15.15a)$$

implying that

$$\mathrm{rk}(GN) = \mathrm{rk}\, N = q. \quad (15.15b)$$

15.2   Finding a Generalized Inverse

Note the following properties associated with the $m \times m$ normal-equations matrix N and its g-inverse G:

1. GN is idempotent (and as such, is a projection matrix), and so is NG. That is, $GN \cdot GN = GN$, and $NG \cdot NG = NG$.

2. $I_m - GN$ is idempotent.

3. $m \geq \operatorname{rk} G \geq \operatorname{rk} N = q$.

Item 3 states that the g-inverse G of N will always have equal or greater rank than that of N itself.

An important subclass of g-inverses is the *reflexive g-inverse*. If G is a g-inverse of N, and if N is also a g-inverse of G, then we say that G is a reflexive g-inverse of N. Considering item 3 above, if $GNG = G$, then $\operatorname{rk} N = q \geq \operatorname{rk} G \Rightarrow \operatorname{rk} G = q$. So, if we are given a g-inverse of rank q, it must be reflexive.

Another important g-inverse subclass is the Moore-Penrose inverse[1], which is also called the *pseudoinverse*. If the following four conditions are met, then the g-inverse G is the pseudoinverse of N denoted as N^+:

$$\left.\begin{aligned} NGN &= N \\ GNG &= G \\ NG &= (NG)^T \\ GN &= (GN)^T \end{aligned}\right\} \Leftrightarrow G = N^+ \tag{15.16}$$

The pseudoinverse is unique, and if N has full rank (i.e. $\operatorname{rk} N = m$), the pseudoinverse is the same as the regular matrix inverse N^{-1}.

Note that g-inverses of N do not need to be symmetric. However, if G is a g-inverse of N, then G^T is as well. This is shown in the following:

$$(NGN)^T = N^T = N = N^T G^T N^T = NG^T N. \tag{15.17}$$

We also note that the pseudoinverse of a symmetric matrix is itself symmetric, and that $N^+ = N^+ N (N^+)^T$ is positive semidefinite (assuming N is singular). However, as already stated, an arbitrary g-inverse G might not be symmetric and also might not be positive semidefinite. However a reflexive symmetric (and therefore positive-semidefinite) g-inverse defined as

$$N_{rs}^- = GNG^T \tag{15.18a}$$

is characterized by

$$\operatorname{rk} N_{rs}^- = \operatorname{rk} N = q \tag{15.18b}$$

and

$$N_{rs}^- = N_{rs}^- N (N_{rs}^-)^T = N_{rs}^- N N_{rs}^-. \tag{15.18c}$$

The g-inverse N_{rs}^- is in the class of reflexive symmetric g-inverses, which is a very important class for the work that follows. We note that a reflexive symmetric g-inverse can always be found from a given arbitrary g-inverse N^- by

$$N_{rs}^- = N^- N (N^-)^T. \tag{15.19}$$

There are many ways to construct a g-inverse of N. We show several examples below. In some of the examples we use a more generic symbol A in order to stress

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[1]According to Cross (1985), this g-inverse was first discovered by Moore in 1920 and then independently by Penrose in 1955.

that the matrix does not have to be symmetric or even square. For the discussion that follows, it is helpful to partition N so that the upper-left $q \times q$ block matrix N_{11} has rank q as follows:

$$\underset{m \times m}{N} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \quad \dim N_{11} = q \times q, \quad \text{rk } N_{11} = q = \text{rk } N. \qquad (15.20)$$

The equations in (15.20) imply that the second column of the partitioned matrix is a linear combination of the first column. Therefore, for some $q \times (m-q)$ matrix L, we have

$$\begin{bmatrix} N_{12} \\ N_{22} \end{bmatrix} = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} \cdot L, \quad \text{or } N_{12} = N_{11}L \text{ and } N_{22} = N_{21}L. \qquad (15.21)$$

In practice, the rows and columns of N might have to be reordered to ensure that N_{11} is full rank as shown in (15.20), but that is usually easy to do. Also note that since N is positive semidefinite, it can be decomposed as follows:

$$N = \begin{bmatrix} U^T DU & U^T DH \\ H^T DU & H^T DH \end{bmatrix} = \begin{bmatrix} U^T \\ H^T \end{bmatrix} D \begin{bmatrix} U & H \end{bmatrix} = \begin{bmatrix} (D^{1/2}U)^T \\ (D^{1/2}H)^T \end{bmatrix} \begin{bmatrix} D^{1/2}U & D^{1/2}H \end{bmatrix}.$$
$$(15.22)$$

Here, U is an upper triangular matrix of size $q \times q$; H is size $q \times (m-q)$, and D is a $q \times q$ diagonal matrix. Also note that $(D^{1/2}U)^T$ is the Cholesky factor of N_{11}.

g-inverse example 1:   (with N defined as in (15.22))

$$N^- = \begin{bmatrix} U^T \\ H^T \end{bmatrix} (UU^T + HH^T)^{-1} D^{-1} (UU^T + HH^T)^{-1} \begin{bmatrix} U & H \end{bmatrix} \qquad (15.23)$$

Check:

$$NN^- = \begin{bmatrix} U^T \\ H^T \end{bmatrix} D \begin{bmatrix} U & H \end{bmatrix} \begin{bmatrix} U^T \\ H^T \end{bmatrix} (UU^T + HH^T)^{-1} D^{-1} (UU^T + HH^T)^{-1} \begin{bmatrix} U & H \end{bmatrix} =$$

$$= \begin{bmatrix} U^T \\ H^T \end{bmatrix} (UU^T + HH^T)^{-1} \begin{bmatrix} U & H \end{bmatrix} \Rightarrow$$

$$NN^-N = \begin{bmatrix} U^T \\ H^T \end{bmatrix} (UU^T + HH^T)^{-1} \begin{bmatrix} U & H \end{bmatrix} \begin{bmatrix} U^T \\ H^T \end{bmatrix} D \begin{bmatrix} U & H \end{bmatrix} =$$

$$= \begin{bmatrix} U^T \\ H^T \end{bmatrix} D \begin{bmatrix} U & H \end{bmatrix} = N$$

Since this g-inverse N^- has rank q, it is reflexive. Obviously it is also symmetric. Therefore, it could also be labeled N_{rs}^-. In this case it also satisfies all the properties of a pseudoinverse.

g-inverse example 2: (see Lütkepohl (1996), section 9.12.3, item (3))

$$N^- = \begin{bmatrix} (D^{1/2}U)^T \\ (D^{1/2}H)^T \end{bmatrix} \left(\begin{bmatrix} D^{1/2}U & D^{1/2}H \end{bmatrix} \begin{bmatrix} (D^{1/2}U)^T \\ (D^{1/2}H)^T \end{bmatrix} \right)^{-2} \begin{bmatrix} D^{1/2}U & D^{1/2}H \end{bmatrix} = N^+$$

(15.24)

The properties shown in (15.16) for the pseudoinverse can be verified for (15.24) after some tedious matrix multiplications.

g-inverse example 3:

$$N^- = \begin{bmatrix} N_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} = N_{rs}^{-1}$$

(15.25)

Check:

$$NN^-N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} N_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ N_{21}N_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} N_{11} & N_{12} \\ N_{21}N_{11}^{-1}N_{11} & N_{21}N_{11}^{-1}N_{12} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{21}N_{11}^{-1}N_{12} \end{bmatrix}$$

From (15.21) we have $N_{12} = N_{11}L$ and $N_{22} = N_{21}L$ so that $N_{21}N_{11}^{-1}N_{12} = N_{21}L = N_{22}$, which completes the check for a g-inverse. Reflexivity is easy to check also.

g-inverse example 4: By rank factorization, the $n \times m$ matrix A may be factored into the product of an $n \times q$ matrix F and a $q \times m$ matrix H, where $\operatorname{rk} A = \operatorname{rk} F = \operatorname{rk} H = q$ and $A = FH$. Then a reflexive g-inverse of A may be obtained by

$$A_r^- = H^T(HH^T)^{-1}(F^TF)^{-1}F^T.$$

(15.26)

15.3 The Singular Value Decomposition

Given a matrix A of size $n \times m$ and $\operatorname{rk} A = q$, the *singular values* of A are the positive square roots of the positive eigenvalues of A^TA or AA^T, which are square, symmetric matrices with real eigenvalues. (Note that only the positive eigenvalues of the matrix products A^TA and AA^T are the same.) Let the diagonal $n \times m$ matrix Λ contain q non-zero elements, being the singular values λ_j of A, where $j = 1, \ldots, q$. Let U be the orthogonal $n \times n$ matrix whose columns are the eigenvectors of AA^T, and let V be the orthogonal $m \times m$ matrix whose columns are the eigenvectors of A^TA. Then

$$A = U\Lambda V^T$$

(15.27)

is the *Singular Value Decomposition* (SVD) of matrix A. Note that if A is symmetric, $U = V$. The g-inverse examples 5 through 8 below are all based on the SVD.

g-inverse example 5:   Define a $q \times q$ diagonal matrix as

$$\Delta^{-1} = \operatorname{diag}(\lambda_i, \ldots, \lambda_q). \tag{15.28}$$

Then there exists a set of g-inverses of A defined by

$$\{A^-\} = \{V \begin{bmatrix} \Delta^{-1} & K \\ L & M \end{bmatrix} U^T \mid K, L, M \text{ arbitrary with suitable size}\}. \tag{15.29}$$

The rank of the block matrix can vary between q and m depending on the choices for K, L, and M.

g-inverse example 6:   Let the matrix M in (15.29) be defined as $M := L\Delta K$, then

$$\{A_r^-\} = \{V \begin{bmatrix} \Delta^{-1} & K \\ L & L\Delta K \end{bmatrix} U^T \mid K, L \text{ arbitrary with suitable size}\} \tag{15.30}$$

represents a set of g-inverses of A.

g-inverse example 7:   Let the matrix M in (15.29) be defined as $M := L\Delta L^T$, then

$$\{A_{rs}^-\} = \{V \begin{bmatrix} \Delta^{-1} & L^T \\ L & L\Delta L^T \end{bmatrix} U^T \mid L \text{ arbitrary with suitable size}\} \tag{15.31}$$

represents a set of g-inverses of A.

g-inverse example 8:

$$A^+ = V \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T \tag{15.32}$$

g-inverse example 9:   Zlobec's formula for the pseudoinverse is

$$N^+ = N(NNN)^- N, \tag{15.33}$$

where the g-inverse can be any g-inverse of N^3. The invariance of N^+ with respect to the choice of the g-inverse in Zlobec's formula is due to the g-inverse's placement between the two occurrences of matrix N. Again we note that the pseudoinverse is unique, but there are a variety of ways to generate it.

Now that we have seen how to generate a g-inverse, the next question regarding our general solution (15.14) is "how do we represent the particular solution $\hat{\boldsymbol{\xi}}_{\text{part}}$?" We claim that $\hat{\boldsymbol{\xi}}_{\text{part}}$ is represented by $N^- \boldsymbol{c}$ (or equivalently $N^- A^T P \boldsymbol{y}$) since $NN^- A^T P \boldsymbol{y} = \boldsymbol{c}$. To validate this claim, we must show that

$$NN^- A^T = A^T, \tag{15.34a}$$

which is done as follows. Because N begins with A^T, we can write

$$\mathcal{R}(NN^-A^T) \subset \mathcal{R}(A^T). \tag{15.34b}$$

Furthermore,

$$\dim \mathcal{R}(NN^-A^T) = \mathrm{rk}(NN^-A^T) \geq \mathrm{rk}([NN^-A^T]PA) = \tag{15.34c}$$

$$= \mathrm{rk}\,NN^-N = \mathrm{rk}\,N = q \geq \mathrm{rk}(NN^-A^T), \tag{15.34d}$$

or

$$\mathrm{rk}(NN^-A^T) \geq q \geq \mathrm{rk}(NN^-A^T), \tag{15.34e}$$

implying that

$$\mathrm{rk}(NN^-A^T) = q. \tag{15.34f}$$

But also $\dim \mathcal{R}(A^T) = q$, which further implies

$$\mathcal{R}(NN^-A^T) = \mathcal{R}(A^T), \tag{15.34g}$$

since if one range space belongs to another and they both have the same dimension, they must be equivalent. Thus, we conclude that

$$NN^-A^T = A^T, \tag{15.35a}$$

and, after transposing,

$$AN^-N = A. \tag{15.35b}$$

Therefore it follows that

$$(NN^-A^T)P\boldsymbol{y} = A^TP\boldsymbol{y} = \boldsymbol{c}. \tag{15.35c}$$

We can now write our general solution (15.14) in terms of $N^-\boldsymbol{c}$ as follows:

$$\boxed{\hat{\boldsymbol{\xi}} = \{N^-\boldsymbol{c} + (I_m - N^-N)\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in \mathbb{R}^m\} \text{ for any chosen g-inverse } N^-,}$$

$$\tag{15.36}$$

where $\boldsymbol{\alpha}$ is an arbitrary, but non-random, $m \times 1$ vector.

From the law of error propagation, we find the dispersion of the general solution to be

$$\boxed{\begin{aligned} D\{\hat{\boldsymbol{\xi}}\} &= D\{N^-\boldsymbol{c} + (I_m - N^-N)\boldsymbol{\alpha}\} = \\ &= N^-D\{\boldsymbol{c}\}(N^-)^T = \sigma_0^2 N^-N(N^-)^T = \sigma_0^2 N_{rs}^-. \end{aligned}} \tag{15.37}$$

We now verify that the dispersion matrix in (15.37) is indeed a reflexive symmetric g-inverse.

1. Obviously the dispersion matrix $\sigma_0^2 N^-N(N^-)^T$ is symmetric.

2. The matrix $N^-N(N^-)^T$ is a g-inverse of N because:

$$N[N^-N(N^-)^T]N = (NN^-N)(N^-)^TN = N(N^-)^TN = N,$$

recalling that if N^- is a g-inverse of N, so is $(N^-)^T$.

3. The matrix N is a g-inverse of $N^- N (N^-)^T$ because:

$$[N^- N (N^-)^T] N [N^- N (N^-)^T] = N^- (N (N^-)^T N)[N^- N (N^-)^T] =$$
$$= N^- N [N^- N (N^-)^T] = N^- (N N^- N)(N^-)^T = N^- N (N^-)^T.$$

Because the dispersion (covariance) matrix in (15.37) is represented by a reflexive symmetric g-inverse of N, we may, without loss of generality, restrict ourselves to reflexive symmetric g-inverses in our search for a general solution $\hat{\boldsymbol{\xi}}$. Furthermore, the covariance matrix cannot have a rank greater than that of N, and thus it is positive semidefinite (cf. (15.15b) and the sentence that precedes it).

We have infinite choices for our particular solution $\hat{\boldsymbol{\xi}}_{\text{part}}$, but one of particular interest is that which is shortest in magnitude (i.e., smallest L_2-norm). This particular solution can be derived by imposing a minimum norm condition on the parameter vector in the least-squares target function; it is thus called MInimum NOrm LEast-Squares Solution (MINOLESS) and is developed formally in the following section.

15.4   Minimum Norm Least-Squares Solution

MINOLESS is an acronym for MInimum NOrm LEast-Squares Solution. We know that, within the rank deficient GMM, $N\hat{\boldsymbol{\xi}} = \boldsymbol{c}$ has many solutions; we seek the shortest (minimum norm) of these. The idea is to minimize the norm (inner product), of $\hat{\boldsymbol{\xi}}$, according to

$$\boldsymbol{\xi}^T \boldsymbol{\xi} = \min_{\boldsymbol{\xi}} \ \text{ such that } N\boldsymbol{\xi} = \boldsymbol{c}. \tag{15.38}$$

Thus, if $\boldsymbol{\xi}$ is an "incremental" parameter vector, as it is under linearization, minimum norm means minimum change from the initial vector $\boldsymbol{\xi}^0$, e.g., the initial Taylor series expansion point.

The Lagrange target function to minimize is written as

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\lambda}) := \boldsymbol{\xi}^T \boldsymbol{\xi} + 2\boldsymbol{\lambda}^T (N\boldsymbol{\xi} - \boldsymbol{c}), \tag{15.39}$$

which must be made made stationary for $\boldsymbol{\xi}$ and $\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ is an $m \times 1$ vector of Lagrange multipliers. Accordingly, the Euler-Lagrange (first-order) necessary conditions are then written as

$$\frac{1}{2} \frac{\partial \Phi}{\partial \boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}^T + \hat{\boldsymbol{\lambda}}^T N \doteq \boldsymbol{0}, \tag{15.40a}$$

$$\frac{1}{2} \frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = \hat{\boldsymbol{\xi}}^T N - \boldsymbol{c}^T \doteq \boldsymbol{0}. \tag{15.40b}$$

The sufficient condition (i.e., that second partial derivative must be positive) is satisfied since $(1/2)\partial^2 \Phi / \partial\boldsymbol{\xi}\partial\boldsymbol{\xi}^T = I_n$, which is positive definite.

Equations (15.40a) and (15.40b) lead to the solution $\hat{\boldsymbol{\xi}}$ as follows:

$$\hat{\boldsymbol{\xi}} = -N\hat{\boldsymbol{\lambda}} \ \text{ and } N\hat{\boldsymbol{\xi}} = \boldsymbol{c}, \tag{15.41a}$$

implying that

$$\boldsymbol{c} = -NN\hat{\boldsymbol{\lambda}} \Rightarrow \hat{\boldsymbol{\lambda}} = -(NN)^-\boldsymbol{c}. \tag{15.41b}$$

Finally, we arrive at the solution for the vector of unknown parameters $\boldsymbol{\xi}$ as

$$\boxed{\hat{\boldsymbol{\xi}}_{\text{MINOLESS}} = N(NN)^-\boldsymbol{c}.} \tag{15.42}$$

Note that $N(NN)^-$ is a particular g-inverse of N, but also note that $(NN)^- \neq N^-N^-$. Equation (15.42) is one expression of MINOLESS. There are others, as will be shown later. Using variance propagation, the dispersion of $\hat{\boldsymbol{\xi}}$ is given by

$$D\{\hat{\boldsymbol{\xi}}\} = N(NN)^- \cdot D\{\boldsymbol{c}\} \cdot (NN)^-N = \sigma_0^2 N(NN)^-N(NN)^-N, \tag{15.43}$$

which implies that

$$N(NN)^-N(NN)^-N = N_{rs}^-. \tag{15.44}$$

Here, we have used the fact that any power of the symmetric matrix N is also symmetric, and the g-inverse of a symmetric matrix is also symmetric. The matrix scaled by σ_0^2 in (15.43) is called *cofactor matrix*. We can always express a LESS as a product of such a cofactor matrix and the normal equation vector $\boldsymbol{c}$. That is,

$$\sigma_0^2 \cdot \hat{\boldsymbol{\xi}} = D\{\hat{\boldsymbol{\xi}}\} \cdot \boldsymbol{c} \text{ for any } \hat{\boldsymbol{\xi}}. \tag{15.45}$$

Therefore, we can also express MINOLESS in terms of the matrix N_{rs}^- in (15.44) by writing

$$\hat{\boldsymbol{\xi}}_{\text{MINOLESS}} = [N(NN)^-N(NN)^-N]\boldsymbol{c}. \tag{15.46}$$

Here, we note that the product AN^-A^T is invariant with respect to the chosen g-inverse N^-. Also, not only is $N(NN)^-$ a g-inverse of N, according to (15.17) its transpose $(NN)^-N$ is also a g-inverse of N. Based on these relations, and by expressing (15.46) alternatively as

$$\hat{\boldsymbol{\xi}} = [A^TPA(NN)^-N(NN)^-N]A^TP\boldsymbol{y}, \tag{15.47}$$

it is seen that (15.46) is unique regardless of the choice of the g-inverse.

Typically, in geodetic-science applications the estimated parameter vector $\hat{\boldsymbol{\xi}}$ is a vector of incremental updates to initial approximations of $\boldsymbol{\xi}$. As noted above, using MINOLESS in this case guarantees that changes from the initial approximations to the estimated values are a minimum, in terms of the L_2-norm. This minimum-norm solution is shown schematically in Figure 15.2, where the origin represents the initial approximation to $\boldsymbol{\xi}$. An infinite number of solution lie on the line labeled $\hat{\boldsymbol{\xi}}_{\text{part}} + \mathcal{N}(A)$. Their vectors originate at the origin; the shortest of these is $\hat{\boldsymbol{\xi}}_{\text{MINOLESS}}$.
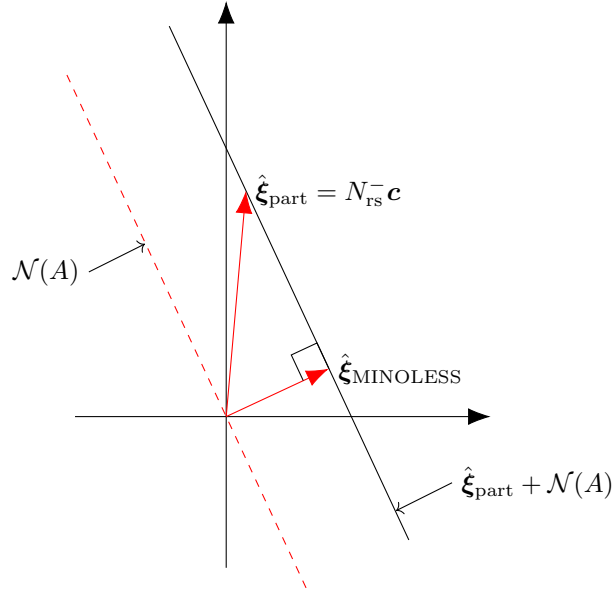
Figure 15.2: Schematic representation of the solution space for MINOLESS

15.5 Partial Minimum Norm Least-Squares Solution

In some cases we may only want a certain subset of the initial parameter vector to change in a minimum-norm sense. For example, we may know the locations of some points in a geodetic network to a high level of accuracy, while locations of the remaining network points may not be known as well or may even be known only approximately. In this case, we may wish to employ partial-MINOLESS, which is based on using a selection matrix S to choose a subset of the parameters for norm minimization.

The minimization problem is then stated as

$$\hat{\boldsymbol{\xi}}^T S \hat{\boldsymbol{\xi}} = \min_{\hat{\boldsymbol{\xi}}} \{N\hat{\boldsymbol{\xi}} = \boldsymbol{c}\}, \quad S := \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}. \tag{15.48}$$

The size of the identity matrix I_s corresponds to the number of selected parameters. Note that we can always construct S with I_s in the upper-left block, as shown in (15.48), by reordering the parameter vector if necessary; otherwise, S would be a diagonal matrix with ones at elements corresponding to selected parameters and zeros elsewhere.

The Lagrange target function to minimize in this case is given by

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\lambda}) := \boldsymbol{\xi}^T S \boldsymbol{\xi} - 2\boldsymbol{\lambda}^T (N\boldsymbol{\xi} - \boldsymbol{c}), \tag{15.49}$$

which must be made stationary with respect to $\boldsymbol{\xi}$ and $\boldsymbol{\lambda}$. Accordingly, The Euler-Lagrange necessary conditions are provided by

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}^T S - \hat{\boldsymbol{\lambda}}^T N \doteq \mathbf{0} \Rightarrow S\hat{\boldsymbol{\xi}} - N\hat{\boldsymbol{\lambda}} \doteq \mathbf{0}, \qquad (15.50a)$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = \hat{\boldsymbol{\xi}}^T N - \boldsymbol{c}^T \doteq \mathbf{0} \Rightarrow N\hat{\boldsymbol{\xi}} - \boldsymbol{c} \doteq \mathbf{0}. \qquad (15.50b)$$

The sufficient condition for minimization is satisfied since $(1/2)\partial^2\Phi/\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}^T = S$, which is positive (semi)definite.

Obviously matrix S is singular, but we choose S so that $S + N$ is invertible, requiring that S selects at least $m-\text{rk}\,A$ parameters (with rk $A := q$), or equivalently requiring rk $S \geq m - q$. Not only that, but it is required that $\text{rk}[N, S] = m$, which implies that, at least in some cases, not just any minimum $m - q$ parameters can be selected, but rather parameters must be selected such that $S + N$ becomes nonsingular. As an example, consider a 3D network adjustment with a datum deficiency of three (i.e., $m - q = 3$) due to the absence of any information about the network's origin. Assuming an XYZ coordinate parameterization, S has to select at least one X, one Y, and one Z coordinate; it could not select only three X-coordinates, for example, as that would not satisfy the rank requirement (i.e., information about the origin of the Y- and Z-axes would still be missing).

The above system of normal equations is solved as follows: By adding equations (15.50a) and (15.50b) we obtain

$$(S + N)\hat{\boldsymbol{\xi}} = N\hat{\boldsymbol{\lambda}} + \boldsymbol{c}, \qquad (15.51a)$$

leading to

$$\hat{\boldsymbol{\xi}} = (S + N)^{-1}(N\hat{\boldsymbol{\lambda}} + \boldsymbol{c}). \qquad (15.51b)$$

Substituting (15.51b) into (15.50b) yields

$$N(S + N)^{-1}(N\hat{\boldsymbol{\lambda}} + \boldsymbol{c}) - \boldsymbol{c} = \mathbf{0},$$

or

$$N(S + N)^{-1}N\hat{\boldsymbol{\lambda}} = \boldsymbol{c} - N(S + N)^{-1}\boldsymbol{c} =$$
$$= [(S + N) - N](S + N)^{-1}\boldsymbol{c} = S(S + N)^{-1}\boldsymbol{c},$$

leading to

$$\hat{\boldsymbol{\lambda}} = [N(S + N)^{-1}N]^- S(S + N)^{-1}\boldsymbol{c} \qquad (15.51c)$$

as an estimate for the vector of Lagrange multipliers. Then, substituting (15.51c) into (15.51a) yields

$$(S + N)\hat{\boldsymbol{\xi}} = \boldsymbol{c} + N[N(S + N)^{-1}N]^- S(S + N)^{-1}\boldsymbol{c}. \qquad (15.51d)$$

We use the identity $NN^-\boldsymbol{c} = \boldsymbol{c}$ to write the equivalent equation

$$(S + N)\hat{\boldsymbol{\xi}} = N[N(S + N)^{-1}N]^- N(S + N)^{-1}\boldsymbol{c} +$$
$$+ N[N(S + N)^{-1}N]^- S(S + N)^{-1}\boldsymbol{c} =$$

$$= N[N(S+N)^{-1}N]^-[N(S+N)^{-1} + S(S+N)^{-1}]\boldsymbol{c} =$$
$$= N[N(S+N)^{-1}N]^-\boldsymbol{c}, \tag{15.51e}$$

finally leading to the partial-MINOLESS

$$\boxed{\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}_{\text{P-MINOLESS}} = (S+N)^{-1}N[N(S+N)^{-1}N]^-\boldsymbol{c}.} \tag{15.51f}$$

Using the law of covariance propagation, we write the partial-MINOLESS dispersion matrix as

$$\boxed{D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2(S+N)^{-1}N[N(S+N)^{-1}N]^-N[N(S+N)^{-1}N]^-N(S+N)^{-1}.}$$
$$\tag{15.52}$$

We may rewrite the partial-MINOLESS solution, replacing the matrix multiplying $\boldsymbol{c}$ in (15.51f) with the cofactor matrix appearing in the dispersion (15.52), resulting in

$$\hat{\boldsymbol{\xi}}_{\text{P-MINOLESS}} = (S+N)^{-1}N[N(S+N)^{-1}N]^-N[N(S+N)^{-1}N]^-N(S+N)^{-1}\boldsymbol{c}. \tag{15.53}$$

Now, what happens if we replace S by I_m, i.e., all parameters are selected for norm minimization? Obviously partial-MINOLESS becomes MINOLESS itself as shown below.

$$\hat{\boldsymbol{\xi}}_{I_m-\text{MINOLESS}} = (I_m+N)^{-1}N[N(I_m+N)^{-1}N]^-N\cdot$$
$$\cdot [N(I_m+N)^{-1}N]^-N(I_m+N)^{-1}\boldsymbol{c} =$$
$$= (I_m+N)^{-1}N[N(I_m+N)^{-1}N]^-\boldsymbol{c} \Rightarrow$$
$$\boxed{\hat{\boldsymbol{\xi}}_{\text{MINOLESS}} = N^+\boldsymbol{c}} \tag{15.54}$$

The dispersion for MINOLESS is computed by

$$\boxed{D\{\hat{\boldsymbol{\xi}}_{\text{MINOLESS}}\} = D\{N^+\boldsymbol{c}\} = N^+D\{\boldsymbol{c}\}N^+ = \sigma_0^2N^+.} \tag{15.55}$$

It is interesting to compare (15.54) with the form of MINOLESS found earlier in (15.42). Once again we note that regardless of the form of MINOLESS (or similarly the form of N^+), the MINOLESS is unique. However, there is no connection between MINOLESS and BLUUE, as there is no unbiased estimate for this LESS. That is, the rank deficient GMM has no unbiased solution for the unknown parameters. This fact is easily demonstrated by attempting to derive a LUUE (Linear Uniformly Unbiased Estimate) as follows.

The Linear Uniformly Unbiased Estimate (LUUE) requires that

$$\hat{\boldsymbol{\xi}} = L\boldsymbol{y},$$

with L being an $m \times n$ matrix. Then, applying the expectation operator,

$$E\{\hat{\boldsymbol{\xi}}\} = LE\{\boldsymbol{y}\} = LA\boldsymbol{\xi},$$

with the size of LA being $m \times m$ and $\text{rk}(LA) \le \text{rk}\,A = q < m$. Thus, LA is singular, and therefore $LA \ne I_m$. Therefore, $\text{LUUE} = \emptyset$; i.e., there is no unbiased solution for $\boldsymbol{\xi}$.

15.6   Best Least-Squares Solution

We expect the least-squares solution to be *best* in a certain class. By best we mean that the trace of its dispersion matrix is minimum. We already found that the dispersion is based on a reflexive symmetric g-inverse, i.e.,

$$D\{\hat{\boldsymbol{\xi}}_{\text{LESS}}\} = \sigma_0^2 N_{rs}^- \text{ for all } N_{rs}^-. \tag{15.56}$$

Our task now is to compare the trace of the dispersion matrix from MINOLESS to that of a general LESS, recalling that the best LESS must satisfy the normal equations $N\hat{\boldsymbol{\xi}} = \boldsymbol{c}$. We start by expressing the estimate as a linear combination of the observations as follows:

$$\hat{\boldsymbol{\xi}} = L\boldsymbol{y}, \text{ with } NL = A^T P, \tag{15.57a}$$

which implies that

$$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 LP^{-1}L^T, \tag{15.57b}$$

permitting us to write

$$\sigma_0^{-2} \operatorname{tr} D\{\hat{\boldsymbol{\xi}}\} = \operatorname{tr}(LP^{-1}L^T) = \min_{L^T}\{L^T N = PA\}. \tag{15.57c}$$

So we see that minimizing the trace of the dispersion matrix is tantamount to minimizing the $m \times n$ matrix L (under the specified conditions) since the weight matrix P is fixed. Analogous to (12.9), we make use of the vec operator and the Kronecker-Zehfuss product to form the following Lagrange target function, where $\boldsymbol{l} := \operatorname{vec} L^T$:

$$\Phi(\boldsymbol{l}, \boldsymbol{\lambda}) := \boldsymbol{l}^T (I_m \otimes P^{-1})\boldsymbol{l} + 2\boldsymbol{\lambda}^T \big[N \otimes I_n \boldsymbol{l} - \operatorname{vec}(PA)\big], \tag{15.58}$$

which must be made stationary with respect to $\boldsymbol{l}$ and $\boldsymbol{\lambda}$. To clarify the form of the target function, we note that the first product comes from applying (A.32) to $\operatorname{tr}(LP^{-1}L^T) = \operatorname{tr}(P^{-1}L^T I_n L)$. The term including the Lagrange multiplier $\boldsymbol{\lambda}$ comes from the constraint $L^T N - PA = 0$ with application of the vec operator such that $\operatorname{vec}(I_n L^T N) - \operatorname{vec}(PA) = \boldsymbol{0}$, followed by the application of (A.31).

The Euler-Lagrange necessary conditions are

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{l}} = (I_m \otimes P^{-1})\hat{\boldsymbol{l}} + (N \otimes I_n)\hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \tag{15.59a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = (N \otimes I_n)\hat{\boldsymbol{l}} - \operatorname{vec}(PA) \doteq \boldsymbol{0}. \tag{15.59b}$$

The sufficient condition for minimization is satisfied since $(1/2) \cdot \partial^2 \Phi / (\partial \boldsymbol{l} \partial \boldsymbol{l}^T) = (I_n \otimes P^{-1})$ is positive definite. The system of normal equations is solved as follows: From (15.59a) we get

$$\hat{\boldsymbol{l}} = -(I_m \otimes P^{-1})^{-1}(N \otimes I_n)\hat{\boldsymbol{\lambda}} = -(N \otimes P)\hat{\boldsymbol{\lambda}}. \tag{15.60a}$$

And, by substituting the preceding equation into (15.59b), we get

$$\operatorname{vec}(PA) = -(NN \otimes P)\hat{\boldsymbol{\lambda}}. \tag{15.60b}$$

We need $(N \otimes P)\hat{\boldsymbol{\lambda}}$; so we exploit the fact that $N(N^2)^- N^2 = N$ by multiplying both sides of (15.60b) by $(N(NN)^- \otimes I_n)$, which gives

$$(N(NN)^- \otimes I_n)\operatorname{vec}(PA) = -(N(NN)^- \otimes I_n)(NN \otimes P)\hat{\boldsymbol{\lambda}} = -(N \otimes P)\hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{l}},$$
$$\text{(15.60c)}$$

leading to

$$\hat{\boldsymbol{l}} = \operatorname{vec}\big[I_n PA(N(NN)^-)^T\big] = \operatorname{vec}(PA(NN)^- N) = \operatorname{vec} L^T, \text{(using (A.31))},$$
$$\text{(15.60d)}$$

implying that

$$L^T = PA(NN)^- N \Rightarrow L = N(NN)^- A^T P. \qquad \text{(15.60e)}$$

Finally, we substitute (15.60e) into (15.57a) to get (compare to (15.42))

$$\hat{\boldsymbol{\xi}} = L\boldsymbol{y} = N(NN)^- A^T P\boldsymbol{y} = N(NN)^- \boldsymbol{c} = \hat{\boldsymbol{\xi}}_{\text{MINOLESS}}. \qquad \text{(15.60f)}$$

Thus we find that MINOLESS is best among all LESS with minimum trace. How can we prove this directly? We start by showing that $(N^+ N)N_{rs}^-(NN^+) = N^+$ and then exploit that relationship in the proof that follows.

$$(N^+ N)N_{rs}^-(NN^+) = N^+(NN_{rs}^- N)N^+ = N^+ NN^+ = N^+$$

implying that

$$\operatorname{tr} N^+ = \operatorname{tr}\big[(N^+ N)N_{rs}^-(NN^+)\big].$$

We continue by exploiting symmetry and applying a cyclic transformation.

$$\operatorname{tr} N^+ = \operatorname{tr}\big[(NN^+)^T(N^+ N)N_{rs}^-\big] =$$
$$= \operatorname{tr}\big[N^+ NN_{rs}^-\big] = \quad \text{(because } N^+ N \text{ is idempotent)}$$
$$= \operatorname{tr}\big[N_{rs}^- - N_{rs}^- + N^+ NN_{rs}^-\big] =$$
$$= \operatorname{tr}\big[N_{rs}^- - (I_m - N^+ N)N_{rs}^-\big] =$$
$$= \operatorname{tr} N_{rs}^- - \operatorname{tr}\big[(I_m - N^+ N)(I_m - N^+ N)N_{rs}^-\big] =$$
$$\quad \text{(because } (I_m - N^+ N) \text{ is idempotent)}$$
$$= \operatorname{tr} N_{rs}^- - \operatorname{tr}\big[(I_m - N^+ N)^T(I_m - N^+ N)N_{rs}^-\big] =$$
$$\quad \text{(transpose due to symmetry)}$$
$$= \operatorname{tr} N_{rs}^- - \operatorname{tr}\big[(I_m - N^+ N)N_{rs}^-(I_m - N^+ N)^T\big] = \quad \text{(cyclic transformation)}$$

But, the triple product is positive semidefinite; therefore we can state that

$$\operatorname{tr} N^+ = \operatorname{tr} N_{rs}^- - \operatorname{tr}\big[(I_m - N^+ N)N_{rs}^-(I_m - N^+ N)^T\big] \leq \operatorname{tr} N_{rs}^-. \qquad \text{(15.61)}$$

Thus we have proved directly that the pseudoinverse N^+ provides a minimum trace in the class of cofactor matrices, i.e.,

$$\operatorname{tr} N^+ \leq \operatorname{tr} N_{rs}^- \quad \text{for all } N_{rs}^-. \qquad \text{(15.62)}$$

Can we make a similar characterization of partial-MINOLESS? Is it *partially best* in terms of having the smallest (partial) trace of the cofactor matrix? In other words, is the sum of the cofactor diagonal elements corresponding to the selected points smallest? The answer is yes, as we show below.

Analogous to (15.57c) we minimize a subset of the trace of the parameter dispersion matrix as

$$\sigma_0^{-2} \operatorname{tr}(SD\{\hat{\boldsymbol{\xi}}\}) = \min_{L^T} \text{ such that } L^T N = PA. \qquad (15.63)$$

From (15.51f) we already know that matrix L should satisfy $L := (S+N)^{-1}N[N(S+N)^{-1}N]^-A^TP$. Analogous to (15.58), we write the following Lagrange target function:

$$\Phi(\boldsymbol{l}, \boldsymbol{\lambda}) := \boldsymbol{l}^T(S \otimes P^{-1})\boldsymbol{l} + 2\boldsymbol{\lambda}^T[(N \otimes I_n)\boldsymbol{l} - \operatorname{vec}(PA)], \qquad (15.64)$$

which must be made stationary with respect to $\boldsymbol{l}$ and $\boldsymbol{\lambda}$. Again we have defined $\boldsymbol{l} := \operatorname{vec} L^T$. Accordingly, the Euler-Lagrange necessary conditions are written as (compare to (15.59a) and (15.59b))

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{l}} = (S \otimes P^{-1})\hat{\boldsymbol{l}} + (N \otimes I_n)\hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0}, \qquad (15.65)$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\lambda}} = (N \otimes I_n)\hat{\boldsymbol{l}} - \operatorname{vec}(PA) \doteq \boldsymbol{0}. \qquad (15.66)$$

Note that $(S \otimes P^{-1})$ is not invertible due to the singularity of S, in general. We solve the system of equations (15.65) and (15.66) as follows: Multiplying (15.65) by $(I_n \otimes P)$ gives

$$(S \otimes I_n)\hat{\boldsymbol{l}} + (N \otimes P)\hat{\boldsymbol{\lambda}} = \boldsymbol{0}. \qquad (15.67a)$$

Adding this to (15.66) yields

$$[(N + S) \otimes I_n]\hat{\boldsymbol{l}} = \operatorname{vec}(PA) - (N \otimes P)\hat{\boldsymbol{\lambda}}, \qquad (15.67b)$$

leading to

$$\hat{\boldsymbol{l}} = [(N + S) \otimes I_n]^{-1}\operatorname{vec}(PA) - [(N + S) \otimes I_n]^{-1}(N \otimes P)\hat{\boldsymbol{\lambda}} =$$
$$= [(N + S)^{-1} \otimes I_n]\operatorname{vec}(PA) - [(N + S)^{-1} \otimes I_n](N \otimes P)\hat{\boldsymbol{\lambda}} =$$
$$= \operatorname{vec}[PA(N + S)^{-1}] - [(N + S)^{-1}N \otimes P]\hat{\boldsymbol{\lambda}}, \text{ using (A.32).} \qquad (15.67c)$$

Now substitute $\hat{\boldsymbol{l}}$ from (15.67c) into (15.66) to obtain

$$\operatorname{vec}(PA) = (N \otimes I_n)\{\operatorname{vec}[PA(N + S)^{-1}] - [(N + S)^{-1}N \otimes P]\hat{\boldsymbol{\lambda}}\} =$$
$$= \operatorname{vec}[PA(N + S)^{-1}N] - [N(N + S)^{-1}N \otimes P]\hat{\boldsymbol{\lambda}}, \text{ using (A.31).} \qquad (15.67d)$$

Now, the product that includes $\hat{\boldsymbol{\lambda}}$ in (15.67a) can be expressed as follows:

$$(N \otimes P)\hat{\boldsymbol{\lambda}} = \{N[N(N + S)^{-1}N]^- \otimes I_n\}[N(N + S)^{-1}N \otimes P]\hat{\boldsymbol{\lambda}}. \qquad (15.67e)$$

Combining (15.67d) and (15.67e) gives

$$(N \otimes P)\hat{\boldsymbol{\lambda}} = -\big\{N[N(N+S)^{-1}N]^- \otimes I_n\big\}[\text{vec}(PA) - \text{vec}[PA(N+S)^{-1}N].$$
$$(15.67\text{f})$$

Multiplying the right side through and using (A.31) yields

$$(N \otimes P)\hat{\boldsymbol{\lambda}} = -\text{vec}\big\{PA[N(N+S)^{-1}N]^-N\big\}+$$
$$+ \text{vec}\big\{PA(N+S)^{-1}N[N(N+S)^{-1}N]^-N\big\}.$$
$$(15.67\text{g})$$

Recalling that $N = A^T PA$, the matrix A in the last term of the preceding line can be replaced by $A = AN^-N$, see (15.35a), which permits reduction of said term to $\text{vec}(PA)$.

$$(N \otimes P)\hat{\boldsymbol{\lambda}} = -\text{vec}\big\{PA[N(N+S)^{-1}N]^-N\big\} + \text{vec}(PA)$$
$$(15.67\text{h})$$

Now we can substitute the preceding line into (15.67b) in order to solve for $\hat{\boldsymbol{l}}$.

$$\text{vec}(PA) - \big[(N+S) \otimes I_n\big]\hat{\boldsymbol{l}} = -\text{vec}\big\{PA[N(N+S)^{-1}N]^-N\big\} + \text{vec}(PA),$$
$$(15.67\text{i})$$

implying that

$$\hat{\boldsymbol{l}} = \big[(N+S)^{-1} \otimes I_n\big]\text{vec}\big\{PA[N(N+S)^{-1}N]^-N\big\} =$$
$$= \text{vec}\big\{PA[N(N+S)^{-1}N]^-N(N+S)^{-1}\big\} = \text{vec}(\hat{L}^T),$$
$$(15.67\text{j})$$

which finally leads to an expression for the $m \times m$ matrix L as

$$\hat{L} = (N+S)^{-1}N[N(N+S)^{-1}N]^-A^T P, \ \text{ with } \ \hat{\boldsymbol{\xi}}_{\text{P-MINOLESS}} = \hat{L}\boldsymbol{y}. \qquad (15.67\text{k})$$

This agrees with our solution in (15.51f) and shows that partial-MINOLESS indeed yields the minimum partial trace of the dispersion matrix among all other estimators; thus it is *partially best*.

15.7   Best Linear Uniformly Minimum Bias Estimate

Analogous to BLUUE in the GMM of full rank, we take a statistical approach here to derive an estimator in the rank-deficient GMM. We already stated that all estimates $\hat{\boldsymbol{\xi}}$ in the rank-deficient GMM are biased by our treatment of the rank deficiency. We wish to minimize this bias by finding the Best Linear Uniformly Minimum Bias Estimate (*BLUMBE*). The attributes of BLUMBE are described below.

(i) Linear: The estimate is required to be linear in the observation vector $\boldsymbol{y}$.

$$\hat{\boldsymbol{\xi}} = L\boldsymbol{y}, \ \text{ where the } m \times n \text{ matrix } L \text{ is to be determined.} \qquad (15.68)$$

(ii) Minimum bias:

$$E\{\hat{\boldsymbol{\xi}}\} = (LA)\boldsymbol{\xi}, \text{ with } \mathrm{rk}(LA) \leq \mathrm{rk}\, A = q \Rightarrow LA \neq I_m. \tag{15.69}$$

We see that the matrix product LA cannot be the identity matrix I_m because it has rank $q < m$. But the product LA would need to be equal to I_m in order for the estimate to be uniformly unbiased. We call the difference $LA - I_m$ the *bias matrix*, and we wish to make it as small as possible by minimizing its Frobenius norm, or rather by minimizing the square of the norm as follows (see the definition of the Frobenius (or Euclidean) norm in (A.27)):

$$\Phi(L) = \|LA - I_m\|_F^2 = \mathrm{tr}\big[(LA - I_m)(LA - I_m)^T\big] = \min_L. \tag{15.70}$$

To fulfill the requirement (15.70), it is necessary that the first partial derivative of $\Phi(L)$ with respect to L^T equal zero (necessary condition), which leads to

$$(AA^T)L^T - A \doteq 0 \text{ or } (LA - I_m)A^T \doteq 0. \tag{15.71}$$

See (A.47) for derivatives of the trace. Of course the sufficiency condition is satisfied for minimization since AA^T is positive semidefinite. From (15.71) we have the geometric interpretation that $\mathcal{R}(A^T L^T - I_m) \subset \mathcal{N}(A)$.

(iii) Best: The trace of dispersion matrix must be minimum.

We wish to minimize the dispersion matrix $D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 L P^{-1} L^T$. Dropping the constant σ_0^2 and considering (ii) leads to the following target function to minimize:

$$\Phi(L^T, \Lambda) = \mathrm{tr}(LP^{-1}L^T) + 2\,\mathrm{tr}(LA - I_m)A^T\Lambda, \tag{15.72}$$

which must be made stationary with respect to L^T and the $n \times n$ matrix of Lagrange multipliers Λ. Accordingly, the Euler-Lagrange necessary conditions are written as

$$\frac{1}{2}\frac{\partial \Phi}{\partial L^T} = P^{-1}\hat{L}^T + AA^T\hat{\Lambda} \doteq 0, \tag{15.73a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \Lambda} = AA^T\hat{L}^T - A \doteq 0. \tag{15.73b}$$

Note that we could also check for the sufficient condition; however, this requires the vec operator and Kronecker products to do so. For the sake of brevity, we simply state that the necessary conditions do indeed lead to a minimization of (15.72). The above system (15.73a) and (15.73b) is solved as follows: From (15.73a)) we can write

$$\hat{L}^T = -PAA^T\hat{\Lambda}. \tag{15.74a}$$

Then, by substituting (15.74a) into (15.73b), we obtain

$$A = -AA^T PAA^T\hat{\Lambda}.$$

Multiplying by $A^T P$ from the left results in

$$N = -NNA^T \hat{\Lambda}.$$

Then multiplying by $N(NN)^-$ from the left gives

$$N(NN)^- N = -N(NN)^- NNA^T \hat{\Lambda}.$$

Noting that $N(NN)^- NN = N$ means

$$N(NN)^- N = -NA^T \hat{\Lambda}. \tag{15.74b}$$

We seek an expression for $PAA^T \hat{\Lambda}$ in terms of known quantities to substitute into (15.74a). So we multiply both sides of (15.74b) by AN^- to get

$$AN^- N(NN)^- N = -AN^- NA^T \hat{\Lambda},$$

which, together with using (15.35b), implies

$$A(NN)^- N = -AA^T \hat{\Lambda}.$$

Then, multiplying on the left by P results in

$$PA(NN)^- N = -PAA^T \hat{\Lambda},$$

and, by substitution into (15.74a), we get

$$\hat{L}^T = PA(NN)^- N,$$

leading to

$$\hat{L} = N(NN)^- A^T P. \tag{15.74c}$$

Finally, upon substituting (15.74c) for $\hat{L}$ into (15.68) and comparing to (15.60f) we get

$$\boxed{\hat{\boldsymbol{\xi}}_{\text{BLUMBE}} = \hat{L}\boldsymbol{y} = N(NN)^- \boldsymbol{c} = \hat{\boldsymbol{\xi}}_{\text{MINOLESS}}.} \tag{15.74d}$$

We have just shown that the BLUMBE and the MINOLESS are equivalent. This equivalency makes these solutions very appealing for the rank deficient GMM, as together they fulfill the following properties:

- Minimum norm of parameter vector.

- Smallest trace of dispersion matrix.

- Smallest norm of bias matrix.

A relevant question at this point is "what is the bias associated with BLUMBE?" The BLUMBE bias vector $\boldsymbol{\beta}$ is derived as follows:

$$\boldsymbol{\beta} := E\{\hat{\boldsymbol{\xi}}\} - \boldsymbol{\xi} = \tag{15.75a}$$

$$= \hat{L}E\{\boldsymbol{y}\} - \boldsymbol{\xi} \stackrel{\text{.}}{=} LA\boldsymbol{\xi} - \boldsymbol{\xi} \Rightarrow \tag{15.75b}$$

$$\boldsymbol{\beta} = (\hat{L}A - I_m)\boldsymbol{\xi} = [N(NN)^- N - I_m]\boldsymbol{\xi}. \tag{15.75c}$$

Equation (15.75c) in the above derivation reveals the bias matrix $\hat{L}A - I_m$ that is minimized by BLUMBE.

We now make a few additional comments about the least-squares solution within the rank-deficient GMM. In addition to the vector of estimated parameters $\hat{\boldsymbol{\xi}}$, we can also generate a predicted residual vector $\tilde{\boldsymbol{e}}$, a vector of adjusted observations $\hat{\boldsymbol{\mu}}_y$, and an estimated variance component $\hat{\sigma}_0^2$. Each of their formulas are summarized below.

$$\hat{\boldsymbol{\xi}} = \big\{ N_{rs}^-\boldsymbol{c} + (I_m - N_{rs}^- N)\boldsymbol{\alpha} \,|\, \boldsymbol{\alpha} \in \mathbb{R}^m \big\} =$$
$$= \big\{ N_{rs}^-\boldsymbol{c} \,|\, N_{rs}^- \text{ is a reflexive symmetric g-inverse of } N \big\} \tag{15.76a}$$

$$\hat{\boldsymbol{\mu}}_y = A\hat{\boldsymbol{\xi}} \tag{15.76b}$$

$$\tilde{\boldsymbol{e}} = \big[I_n - (AN_{rs}^- A^T)P\big]\boldsymbol{y}, \tag{15.76c}$$

where $AN_{rs}^- A^T$ is invariant with respect to the chosen g-inverse

$$\hat{\sigma}_0^2 = \tilde{\boldsymbol{e}}^T P\tilde{\boldsymbol{e}}/(n - \operatorname{rk} A). \tag{15.76d}$$

The denominator in (15.76d) is the redundancy of the model. It is equal to the number of observations minus the number of estimable parameters. It is only equivalent to the number of observations minus the number of parameters, or unknowns, if the design matrix A has full column rank, in which case the redundancy is $n - m$.

An important point to make is that $\tilde{\boldsymbol{e}}$, $\hat{\boldsymbol{\mu}}_y$, and $\hat{\sigma}_0^2$ are all unique. That is, they do not depend on the chosen g-inverse, N_{rs}^-, for the solution (15.76a). However, $\hat{\boldsymbol{\xi}}$ itself is not unique; that is, it does depend directly on the chosen g-inverse. The following expectations reflect these properties (where (15.35a), (15.35b), and $E\{\boldsymbol{y}\} = A\boldsymbol{\xi}$ have been used):

$$E\{\hat{\boldsymbol{\xi}}\} = E\{N_{rs}^-\boldsymbol{c}\} = N_{rs}^- A^T PE\{\boldsymbol{y}\} = N_{rs}^- A^T PA\boldsymbol{\xi} = N_{rs}^- N\boldsymbol{\xi} \neq \boldsymbol{\xi}, \tag{15.77a}$$

$$E\{\hat{\boldsymbol{\mu}}_y\} = E\{A\hat{\boldsymbol{\xi}}\} = AN_{rs}^- N\boldsymbol{\xi} = A\boldsymbol{\xi} = E\{\boldsymbol{y}\}, \tag{15.77b}$$

$$E\{\tilde{\boldsymbol{e}}\} = \big[I_n - (AN_{rs}^- A^T)P\big]E\{\boldsymbol{y}\} = \big[A - AN_{rs}^- N\big]\boldsymbol{\xi} = \big[A - A\big]\boldsymbol{\xi} = \boldsymbol{0} = E\{\boldsymbol{e}\}. \tag{15.77c}$$

15.7.1 *S*-Transformations Introduced

Suppose we have two different elements of the solution space, namely $\hat{\boldsymbol{\xi}}^{(1)}$ and $\hat{\boldsymbol{\xi}}^{(2)}$, that take the forms

$$\hat{\boldsymbol{\xi}}^{(1)} = G_1\boldsymbol{c} \text{ and } \hat{\boldsymbol{\xi}}^{(2)} = G_2\boldsymbol{c} \text{ where } G_1, G_2 \in \{N_{rs}^-\}, \tag{15.78}$$

i.e., G_1 and G_2 are different g-inverses, both of which are reflexive symmetric.

Now suppose we would like to transform from one solution to the other. The transformations are written as

$$\hat{\boldsymbol{\xi}}^{(1)} = (G_1 N)\hat{\boldsymbol{\xi}}^{(2)} \text{ and } \hat{\boldsymbol{\xi}}^{(2)} = (G_2 N)\hat{\boldsymbol{\xi}}^{(1)}, \tag{15.79}$$

with respective dispersion matrices

$$D\{\hat{\boldsymbol{\xi}}^{(1)}\} = (G_1 N) \cdot D\{\hat{\boldsymbol{\xi}}^{(2)}\} \cdot (G_1 N)^T \ \text{ and } \ D\{\hat{\boldsymbol{\xi}}^{(2)}\} = (G_2 N) \cdot D\{\hat{\boldsymbol{\xi}}^{(1)}\} \cdot (G_2 N)^T. \tag{15.80}$$

The relations in (15.79) hold because the normal equations $N\hat{\boldsymbol{\xi}}^{(i)} = \boldsymbol{c}$ are fulfilled for all $\hat{\boldsymbol{\xi}}^{(i)}$. These transformations are called "S-transformations." They have practical use in datum transformation problems, and they are discussed further in section 15.9.

It is often costly to compute N_{rs}^-. How then can we represent the product $G_i N$? The only difference between various $G_i N$ is in the dimension of their nullspaces. Thus, in the following we look at different bases for the nullspace of A (or, equivalently, the nullspace of N) to solve our rank deficient problem.

15.8 Minimum and Inner Constraints

In the context of minimum constraints, the term "minimum" is used to mean the minimum number of constraints required to overcome the rank deficiency of the model. This is in contrast to an over-constrained model, where more constraints than are necessary are provided, in which case the residual vector will be impacted by the constraints. The constraints are given in the form of linear equations in the unknown parameters. The term "inner constraints" adjustment is used by some authors in the context of networks with datum deficiency to denote the MINOLESS shown Section 15.8.3. Among these authors are Leick et al. (2015) and (Blaha, 1971, p. 3, §7.3, §8), the latter of which refers to the constraints as inner adjustment constraints.[2]

15.8.1 Restricted LEast-Squares Solution (RLESS)

The minimal constraint equation is written as

$$K\boldsymbol{\xi} = \boldsymbol{\kappa}_0. \tag{15.81}$$

where, K is an $l \times m$ matrix whose row space relates to that of matrix A by

$$\mathcal{R}(K^T) \cap \mathcal{R}(A^T) = \emptyset \ \text{ and } \ \mathcal{R}(K^T) \cup \mathcal{R}(A^T) = \mathbb{R}^m, \tag{15.82a}$$

which also means the direct sum of the row spaces results in

$$\mathcal{R}(K^T) \oplus \mathcal{R}(A^T) = \mathbb{R}^m. \tag{15.82b}$$

Equivalently, the rank condition

$$\text{rk}\big[A^T, \, K^T\big] = \text{rk}\, A + \text{rk}\, K = m, \tag{15.82c}$$

holds, implying that

$$\text{rk}\, K = m - q, \tag{15.82d}$$

_____

[2]The term "inner constraint" stems from the work of Meissl (1969).

leading to the "restriction"

$$\text{rows of } K = l = m - q = \text{datum deficiency}, \tag{15.82e}$$

for the number of *minimal constraints* needed to overcome the rank deficiency of matrix A (Schaffrin, 1985, p. 552). This restriction is what the 'R' in RLESS (Restricted LESS) denotes.

Both the *constraint matrix* K and the right-side vector $\boldsymbol{\kappa}_0$ are known, constant quantities. In practice, $\boldsymbol{\kappa}_0$ is often a vector of zeros, especially in the case of linearized observation equations. The above equations tell us that the row space of matrix A combined with the row space of K (i.e., their union) span all of $\mathbb{R}^m$. Even more, the union forms a basis for $\mathbb{R}^m$.

Combining equation (15.81) with the observation equations (11.3), allows us to write the following system of extended normal equations:

$$\begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{\kappa}_0 \end{bmatrix}. \tag{15.83}$$

The extended normal-equation matrix on the left side is indeed regular (non-singular) due to the rank relations of (15.82c). The normal equations can be solved as follows: Adding $K^T \times$ row 2 to row 1 results in

$$(N + K^T K)\hat{\boldsymbol{\xi}} = \boldsymbol{c} + K^T(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}), \tag{15.84a}$$

leading to

$$\hat{\boldsymbol{\xi}} = (N + K^T K)^{-1}\boldsymbol{c} + (N + K^T K)^{-1}K^T(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}). \tag{15.84b}$$

Now we combine the preceding line with row 2 to obtain

$$\boldsymbol{\kappa}_0 = K\hat{\boldsymbol{\xi}} = K(N + K^T K)^{-1}\boldsymbol{c} + K(N + K^T K)^{-1}K^T(\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}}), \tag{15.84c}$$

which leads to

$$\boldsymbol{\kappa}_0 - \hat{\boldsymbol{\lambda}} = \big[K(N + K^T K)^{-1}K^T\big]^{-1}\big[\boldsymbol{\kappa}_0 - K(N + K^T K)^{-1}\boldsymbol{c}\big]. \tag{15.84d}$$

Finally, upon substituting (15.84d) into (15.84b), we can write the Restricted Least-Squares Solution (RLESS) as

$$\boxed{\begin{aligned} \hat{\boldsymbol{\xi}}_{\text{RLESS}} = {}& (N + K^T K)^{-1}\boldsymbol{c} + (N + K^T K)^{-1}K^T \cdot \\ & \cdot \big[K(N + K^T K)^{-1}K^T\big]^{-1}\big[\boldsymbol{\kappa}_0 - K(N + K^T K)^{-1}\boldsymbol{c}\big]; \end{aligned}} \tag{15.85}$$

see (15.104) for a simplified expression.

If $\boldsymbol{\kappa}_0$ turns out to be zero, and if we factor out the vector $\boldsymbol{c}$, the solution (15.85) reduces to

$$\begin{aligned} \hat{\boldsymbol{\xi}}_{\text{RLESS}} = {}& \big[(N + K^T K)^{-1} - (N + K^T K)^{-1}K^T\big[K(N + K^T K)^{-1}K^T\big]^{-1} \cdot \\ & \cdot K(N + K^T K)^{-1}\big]\boldsymbol{c}, \ \text{if} \ \boldsymbol{\kappa}_0 = \boldsymbol{0}. \end{aligned} \tag{15.86}$$

Now, for convenience in further analysis, denote the matrix on the right side of 15.86 as G, viz.

$$G := \left[(N + K^T K)^{-1} - (N + K^T K)^{-1} K^T \left[K(N + K^T K)^{-1} K^T \right]^{-1} K(N + K^T K)^{-1} \right]. \tag{15.87}$$

15.8.2   Reflexive Symmetric G-Inverse

Question: is the matrix G in (15.87) a reflexive symmetric g-inverse of N? We claim that it is. The proof that follows is rather lengthy, but out of it comes a representation of matrix K that leads to the so called inner-constraint solution. The symmetry of G is obvious from inspection. The reflexivity can be confirmed by checking the rank. Because the rank of G will not change when premultiplied by the full-rank matrix $N + K^T K$, and because this multiplication results in a idempotent matrix $(N + K^T K)G$, we make use of this multiplication as follows:

$$\operatorname{rk} G = \operatorname{rk}\left[(N + K^T K)G \right] =$$
$$= \operatorname{tr}\left[(N + K^T K)G \right] = \quad \text{(due to its idempotent property)}$$
$$= \operatorname{tr} I_m - \operatorname{tr}\left\{ [K(N + K^T K)^{-1} K^T]^{-1} K(N + K^T K)^{-1} K^T \right\} =$$
$$= m - (m - q) = q = \operatorname{rk} N.$$

Since $\operatorname{rk} G = \operatorname{rk} N$, if the symmetric matrix G is a g-inverse of N, it is also a reflexive symmetric g-inverse. What is left is to show that G is indeed a g-inverse of N. We start by forming the product NGN.

$$NGN = N(N + K^T K)^{-1} N - N(N + K^T K)^{-1} K^T \cdot$$
$$\cdot \left[K(N + K^T K)^{-1} K^T \right]^{-1} K(N + K^T K)^{-1} N \tag{15.88}$$

Our aim is to show that $K(N + K^T K)^{-1} N = 0$, which would cancel what follows the minus sign on the right side of (15.88). Then we must show that the remaining term $N(N + K^T K)^{-1} N$ equals N, which implies that $(N + K^T K)^{-1}$ is a g-inverse of N and thereby proves that G is as well. This is done in the following section.

15.8.3   (Partial) <u>Mi</u>nimum <u>No</u>rm <u>Le</u>ast-<u>S</u>quares Solution (MINOLESS)

Recalling that m is the number of unknown parameters and l is the number of required constraints, we introduce an $l \times m$ matrix E to form the constraint equations

$$E\boldsymbol{\xi} = \mathbf{0}, \tag{15.89a}$$

such that

$$AE^T = 0, \tag{15.89b}$$

and

$$\mathcal{R}(A^T) \overset{\perp}{\oplus} \mathcal{R}(E^T) = \mathbb{R}^m, \tag{15.89c}$$

implying that

$$\operatorname{rk} E = m - q = l. \tag{15.89d}$$

The above relations mean that the columns of E^T (or rows of E) form a basis for the nullspace of A, and thus also for the nullspace of N. Every row of E is perpendicular to every row of A, and though the rows of E do not have to be perpendicular to one another, they are linearly independent of each other. So, we could construct matrix E with eigenvectors corresponding to the zero eigenvalues of N. But this is only one choice for constructing E; the matrix E is not unique.

Considering the constraint matrix K introduced in (15.81), because of (15.89b) we have the relation

$$(N + K^T K)E^T = K^T (KE^T). \tag{15.90}$$

Now we assert that the $l \times l$ matrix KE^T is invertible.

Proof: Suppose KE^T is not invertible. This implies that there exists a linear combination of the rows of K that is orthogonal to a column of E^T; or in mathematical terms $\mathcal{R}(K^T) \subset \mathcal{R}(E^T)^\perp$. This would mean that a vector in $\mathcal{R}(K^T)$ is contained in $\mathcal{R}(A^T)$ since $\mathcal{R}(A^T) = \mathcal{R}(E^T)^\perp$. But this contradicts the direct sum in (15.82b). In other words, there exists no linear combination of the rows of K that is perpendicular to a column of E^T, and therefore, KE^T is invertible. We continue by pre- and post-multiplying (15.90) by appropriate inverses as follows:

$$E^T (KE^T)^{-1} = (N + K^T K)^{-1} K^T, \tag{15.91a}$$

implying that

$$N(N + K^T K)^{-1} K^T = NE^T (KE^T)^{-1}. \tag{15.91b}$$

But, $NE^T = 0$ due to (15.89b), therefore

$$N(N + K^T K)^{-1} K^T = 0. \tag{15.91c}$$

Thus (15.88) does reduce to $NGN = N(N + K^T K)^{-1}N$. Now, using two successive applications of the rule for the inverse of a sum (see equation (A.6a) in the appendix) we can check to see if this product further reduces to N.

$$N(N + K^T K)^{-1}N = N\big[N^- - N^- K^T (I_l + KN^- K^T)^{-1}KN^-\big]N = \tag{15.92a}$$

$$= NN^- N - NN^- K^T \big[I_l - K(N + K^T K)^{-1}K^T\big]KN^- N = \tag{15.92b}$$

$$= N - NN^- K^T (I_l - I_l)KN^- N = N \tag{15.92c}$$

Here we have used the relationship $K(N + K^T K)^{-1}K^T = I_l$, which is obvious from (15.91a).

Thus we have shown that the matrix G of (15.87) is indeed a reflexive symmetric g-inverse for N, given any arbitrary matrix K satisfying the conditions (15.82a)–(15.82d). We summarize by listing three important relations between the normal-equations matrix N and the minimal-constraint matrix K.

$$N(N + K^T K)^{-1}N = N \tag{15.93a}$$

$$N(N + K^T K)^{-1} K^T = 0 \qquad\qquad (15.93\text{b})$$

$$K(N + K^T K)^{-1} K^T = I_l \qquad\qquad (15.93\text{c})$$

As we have said already, the minimum-constraint matrix K must satisfy conditions (15.82a)–(15.82d); the matrix K is otherwise arbitrary. The matrix E introduced above satisfies these conditions and may be used in place of K. In this case we get MINOLESS. Rewriting (15.86), which used $\boldsymbol{\kappa}_0 = \mathbf{0}$, with E instead of K gives

$$\hat{\boldsymbol{\xi}}_{\text{MINOLESS}} = \big\{ (N + E^T E)^{-1} - (N + E^T E)^{-1} E^T \cdot$$
$$\cdot \big[E(N + E^T E)^{-1} E^T \big]^{-1} E(N + E^T E)^{-1} \big\} \boldsymbol{c}. \qquad (15.94)$$

Using relations (15.91a) and (15.93c), with K replaced by E, allows us to rewrite (15.94) as

$$\boxed{ \hat{\boldsymbol{\xi}}_{\text{MINOLESS}} = \big[(N + E^T E)^{-1} - E^T (EE^T)^{-1} (EE^T)^{-1} E \big] \boldsymbol{c}. } \qquad (15.95)$$

The diagram in Figure 15.3 illustrates the geometric relationships between the range spaces of A^T, E^T, and K^T, together with RLESS and MINOLESS.
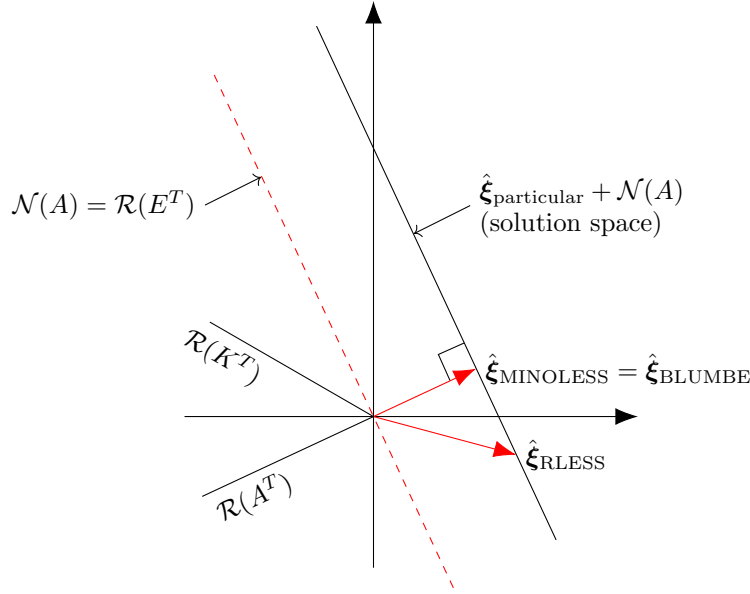
Figure 15.3: Schematic representation of the solution space with RLESS and MINOLESS (BLUMBE) as particular solutions

Now we prove that (15.95), or equivalently (15.94), is in fact MINOLESS. To do so we must show that the matrix on the right side of (15.95), which we define here as G, is the pseudoinverse N^+ of N.

$$G := \big[(N + E^T E)^{-1} - E^T (EE^T)^{-1} (EE^T)^{-1} E \big] \overset{?}{=} N^+. \qquad (15.96)$$

We already know that $G \in N_{rs}^-$ from the above derivation of RLESS. We only have to show the two remaining properties of the pseudoinverse; see (15.16).

Note that $NG = N(N + E^T E)^{-1}$, because NE^T contains the product AE^T, which is zero by (15.89b). If G is in fact equal to N^+, then NG must satisfy $NG = (NG)^T$ according to (15.16). This is checked as follows:

$$NG = N(N + E^T E)^{-1} =$$
$$= (N + E^T E - E^T E)(N + E^T E)^{-1} =$$
$$= (N + E^T E)(N + E^T E)^{-1} - E^T E(N + E^T E)^{-1}$$

And now using the transpose of (15.91a) with K replaced by E leads to

$$NG = I_m - E^T(EE^T)^{-1}E. \tag{15.97}$$

The matrix in (15.97) is obviously symmetric so that $NG = (NG)^T$. Also, since G and N are both symmetric, $NG = (NG)^T = G^T N^T = GN$ so that all conditions for the pseudoinverse have been satisfied, and thus it is proved that (15.94) is indeed MINOLESS. Note also that due to the orthogonality relation (15.89b), we can write

$$\boxed{\hat{\boldsymbol{\xi}}_{\text{MINOLESS}} = (N + E^T E)^{-1}\boldsymbol{c} = N^+\boldsymbol{c}.} \tag{15.98}$$

Note, however, that $(N + E^T E)^{-1} \neq N^+$. The solution for $\boldsymbol{\xi}$ based on matrix E is a particular type of minimum-constraint solution, which has been called the *inner-constraint* solution by some authors as noted in Section 15.8. Note that the constraint equation (15.81) has, in essence, been replaced by $E\boldsymbol{\xi} = 0$ and that MINOLESS can actually be obtained by the following extended normal equations system, analogously to (15.83), viz.

$$\begin{bmatrix} N & E^T \\ E & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix}. \tag{15.99}$$

One form of the dispersion matrix for MINOLESS was already shown in (15.55). Applying covariance propagation to (15.98) leads to the equivalent formula

$$\boxed{D\{\hat{\boldsymbol{\xi}}_{\text{MINOLESS}}\} = \sigma_0^2 (N + E^T E)^{-1}N(N + E^T E)^{-1} = \sigma_0^2 N^+.} \tag{15.100}$$

Also, analogous to (15.75c), we write the bias vector for the inner constraint solution (15.98) as

$$\boldsymbol{\beta} = [(N + E^T E)^{-1}N - I_m]\boldsymbol{\xi}. \tag{15.101}$$

By introduction of the selection matrix S defined in Section 15.8.3 into the extended normal equations (15.99), so that the constraint equation (15.89a) becomes

$$ES\boldsymbol{\xi} = 0, \tag{15.102}$$

one may also derive partial MINOLESS and its dispersion matrix as

$$\hat{\boldsymbol{\xi}}_{\text{P-MINOLESS}} = (N + SE^TES)^{-1}\boldsymbol{c}, \tag{15.103a}$$

$$D\{\hat{\boldsymbol{\xi}}_{\text{P-MINOLESS}}\} = \sigma_0^2(N + SE^TES)^{-1}N(N + SE^TES)^{-1}. \tag{15.103b}$$

We end this section by noting that as a consequence of equations (15.93a) through (15.93c), and because the leading N in (15.93b) can be replaced by A, the formula (15.85) for RLESS can be rewritten in the following simplified form:

$$\hat{\boldsymbol{\xi}}_{\text{RLESS}} = (N + K^TK)^{-1}(\boldsymbol{c} + K^T\boldsymbol{\kappa}_0). \tag{15.104}$$

Applying covariance propagation to equation (15.104) yields an alternate form for the RLESS dispersion as

$$D\{\hat{\boldsymbol{\xi}}_{\text{RLESS}}\} = \sigma_0^2(N + K^TK)^{-1}N(N + K^TK)^{-1}. \tag{15.105}$$

15.8.4   Summary Formulas for Minimally-Constrained LESS's

Regarding partial MINOLESS, if the selection matrix S is the identity matrix, all parameters are selected, and partial MINOLESS becomes MINOLESS. On the other hand, if S selects only the minimum number of parameters necessary to overcome the datum deficiency, then partial MINOLESS is equivalent to RLESS (if $\boldsymbol{\kappa}_0 = \boldsymbol{0}$). The following table list commonly used formulas for the three minimally-constrained solutions RLESS, MINOLESS, and partial MINOLESS.

Table 15.1: Summary of formulas for minimally-constrained least-squares estimators

Type	Estimator	Dispersion matrix
RLESS	$\hat{\boldsymbol{\xi}} = (N+K^TK)^{-1}(\boldsymbol{c}+K^T\boldsymbol{\kappa}_0)$	$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2(N + K^TK)^{-1}N(N + K^TK)^{-1}$
MINO-LESS	$\hat{\boldsymbol{\xi}} = (N + E^TE)^{-1}\boldsymbol{c} = N^+\boldsymbol{c}$	$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2(N + E^TE)^{-1}N(N + E^TE)^{-1}$
Partial MINO-LESS	$\hat{\boldsymbol{\xi}} = (N + SE^TES)^{-1}\boldsymbol{c}$	$D\{\hat{\boldsymbol{\xi}}\}=\sigma_0^2(N+SE^TES)^{-1}N(N+SE^TES)^{-1}$

15.9   More on *S*-Transformations

In equation (15.79) we introduced the so called S-transformation. We now express the S-transformation in terms of the minimum-constraint matrix K from (15.81)

and the inner-constraint matrix E from (15.89b). From (15.86) we have the following reflexive symmetric g-inverse for the (singular) normal-equations matrix N:

$$N_{rs}^- = \left\{ (N + K^T K)^{-1} - (N + K^T K)^{-1} K^T \left[K(N + K^T K)^{-1} K^T \right]^{-1} \cdot \right.$$
$$\left. \cdot K(N + K^T K)^{-1} \right\} =$$
$$= (N + K^T K)^{-1} - (N + K^T K)^{-1} K^T K(N + K^T K)^{-1} = \quad \text{(using (15.93c))}$$
$$= (N + K^T K)^{-1} \left[(N + K^T K) - K^T K \right] (N + K^T K)^{-1} \Rightarrow$$

$$\boxed{N_{rs}^- = (N + K^T K)^{-1} N (N + K^T K)^{-1}} \qquad (15.106)$$

Now according to (15.79) we must multiply N_{rs}^- on the right by N to form an S-transformation. Doing so yields

$$N_{rs}^- N = (N + K^T K)^{-1} N (N + K^T K)^{-1} N = (N + K^T K)^{-1} N = \quad \text{(using (15.93a))}$$
$$= (N + K^T K)^{-1} (N + K^T K - K^T K) = I_m - (N + K^T K)^{-1} K^T K \Rightarrow$$
$$N_{rs}^- N = I_m - E^T (KE^T)^{-1} K \quad \text{(using (15.91a)).} \qquad (15.107)$$

Thus, given any RLESS solution $\hat{\boldsymbol{\xi}}^{(2)}$ we can compute a different RLESS solution $\hat{\boldsymbol{\xi}}^{(1)}$ that is based on *its* associated constraint matrix K using (15.107) as follows:

$$\boxed{\hat{\boldsymbol{\xi}}^{(1)} = N_{rs}^- N \hat{\boldsymbol{\xi}}^{(2)} = \left[I_m - E^T (KE^T)^{-1} K \right] \hat{\boldsymbol{\xi}}^{(2)}.} \qquad (15.108a)$$

To be clear, the matrix K used in (15.108a) is the matrix that would have been used in the solution for $\hat{\boldsymbol{\xi}}^{(1)}$ had it been computed directly, not the one that was used in the solution for $\hat{\boldsymbol{\xi}}^{(2)}$. Note that the matrix to invert in (15.108a) is size $l \times l$, which will almost certainly be small compared to the dimension of N. What's more, since the relations (15.93a) to (15.93c) hold when K is replaced by E (for MINOLESS) or ES (for partial MINOLESS), (15.108a) can be used to convert between any two minimally constrained solutions. Using the law of error propagation, the dispersion matrix for $\hat{\boldsymbol{\xi}}^{(1)}$ is provided by

$$\boxed{D\{\hat{\boldsymbol{\xi}}^{(1)}\} = \left[I_m - E^T (KE^T)^{-1} K \right] \cdot D\{\hat{\boldsymbol{\xi}}^{(2)}\} \cdot \left[I_m - E^T (KE^T)^{-1} K \right]^T.}$$
$$(15.108b)$$

15.9.1   Example Use of an S-Transformation

Here, an example is presented where the preservation of sparsity of the normal equation matrix $N := A^T P A$ is the motivation to use an S-transformation. Consider the case of a 3D network adjustment comprised of GPS vectors as the only type of observation. Such observations provide scale and orientation information about the network datum, but they provide no information about its origin. Thus, the network adjustment problem has a datum deficiency of three, which also means the normal equation matrix N has a rank deficiency of three. Now, further suppose that

MINOLESS is the type of adjustment that must be computed. Then, the $3 \times m$ matrix E, $AE^T = 0$, would be defined by

$$E := [I_3 \ \cdots \ I_3], \tag{15.109a}$$

and therefore the $m \times m$ product $E^T E$ would result in

$$E^T E = \begin{bmatrix} I_3 & \dots & I_3 \\ \vdots & \ddots & \vdots \\ I_3 & \dots & I_3 \end{bmatrix}. \tag{15.109b}$$

Here, the number of unknown parameters m is an integer multiple of 3, and we assume they are (incremental) point coordinates ordered by $\boldsymbol{\xi} =: [x_1 \ y_1 \ z_1 \ \cdots \ x_k \ y_k \ z_k]^T$ for $k := m/3$ points.

Obviously, adding $E^T E$ to N in this case may greatly reduce the sparsity of N, especially if N had most of its nonzero elements near its diagonal. On the other hand, if partial MINOLESS is computed with the $m \times m$ selection matrix S defined by

$$S := \mathrm{diag}(I_3 \ 0_3 \ \cdots \ 0_3), \tag{15.110}$$

then the sparsity of $N + SE^T ES$ would be the same as that of N itself. Moreover, this expression of S results in the minimally-constrained adjustment RLESS. Thus, we may take ES as our matrix K; i.e., $K := ES$, and therefore $N + SE^T ES \to N + K^T K$. Now, let $\hat{\boldsymbol{\xi}}^{(2)}$ be the partial MINOLESS computed using S as just defined. That solution can be transformed to MINOLESS by substituting E for K in (15.108a), resulting in

$$\hat{\boldsymbol{\xi}}^{(1)} = \big[I_m - E^T (EE^T)^{-1} E\big]\hat{\boldsymbol{\xi}}^{(2)}. \tag{15.111}$$

Then, we must only invert the 3×3 matrix EE^T to convert partial MINOLESS $\hat{\boldsymbol{\xi}}^{(2)}$ to MINOLESS $\hat{\boldsymbol{\xi}}^{(1)}$. Incidentally, an inspection of the structure of $[I_m - E^T (EE^T)^{-1} E]$ reveals that its density of non-zero elements is $1/3$.

We saw already what $E^T E$ looks like in this example, but what about EE^T? It turns out to be a diagonal matrix that is a scalar multiple of the identity matrix I_l, where the scalar is simply m/l. Thus, $(EE^T)^{-1} = (l/m) \cdot I_l$. This means that $E^T (EE^T)^{-1} E = (l/m)E^T E$, and, therefore, using (15.109b) and defining $a := 1 - l/m$ and $b := a - 1$, we have

$$\big[I_m - E^T (EE^T)^{-1} E\big] = \begin{bmatrix} aI_l & bI_l & \cdots \\ bI_l & aI_l & \\ \vdots & & \ddots \end{bmatrix}_{m \times m} = \begin{bmatrix} a & b & \cdots \\ b & a & \\ \vdots & & \ddots \end{bmatrix}_{(m/l) \times (m/l)} \otimes I_l. \tag{15.112}$$

So we see that the expression $[I_m - E^T(EE^T)^{-1}E]$ can be formed directly without any mathematical operations. A closer inspection reveals a very simple way to compute the S-transformation in this example. Let $(\hat{x}_i^{(2)}, \hat{y}_i^{(2)}, \hat{z}_i^{(2)})$ denote the ith estimated (incremental) coordinate triple from $\hat{\boldsymbol{\xi}}^{(2)}$. Then the MINOLESS estimates of the ith triple of coordinates $(\hat{x}_i^{(1)}, \hat{y}_i^{(1)}, \hat{z}_i^{(1)})$ can be computed by

$$\hat{x}_i^{(1)} = a \cdot \hat{x}_i^{(2)} + b \cdot \sum_{\substack{j=1 \\ j \neq i}}^{m/3} \hat{x}_j^{(2)} = \hat{x}_i^{(2)} + b \cdot \sum_{j=1}^{m/3} \hat{x}_j^{(2)}, \ \ \text{etc.} \tag{15.113}$$

Equation (15.113) shows MINOLESS as a weighted average of RLESS (or any other minimally-constrained LESS), where the weights are a function of the number of unknowns m and the size of the datum deficiency l. It is stressed that the MINOLESS $\hat{\boldsymbol{\xi}}^{(1)}$ is unique regardless of the datum choice for $\hat{\boldsymbol{\xi}}^{(2)}$.

But, what is really important, is that the solution $\hat{\boldsymbol{\xi}}^{(2)}$ can first be computed using (15.104) with the sparsity of $N + K^T K$ being the same as that of N. This is important, because large systems of equations can be solved more efficiently when the matrix to "invert" (or compute the Cholesky factor of) is sparse, especially if the parameters have been ordered in a way to keep the nonzero elements of N close to its diagonal. For reordering algorithms that accomplish that, see, for example, the Banker's ordering algorithm presented by Snay (1979) and the Approximate Minimum Degree (AMD) ordering algorithm described by Amestoy et al. (1996).

Obviously, the corresponding dispersion matrix could be computed by

$$D\{\hat{\boldsymbol{\xi}}^{(1)}\} = \left[I_m - E^T(EE^T)^{-1}E\right] \cdot D\{\hat{\boldsymbol{\xi}}^{(2)}\} \cdot \left[I_m - E^T(EE^T)^{-1}E\right]. \tag{15.114}$$

Here we see that extra work involving a triple product in three $m \times m$ matrices is required to transform the dispersion matrix $D\{\hat{\boldsymbol{\xi}}^{(2)}\}$. However, this extra work is likely offset by the time saved in computing an inverse, or Cholesky factor, for a less dense matrix involved in computing $\hat{\boldsymbol{\xi}}^{(2)}$, as already discussed. Also, the nonzero elements of the matrix in (15.112) take on only two different values; so, some clever, cost-saving implementation of an algorithm for the triple product could be employed. In fact, the matrix product $[I_m - E^T(EE^T)^{-1}E](N + K^T K)^{-1}$ can be computed efficiently by noting that multiplying by $E^T(EE^T)^{-1}E$ in this case results in summing every third row of $(N + K^T K)^{-1}$, three times, and then multiplying all three of the sums by l/m, i.e., rows 1, 4, 7, ... are summed; rows 2, 5, 8, ... are summed; and rows 3, 6, 9, ... are summed.

What if the S-transformation should transform $\hat{\boldsymbol{\xi}}^{(1)}$ to partial MINOLESS rather than MINOLESS? In that case, one only needs to skip over the non-selected points in the summation in (15.113) and change the denominator from m to the number of selected parameters in the definition of a. Furthermore, the matrix $[I_m - E^T(KE^T)^{-1}K]$ may have a much smaller density of non-zero elements than $[I_m - E^T(EE^T)^{-1}E]$, depending on the number of parameters selected by S. If that number is denoted by n_s, then apparently the density can be computed by $[(m - n_s) + m \cdot n_s/3]/m^2$, which turns out to be $1/3$ in the case of MINOLESS ($n_s = m$) but smaller otherwise.

Finally, the respective residual vectors corresponding to $\hat{\boldsymbol{\xi}}^{(1)}$ and $\hat{\boldsymbol{\xi}}^{(2)}$ are identical, a property of minimally-constrained solutions reiterated in the next section.

15.10   Concluding Remarks about the Restricted Least-Squares Solution

In addition to (15.14) and (15.36), we have an alternative way to represent the solution space within the rank deficient GMM via RLESS.

$$\hat{\boldsymbol{\xi}} = \{\hat{\boldsymbol{\xi}} \mid \hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}_{\mathrm{RLESS}} \text{ subject to minimum constraints } K\boldsymbol{\xi} = \boldsymbol{\kappa}_0\} \qquad (15.115)$$

For convenience we have set $\boldsymbol{\kappa}_0 := K\boldsymbol{\xi} = \mathbf{0}$ in some of the derivations above. The zero-vector always applies to the case where the observation equations have been linearized, as $\hat{\boldsymbol{\xi}}$ becomes a vector of estimated updates to the initial, or approximate, parameter values in that case. For purely linear observation equations (15.1), we may have a non-zero vector $\boldsymbol{\kappa}_0$, in which case it must be included in the solution formula as in (15.93b).

If we base the reflexive symmetric matrix N_{rs}^- for RLESS on the singular value decomposition of N, as in (15.31), we only need to replace the arbitrary matrix L in that formula with the constraint matrix K in order to reach a minimum-constraint solution satisfying $K\boldsymbol{\xi} = \boldsymbol{\kappa}_0$.

Finally, we reiterate that no matter which minimum-constraint conditions we impose, the residual vector will turn out the same. Incidentally, that is a necessary condition for two adjustment models to be the same; i.e, they will lead to the same residual vector. This is true because the term $AN_{rs}^-A^T$ in the equation for the predicted random errors (residuals), viz.

$$\boxed{\tilde{\boldsymbol{e}} = (I_n - AN_{rs}^-A^TP)\boldsymbol{y},} \qquad (15.116)$$

is invariant with respect to the choice of N_{rs}^-. Applying covariance propagation to (15.116) leads to the dispersion matrix

$$\boxed{D\{\tilde{\boldsymbol{e}}\} = \sigma_0^2(P^{-1} - AN_{rs}^-A^T),} \qquad (15.117)$$

which is also unique among all minimally constrained adjustments.

Chapter 16

The Variance Component Model

16.1 Introduction of the Variance Component Model

The variance component model (VCM) allows for multiple variance components in the covariance matrix Σ of the random error vector $\boldsymbol{e}$. The functional part (observation equations) of the VCM looks like that of the Gauss-Markov Model (GMM), but the covariance matrix is expressed as a linear combination of *known* cofactor matrices Q_i, each multiplied by a unique, *unknown* variance component σ_i^2 (or covariance component σ_{ij}). In fact, Koch (1999, p. 226) refers to the VCM as the GMM with unknown variance and covariance components. The VCM is written as

$$\boldsymbol{y} = \underset{n \times m}{A}\, \boldsymbol{\xi} + \boldsymbol{e}, \quad \operatorname{rk} A \leq m < n, \tag{16.1a}$$

$$\boldsymbol{e} \sim (\boldsymbol{0}, \Sigma = \sigma_1^2 Q_1 + \sigma_{12} Q_2 + \cdots + \sigma_l^2 Q_k), \quad k = l(l+1)/2. \tag{16.1b}$$

If the covariance matrix Σ is diagonal, and possibly if it is block diagonal, then no covariance components will be involved, and the number of variance components l will be equal to the number of cofactor matrices k.

If matrix A has full column rank, the Best Linear Uniformly Unbiased Estimate (BLUUE) of the unknown parameters $\boldsymbol{\xi}$ within the VCM (16.1) is given by

$$\hat{\boldsymbol{\xi}}_{\mathrm{BLUUE}} = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1} \boldsymbol{y} =$$
$$= \left[A^T (\sigma_1^2 Q_1 + \sigma_{12} Q_2 + \ldots + \sigma_l^2 Q_k)^{-1} A \right]^{-1} A^T (\sigma_1^2 Q_1 + \sigma_{12} Q_2 + \ldots + \sigma_l^2 Q_k)^{-1} \boldsymbol{y}. \tag{16.2}$$

We see from (16.2) that the parameter estimates $\hat{\boldsymbol{\xi}}_{\mathrm{BLUUE}}$ depend on the unknown variance components, and thus we cannot actually compute the BLUUE for the

VCM. So we are left with the option of replacing the unknown variance components with their estimates $\hat{\sigma}_1^2$, $\hat{\sigma}_{12}$, ..., $\hat{\sigma}_l^2$. But then we have a functional dependency viz. $\hat{\boldsymbol{\xi}} := \hat{\boldsymbol{\xi}}(\hat{\sigma}_1^2, \hat{\sigma}_{12}, \ldots, \hat{\sigma}_l^2)$.

At this point, one may naturally ask whether such an estimator retains the *linear* and *unbiased* properties of BLUUE. To be sure, $\hat{\boldsymbol{\xi}}(\hat{\sigma}_1^2, \hat{\sigma}_{12}, \ldots, \hat{\sigma}_l^2)$ is nonlinear in $\boldsymbol{y}$, since the unknown variance components are functions of $\boldsymbol{y}$ (as we shall soon see) and they also multiply $\boldsymbol{y}$ in the solution (16.2) of the normal equations. Therefore, the estimator $\hat{\boldsymbol{\xi}}$ is not BLUUE. However, under certain assumptions, it can be shown that $\hat{\boldsymbol{\xi}}$ is unbiased. In order to show this we make use of the concept of E-D correspondence (see chapter 14). Let us now proceed with the derivations for variance component estimation, leading to formulas for the parameter estimator $\hat{\boldsymbol{\xi}}(\hat{\sigma}_1^2, \hat{\sigma}_{12}, \ldots, \hat{\sigma}_l^2)$.

16.2   A Model Linear in the Unknown Variance Components

Our approach will be to develop a model linear in the unknown variance components that has the same structure as the Gauss Markov Model (GMM). We will then derive estimators for the unknown parameters of the model, i.e., the variance components, by application of a least-squares solution (LESS), as was done within the GMM. *A key concept of variance component estimation is that of invariance, which makes the estimation of variance components entirely independent of the estimation of the parameters $\boldsymbol{\xi}$.*

In the developments of this section we omit covariance components σ_{ij} and limit the number of variance components to two for simplicity. In theory, any number of variance components could be included in the VCM, but the presence of very many of them would require a sufficiently large model redundancy to avoid numerical instabilities that could occur in algorithms for their estimation. We also assume in the following that matrix A has full column rank so that the inverse of $A^T\Sigma^{-1}A$ exists. Later in the chapter we comment on how to handle the problem if matrix A is rank deficient.

We begin by introducing approximations (or initial values) to Σ, σ_1^2, and σ_2^2 as Σ_0, $\sigma_{1,0}^2$, and $\sigma_{2,0}^2$, respectively, relating them as follows:

$$\Sigma_0 := \sigma_{1,0}^2 Q_1 + \sigma_{2,0}^2 Q_2, \tag{16.3a}$$

implying that

$$\underbrace{\text{vec}\,\Sigma_0}_{n^2 \times 1} = \begin{bmatrix} \text{vec}\,Q_1 & \text{vec}\,Q_2 \end{bmatrix} \begin{bmatrix} \sigma_{1,0}^2 \\ \sigma_{2,0}^2 \end{bmatrix} =: V\boldsymbol{\vartheta}_0. \tag{16.3b}$$

The transformation of (16.3a) into (16.3b) by use of the vec operator is key to arriving at a model that is linear in the unknown variance components.

Using the above approximations, and continuing to assume that matrix A has full column rank, leads to the following vectors of estimated parameters and predicted

random errors (residuals):

$$\hat{\boldsymbol{\xi}}_0 = \hat{\boldsymbol{\xi}}(\sigma_{1,0}^2, \sigma_{2,0}^2) = (A^T \Sigma_0^{-1} A)^{-1} A^T \Sigma_0^{-1} \boldsymbol{y}, \tag{16.4a}$$

$$\tilde{\boldsymbol{e}}_0 = \boldsymbol{y} - A\hat{\boldsymbol{\xi}}_0 = \big[I_n - A(A^T \Sigma_0^{-1} A)^{-1} A^T \Sigma_0^{-1}\big]\boldsymbol{y} = \big[I_n - S_0\big]\boldsymbol{e}. \tag{16.4b}$$

Here, we define the similarity-transformation matrix $S_0 := A(A^T \Sigma_0^{-1} A)^{-1} A^T \Sigma_0^{-1}$ and note that the columns of A are in the nullspace of $[I_n - S_0]$, which is apparent from

$$[I_n - S_0]A = [I_n - A(A^T \Sigma_0^{-1} A)^{-1} A^T \Sigma_0^{-1}]A = 0. \tag{16.5a}$$

Thus, we can replace $\boldsymbol{y}$ with the true random error vector $\boldsymbol{e}$ to arrive at

$$[I_n - S_0]\boldsymbol{y} = [I_n - S_0](A\boldsymbol{\xi} + \boldsymbol{e}) = [I_n - S_0]\boldsymbol{e}, \tag{16.5b}$$

which is theoretically accurate even though $\boldsymbol{e}$ is unknown and thus cannot be computed. From here we can write a new model based on the Kronecker product $\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0$ of the residual vectors, viz.

$$E\{\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0\} = \big[(I_n - S_0) \otimes (I_n - S_0)\big] \cdot E\{\boldsymbol{e} \otimes \boldsymbol{e}\}. \tag{16.6}$$

Again, the motivation for this new model is to eventually arrive at a model that is linear in the unknown variance components σ_1^2 and σ_2^2. Using (12.7), we have $\boldsymbol{e} \otimes \boldsymbol{e} = \mathrm{vec}(\boldsymbol{e}\boldsymbol{e}^T)$, which implies that

$$E\{\boldsymbol{e} \otimes \boldsymbol{e}\} = E\{\mathrm{vec}(\boldsymbol{e}\boldsymbol{e}^T)\} = \mathrm{vec}\, E\{\boldsymbol{e}\boldsymbol{e}^T\} = \mathrm{vec}\, D\{\boldsymbol{e}\} = \mathrm{vec}\, \Sigma. \tag{16.7}$$

This means that (16.6) can be rewritten as

$$E\{\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0\} = \big[(I_n - S_0) \otimes (I_n - S_0)\big]V\boldsymbol{\vartheta}, \tag{16.8}$$

where $\boldsymbol{\vartheta}$ is analogous to $\boldsymbol{\vartheta}_0$ but is based on the true (unknown) variance components rather than the approximate ones.

Equation (16.8) is now a linear form in $\boldsymbol{\vartheta} = [\sigma_1^2, \sigma_2^2]^T$, which is precisely the quantity that we want to estimate. Keep in mind that the initial approximation $\boldsymbol{\vartheta}_0$ is used in the computation of $\tilde{\boldsymbol{e}}_0$. This is because $\tilde{\boldsymbol{e}}_0$ is defined through S_0, and S_0 depends on Σ_0, which depends on the approximation $\boldsymbol{\vartheta}_0$. Numerically, this means that we must iterate the solution of $\boldsymbol{\vartheta}_0$ until it converges to $\hat{\boldsymbol{\vartheta}}$. This is the so-called *reproducing property*.

Now we show the dispersion of the Kronecker product $\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0$, which, under the assumption of quasi-normality, reads

$$D\{\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0\} = (I_{n^2} + K)\big[(I_n - S_0)\Sigma(I_n - S_0)^T \otimes (I_n - S_0)\Sigma(I_n - S_0)^T\big], \tag{16.9a}$$

where K is a commutation matrix. Equation (16.9a) is more complicated than (14.16) within the GMM, since it is based on both the true matrix Σ and the

approximate matrix S_0. However, by substituting the approximation Σ_0 and exploiting the symmetry of $\Sigma_0(I_n - S_0)^T$ and the idempotent property of $(I_n - S_0)$, we can write an approximate dispersion matrix as

$$D_0\{\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0\} = (I_{n^2} + K)\big[(I_n - S_0)\Sigma_0 \otimes (I_n - S_0)\Sigma_0\big]. \tag{16.9b}$$

Combining (16.4b) and (16.8), and including (16.9b), the analogy of the model (16.6) to the GMM (i.e., $E\{\boldsymbol{y}\} = A\boldsymbol{\xi}$, $D\{\boldsymbol{y}\} = \Sigma = \sigma_0^2 Q$) is shown in the following schematic:

New model—linear in $\boldsymbol{\vartheta} = [\sigma_1^2,\ \sigma_2^2]^T$

Expectation:

$$E\{\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0\} = E\{\underbrace{(I_n - S_0)\boldsymbol{y} \otimes (I_n - S_0)\boldsymbol{y}}_{\text{Analogous to } \boldsymbol{y} \text{ in the GMM}}\} =$$

$$= \underbrace{\big[(I_n - S_0) \otimes (I_n - S_0)\big]V}_{\text{Analogous to } A \text{ in GMM}}\ \underbrace{\boldsymbol{\vartheta}}_{\text{Analogous to } \boldsymbol{\xi} \text{ in the GMM}} \tag{16.10a}$$

Dispersion:

$$D_0\{\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0\} = \underbrace{(I_{n^2} + K)\big[(I_n - S_0)\Sigma_0 \otimes (I_n - S_0)\Sigma_0\big]}_{\text{Analogous to } \Sigma = \sigma_0^2 Q \text{ in the GMM}} \tag{16.10b}$$

Based on the analogy to the GMM, we require a weight matrix G_0 (analogous to P in the GMM) to compute the weighted LEast-Squares Solution (LESS) of the variance component vector $\boldsymbol{\vartheta}$. In full analogy to (14.17), we define an "approximate weight matrix" as follows:

$$G_0 = (I_{n^2} + K)\big[\Sigma_0^{-1}(I_n - S_0) \otimes \Sigma_0^{-1}(I_n - S_0)\big]. \tag{16.11}$$

Note that up to a factor of $1/4$, G_0 turns out to be a g-inverse of the approximate dispersion matrix (16.9b) (cf. (14.16) and (14.17)). Once again, we note that K is a commutation matrix. We can now write the normal equations for the weighted LESS. First we form the right-side vector, analogously to $A^T P\boldsymbol{y}$ in the GMM.

Right-side:

$$V^T\big[(I_n - S_0)^T \otimes (I_n - S_0)^T\big](I_{n^2} + K)\big[\Sigma_0^{-1}(I_n - S_0) \otimes \Sigma_0^{-1}(I_n - S_0)\big]\cdot$$
$$\cdot \big[(I_n - S_0)\boldsymbol{y} \otimes (I_n - S_0)\boldsymbol{y}\big] = \tag{16.12a}$$

$$V^T(I_{n^2} + K)\big[(I_n - S_0)^T \otimes (I_n - S_0)^T\big]\big[\Sigma_0^{-1}(I_n - S_0) \otimes \Sigma_0^{-1}(I_n - S_0)\big]\cdot$$
$$\cdot \big[(I_n - S_0)\boldsymbol{y} \otimes (I_n - S_0)\boldsymbol{y}\big] = \tag{16.12b}$$

$$= V^T(I_{n^2} + K)\big[(I_n - S_0)^T\Sigma_0^{-1}(I_n - S_0) \otimes (I_n - S_0)^T\Sigma_0^{-1}(I_n - S_0)\big]\cdot$$
$$\cdot \big[(I_n - S_0)\boldsymbol{y} \otimes (I_n - S_0)\boldsymbol{y}\big] = \tag{16.12c}$$

$$= V^T(I_{n^2} + K)\big[(I_n - S_0)^T\Sigma_0^{-1}(I_n - S_0)^2\boldsymbol{y} \otimes (I_n - S_0)^T\Sigma_0^{-1}(I_n - S_0)^2\boldsymbol{y}\big] = \tag{16.12d}$$

$$= V^T(I_{n^2} + K)\big[(I_n - S_0)^T\Sigma_0^{-1}(I_n - S_0)\boldsymbol{y} \otimes (I_n - S_0)^T\Sigma_0^{-1}(I_n - S_0)\boldsymbol{y}\big].$$
$$(16.12\text{e})$$

We used (A.40) in going from (16.12a) to (16.12b), and the idempotency of $(I_n - S_0)$ was exploited from (16.12d) to (16.12e). Note that $(I_n - S_0)^T\Sigma_0^{-1}$ is symmetric, so by using its transpose and considering that $I_n - S_0$ is idempotent, we can further reduce the right side to

$$V^T(I_{n^2}+K)\big[\Sigma_0^{-1}(I_n-S_0)\boldsymbol{y}\otimes\Sigma_0^{-1}(I_n-S_0)\boldsymbol{y}\big] = V^T(I_{n^2}+K)\big[\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0\otimes\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0\big] =$$
$$= 2V^T\big(\Sigma_0^{-1} \otimes \Sigma_0^{-1}\big)(\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0). \quad (16.12\text{f})$$

In the second line we used the fact that K is a vec permutation matrix, so that with symmetric Q_i ($i \in \{1, 2\}$ in this case) and use of (A.41), we can rewrite $V^T(I_{n^2}+K)$ as

$$V^T(I_{n^2} + K) = \begin{bmatrix} (\text{vec}\, Q_1)^T \\ (\text{vec}\, Q_2)^T \end{bmatrix} (I_{n^2} + K) = 2V^T. \qquad (16.12\text{g})$$

Note that the factor of 2 is independent of the number of variance components. Now we can successively apply (A.31) in its transposed form to the last line of (16.12f), resulting in

$$2\begin{bmatrix} [\text{vec}(\Sigma_0^{-1}Q_1\Sigma_0^{-1})]^T \\ [\text{vec}(\Sigma_0^{-1}Q_2\Sigma_0^{-1})]^T \end{bmatrix}(\tilde{\boldsymbol{e}}_0 \otimes \tilde{\boldsymbol{e}}_0) = 2\begin{bmatrix} [\text{vec}(\tilde{\boldsymbol{e}}_0^T\Sigma_0^{-1}Q_1\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0)]^T \\ [\text{vec}(\tilde{\boldsymbol{e}}_0^T\Sigma_0^{-1}Q_2\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0)]^T \end{bmatrix} =$$
$$(16.12\text{h})$$
$$= 2\begin{bmatrix} \tilde{\boldsymbol{e}}_0^T\Sigma_0^{-1}Q_1\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0 \\ \tilde{\boldsymbol{e}}_0^T\Sigma_0^{-1}Q_2\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0 \end{bmatrix}.$$

Finally, by use of (16.4b) and introduction of the singular matrix $W_0 := \Sigma_0^{-1}(I_n - S_0)$, we may write

$$2\begin{bmatrix} \tilde{\boldsymbol{e}}_0^T\Sigma_0^{-1}Q_1\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0 \\ \tilde{\boldsymbol{e}}_0^T\Sigma_0^{-1}Q_2\Sigma_0^{-1}\tilde{\boldsymbol{e}}_0 \end{bmatrix} = 2\begin{bmatrix} \boldsymbol{y}^T(I_n - S_0)\Sigma_0^{-1}Q_1\Sigma_0^{-1}(I_n - S_0)\boldsymbol{y} \\ \boldsymbol{y}^T(I_n - S_0)\Sigma_0^{-1}Q_2\Sigma_0^{-1}(I_n - S_0)\boldsymbol{y} \end{bmatrix} = 2\begin{bmatrix} \boldsymbol{y}^TW_0Q_1W_0\boldsymbol{y} \\ \boldsymbol{y}^TW_0Q_2W_0\boldsymbol{y} \end{bmatrix}$$
$$(16.12\text{i})$$

as an expression of the right side of the normal equations.

Now we work out the left side of the normal equations, analogous to $(A^TPA)\hat{\boldsymbol{\xi}}$ in the GMM. Much of this work has already been done since the left side begins with the same terms as the right side (analogous to A^TP in the GMM); these steps will not be repeated.

Left side:

$$V^T\big[(I_n - S_0)^T \otimes (I_n - S_0)^T\big](I_{n^2} + K)\big[\Sigma_0^{-1}(I_n - S_0) \otimes \Sigma_0^{-1}(I_n - S_0)\big]\cdot$$
$$\cdot \big[(I_n - S_0) \otimes (I_n - S_0)\big]V\hat{\boldsymbol{\vartheta}} =$$
$$(16.13\text{a})$$

$$= 2V^T \left[\Sigma_0^{-1}(I_n - S_0) \otimes \Sigma_0^{-1}(I_n - S_0) \right] V \hat{\boldsymbol{\vartheta}} = \tag{16.13b}$$

(see (16.12) for more details)

$$= 2 \left[\begin{array}{c|c} \mathrm{tr}\left[\Sigma_0^{-1}(I_n - S_0)Q_1\Sigma_0^{-1}(I_n - S_0)Q_1\right] & \mathrm{tr}\left[\Sigma_0^{-1}(I_n - S_0)Q_1\Sigma_0^{-1}(I_n - S_0)Q_2\right] \\ \hline \mathrm{tr}\left[\Sigma_0^{-1}(I_n - S_0)Q_2\Sigma_0^{-1}(I_n - S_0)Q_1\right] & \mathrm{tr}\left[\Sigma_0^{-1}(I_n - S_0)Q_2\Sigma_0^{-1}(I_n - S_0)Q_2\right] \end{array} \right] \begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix} = \tag{16.13c}$$

$$= 2 \left[\begin{array}{c|c} \mathrm{tr}[W_0Q_1W_0Q_1] & \mathrm{tr}[W_0Q_1W_0Q_2] \\ \hline \mathrm{tr}[W_0Q_2W_0Q_1] & \mathrm{tr}[W_0Q_2W_0Q_2] \end{array} \right] \begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix}, \tag{16.13d}$$

with

$$W_0 := \Sigma_0^{-1}(I_n - S_0) = \Sigma_0^{-1} - \Sigma_0^{-1}A(A^T\Sigma_0^{-1}A)^{-1}A^T\Sigma_0^{-1}. \tag{16.13e}$$

Before combining the left (16.12i) and right (16.13e) sides into one system of equations we introduce subscripts to express the dependence of the $(j+1)$th solution on the jth solution, and we drop the leading factor of 2 from both sides. Then the system of normal equations for the $(j + 1)$th solution is given by

$$\left[\begin{array}{c|c} \mathrm{tr}[W_jQ_1W_jQ_1] & \mathrm{tr}[W_jQ_1W_jQ_2] \\ \hline \mathrm{tr}[W_jQ_2W_jQ_1] & \mathrm{tr}[W_jQ_2W_jQ_2] \end{array} \right] \begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix}^{(j+1)} = \begin{bmatrix} \boldsymbol{y}^T W_jQ_1W_j\boldsymbol{y} \\ \boldsymbol{y}^T W_jQ_2W_j\boldsymbol{y} \end{bmatrix}. \tag{16.14}$$

The solution for (16.14) is usually iterated until, for some prescribed level of precision δ, we arrive at

$$\left\| \hat{\boldsymbol{\vartheta}}_{j+1} - \hat{\boldsymbol{\vartheta}}_j \right\|^2 < \delta^2. \tag{16.15}$$

Thus, the solution $\hat{\boldsymbol{\vartheta}}$ is called the reproducing Best Invariant Quadratic Uniformly Unbiased Estimate (reproBIQUUE) of $\boldsymbol{\vartheta}$.

There is always a solution to the system of equations (16.14) since they represent normal equations. However, the solution may not be unique, and often it is not. As stated previously, we consider the system to be nonlinear in $\hat{\boldsymbol{\vartheta}}$. In summary, we write the normal equations for the weighted LESS of $\boldsymbol{\vartheta}$ in its nonlinear form without iteration subscripts, bearing in mind that $\sigma_{1,0}^2$ and $\sigma_{2,0}^2$ are approximated by $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ from the previous iteration.

$$\boxed{ \begin{array}{c} \left[\begin{array}{c|c} \mathrm{tr}(W_0Q_1W_0Q_1) & \mathrm{tr}(W_0Q_1W_0Q_2) \\ \hline \mathrm{tr}(W_0Q_2W_0Q_1) & \mathrm{tr}(W_0Q_2W_0Q_2) \end{array} \right] \begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}^T W_0Q_1W_0\boldsymbol{y} \\ \boldsymbol{y}^T W_0Q_2W_0\boldsymbol{y} \end{bmatrix} \quad (16.16a) \\ W_0 := \Sigma_0^{-1} - \Sigma_0^{-1}A(A^T\Sigma_0^{-1}A)^{-1}A^T\Sigma_0^{-1}. \quad (16.16b) \\ \Sigma_0 = \sigma_{1,0}^2 Q_1 + \sigma_{2,0}^2 Q_2 \quad (16.16c) \end{array} }$$

Because of the invariant property of BIQUUE, the vector $\boldsymbol{y}$ in (16.16a) can be replaced by $\tilde{\boldsymbol{e}} = \boldsymbol{y} - A\hat{\boldsymbol{\xi}}$, which is also apparent since the columns of matrix A are in the nullspace of W_0, i.e., $W_0 A = 0$. Thus, we also have

$$
\begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix} = \left[\begin{array}{c|c} \mathrm{tr}(W_0 Q_1 W_0 Q_1) & \mathrm{tr}(W_0 Q_1 W_0 Q_2) \\ \hline \mathrm{tr}(W_0 Q_2 W_0 Q_1) & \mathrm{tr}(W_0 Q_2 W_0 Q_2) \end{array} \right]^{-1} \begin{bmatrix} \tilde{\boldsymbol{e}}^T W_0 Q_1 W_0 \tilde{\boldsymbol{e}} \\ \tilde{\boldsymbol{e}}^T W_0 Q_2 W_0 \tilde{\boldsymbol{e}} \end{bmatrix} . \quad (16.17)
$$

The corresponding estimator for the unknown parameters $\boldsymbol{\xi}$ is then provided by

$$
\boxed{
\begin{aligned}
\hat{\boldsymbol{\xi}} &= (A^T \hat{\Sigma}^{-1} A)^{-1} A^T \hat{\Sigma}^{-1} \boldsymbol{y} = \\
&= \left[A^T (\hat{\sigma}_1^2 Q_1 + \hat{\sigma}_2^2 Q_2)^{-1} A \right]^{-1} A^T (\hat{\sigma}_1^2 Q_1 + \hat{\sigma}_2^2 Q_2)^{-1} \boldsymbol{y}.
\end{aligned}
} \quad (16.18)
$$

16.2.1 Extension to k Variance Components

It's easy enough to imagine what (16.16a) to (16.18) would look like for more than two variance components, say $k = l(l+1)/2$ of them (including covariance components). To inspire the imagination, we show that larger system here:

$$
\begin{bmatrix} \mathrm{tr}(W_0 Q_1 W_0 Q_1) & \mathrm{tr}(W_0 Q_1 W_0 Q_2) & \cdots & \mathrm{tr}(W_0 Q_1 W_0 Q_k) \\ \mathrm{tr}(W_0 Q_2 W_0 Q_1) & \mathrm{tr}(W_0 Q_2 W_0 Q_2) & & \\ \vdots & & \ddots & \\ \mathrm{tr}(W_0 Q_k W_0 Q_1) & \cdots & & \mathrm{tr}(W_0 Q_k W_0 Q_k) \end{bmatrix} \begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_{12} \\ \vdots \\ \hat{\sigma}_l^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}^T W_0 Q_1 W_0 \boldsymbol{y} \\ \boldsymbol{y}^T W_0 Q_2 W_0 \boldsymbol{y} \\ \vdots \\ \boldsymbol{y}^T W_0 Q_k W_0 \boldsymbol{y} \end{bmatrix},
$$
$$
\text{with} \quad W_0 = \Sigma_0^{-1} - \Sigma_0^{-1} A (A^T \Sigma_0^{-1} A)^- A^T \Sigma_0^{-1}. \quad (16.19)
$$

Then, assuming matrix A has full column rank, the corresponding estimator for the unknown parameters $\boldsymbol{\xi}$ reads

$$
\hat{\boldsymbol{\xi}} = (A^T \hat{\Sigma}^{-1} A)^{-1} A^T \hat{\Sigma}^{-1} \boldsymbol{y}, \quad (16.20a)
$$
$$
\hat{\Sigma} = \hat{\sigma}_1^2 Q_1 + \hat{\sigma}_{12} Q_2 + \cdots + \hat{\sigma}_l^2 Q_k, \quad \text{with} \ k = l(l+1)/2, \quad (16.20b)
$$

if Σ_0 has converged to $\hat{\Sigma}$ by iterative computation.

16.2.2 Practical Matters

Should we solve the problem by aiming for a local BIQUUE at every iteration step? This is an open question. Prof. Schaffrin said that he does not believe it is the best way, but it is the way it is often done in practice. The best algorithm may not produce a local minimum at each iteration, but we are not interested in these local minimums. Our objective is to convergence to a global minimum.

The solutions may or may not depend on the initial approximations. Usually we know which solution to choose if we do find multiple solutions. The larger

problem is that the system is "blind" to the non-negativity requirement of the estimates (i.e. the variance components must be positive). In practice, the cofactor matrices, Q_i, are usually revised if the solution yields negative variance component estimates. This is because the negative values are likely an indicator that there is something wrong with the model, i.e., the model is not consistent with the the observations. And we would not change the observations. However, we may be inclined to disregard a few observations (if we deem them to be outliers). Another approach would be to introduce an additional variance component to estimate.

Another question that one might ask is why the variance component estimates sometimes turn out negative. It is easy to see this in our case of two variance components. The normal matrix in (16.16a) is positive in each block. Therefore, the off-diagonal elements of its inverse are negative (think of the familiar formula for the inverse of a 2×2 matrix). So depending on the relative magnitudes of Q_1 and Q_2, we may or may not end up with positive estimates.

Estimated dispersion of variance components   What about the precision of our estimates? The estimated dispersion matrix is simply the inverse of the matrix on the left side of 16.16a, which for k variance components is

$$
\hat{D}\{\hat{\boldsymbol{\vartheta}}\} =
\begin{bmatrix}
\text{tr}(W_0Q_1W_0Q_1) & \text{tr}(W_0Q_1WQ_2) & \cdots & \\
\text{tr}(W_0Q_2W_0Q_1) & \text{tr}(W_0Q_2WQ_2) & & \\
\vdots & & \ddots & \\
\text{tr}(W_0Q_kW_0Q_1) & \cdots & & \text{tr}(W_0Q_kW_0Q_k)
\end{bmatrix}^{-1} . \quad (16.21)
$$

This is already the estimated dispersion of $\hat{\boldsymbol{\vartheta}}$ (hence the hat over the dispersion operator D) due to the estimated variance components involved in matrix W. It is hard to express the true dispersion $D\{\hat{\boldsymbol{\vartheta}}\}$ because of the iteration process.

Assume a single variance component   Let us check (16.21) for the case of a single variance component, which we simply label as $\hat{\sigma}_0^2$.

Start with $\text{tr}(\hat{W}Q\hat{W}Q)$, where

$$
\hat{W} = \hat{\Sigma}^{-1} - \hat{\Sigma}^{-1}A(A^T\hat{\Sigma}^{-1}A)^{-1}A^T\hat{\Sigma}^{-1} = (\hat{\sigma}_0^2)^{-1}(P - PAN^{-1}A^TP). \quad (16.22a)
$$

Multiplication by the cofactor matrix Q from the right yields

$$
\hat{W}Q = (\hat{\sigma}_0^2)^{-1}(I_n - PAN^{-1}A^T), \quad (16.22b)
$$

which implies that

$$
\begin{aligned}
\text{tr}(\hat{W}Q\hat{W}Q) &= (\hat{\sigma}_0^2)^{-2}\,\text{tr}\big[(I_n - PAN^{-1}A^T)^2\big] = \\
&= (\hat{\sigma}_0^2)^{-2}\,\text{tr}\big(I_n - PAN^{-1}A^T\big) = (\hat{\sigma}_0^2)^{-2}(n - m),
\end{aligned} \quad (16.22c)
$$

from which it follows

$$2\big[\mathrm{tr}(\hat{W}Q\hat{W}Q)\big]^{-1} = 2(\hat{\sigma}_0^2)^2/(n-m). \qquad (16.22\mathrm{d})$$

Note that (16.22d) is identical to (13.23), showing the consistency of univariate and multivariate variance component estimators. In general, we should replace m with rk A to account for possible rank deficiency in matrix A. Compare this result to the solution found in (13.23).

Redundancy matters   In general, variance component estimation requires a relatively large redundancy in the observational model. For comparison, when estimating the $m \times 1$ vector of unknowns $\boldsymbol{\xi}$ in the GMM we might like to have a redundancy of about m. However, for variance component estimation we probably would like to have roughly the square of m. It may even require a redundancy of over 100 to estimate as few as five variance components.

Unbiasedness matters   Now we return to our earlier question (Section 16.1) regarding the unbiasedness of the parameter estimates. Specifically, is $\hat{\boldsymbol{\xi}}$ still unbiased when we replace the "true" variance components σ_i^2, $i \in \{1, \dots, k\}$, with their reproBIQUUE estimates $\hat{\sigma}_i^2$?

Formally we can equate the vectors of quadratic products

$$\begin{bmatrix} \boldsymbol{y}^T \hat{W} Q_1 \hat{W} \boldsymbol{y} \\ \vdots \\ \boldsymbol{y}^T \hat{W} Q_k \hat{W} \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}^T \hat{W} Q_1 \hat{W} \boldsymbol{e} \\ \vdots \\ \boldsymbol{e}^T \hat{W} Q_k \hat{W} \boldsymbol{e} \end{bmatrix}, \qquad (16.23)$$

though we cannot actually compute the right side because of the unknown random error vector $\boldsymbol{e}$. However, let us assume that $\boldsymbol{e}$ is symmetrically distributed with $E\{\boldsymbol{e}\} = \boldsymbol{0}$. This assumption means that we have an equal chance of any element of $\boldsymbol{e}$ being positive or negative. So the terms $\hat{\sigma}_i^2$ do not change when $+\boldsymbol{e}$ is replaced by $-\boldsymbol{e}$, because we base our estimation on a quadratic form in $\boldsymbol{e}$. Formally we can write a difference between the estimate and the true parameter vector as follows:

$$\hat{\boldsymbol{\xi}} - \boldsymbol{\xi} = (A^T \hat{\Sigma}^{-1} A)^{-1} A^T \hat{\Sigma}^{-1} (\boldsymbol{y} - A\boldsymbol{\xi}) = \big[(A^T \hat{\Sigma}^{-1} A)^{-1} A^T \hat{\Sigma}^{-1}\big] \boldsymbol{e}. \qquad (16.24)$$

We see that the difference $\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}$ is linear in $\boldsymbol{e}$ and note that the term multiplying $\boldsymbol{e}$ will not change in sign when $\boldsymbol{e}$ does. Due to our symmetric distribution assumption, there is an equal chance of $\boldsymbol{e}$ being positive or negative; therefore there is also an equal chance of $\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}$ being positive or negative. As a formality, we also assume that $E\{\hat{\boldsymbol{\xi}}\}$ exists. And since $\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}$ changes sign whenever $\boldsymbol{e}$ does, this implies that

$$E\{\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\} = \boldsymbol{0} \Rightarrow E\{\hat{\boldsymbol{\xi}}\} = \boldsymbol{\xi}, \qquad (16.25)$$

which means that $\hat{\boldsymbol{\xi}}$ is uniformly unbiased under reproBIQUUE.

Rank deficient matrix A In the model (16.1), the coefficient matrix A is shown to have full column rank, implying that the matrix product $A^T \Sigma_0^{-1} A$ appearing in (16.16b) is invertible. But many problems in geodesy involve a rank-deficient system of equations. How is the matrix W_0 shown in (16.16b) computed in that case? Koch (1999, p. 230) answers this question tacitly by simply using the sign for a generalized inverse instead of the regular inverse, writing

$$W_0 := \Sigma_0^{-1} - \Sigma_0^{-1} A (A^T \Sigma_0^{-1} A)^- A^T \Sigma_0^{-1} \tag{16.26}$$

instead of (16.16b). For network adjustment problems with rank deficiencies, one could replace $(A^T \Sigma_0^{-1} A)^-$ with $(A^T \Sigma_0^{-1} A + K^T K)^{-1}$, where K is the datum matrix associated with RLESS, which was shown to be a generalized inverse for the normal-equation matrix N in Section 15.8.2. Regardless of the choice for the generalized inverse, the matrix W_0 is unique since the generalized inverse is multiplied on the left by A and on the right by A^T, a property stated already following (15.47).

16.3   Variance Component Estimation in the Gauss-Markov Model with Constraints

Suppose instead of the Gauss Markov Model (GMM) of (16.1) we must estimate variance components within the GMM with constraints (Chapter 5) or within the GMM with stochastic constraints (Chapter 6). It turns out that nothing different needs to be done for these problems for the reasons explained in the following.

Because BIQUUE is invariant with respect to a translation within the range space of the coefficient matrix A, we found out that the solutions to (16.16a) and (16.17) are identical. Therefore, we can say that the residual vector $\tilde{e}$ computed by $\tilde{e} = y - A\hat{\xi}$ for any $\hat{\xi}$ will result in the same estimates for the variance components, meaning that we may use the residual vector computed from any LESS in (16.17). Based on this conclusion, we can say that the variance components estimated within GMM for which the RLESS, partial MINOLESS, or MINOLESS is computed should be one and the same as those used within an overly constrained GMM for which LESS is computed. What is important to bear in mind is that for the overly constrained cases, the inverse of $A^T \Sigma_0^{-1} A$, or a generalized inverse for it, should still be used in the computation of W_0 (e.g, in (16.19)). Replacing it with the inverse of an extended normal equations matrix used to compute the parameter estimates within those models would not lead to the correct values for BIQUUE. An exception to this statement is discussed in the following section.

16.3.1   Variance Component Estimation When a Variance Component for Stochastic Constraints Must Also be Estimated

If variance components within the GMM with stochastic constraints (16.17) must be estimated, *including the one associated with the cofactor matrix for the con-*

straints Q_0, where $Q_0 = P_0^{-1}$ and P_0 is defined in (6.1c), then the following modifications should be made to (16.19) in accordance with Smith et al. (2018).

With the variables described in (6.1), define the following extended matrices and vectors:

$$A' := \underset{(n+l) \times m}{\begin{bmatrix} A \\ K \end{bmatrix}}, \quad \boldsymbol{y}' := \underset{(n+l) \times 1}{\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z}_0 \end{bmatrix}}, \quad \tilde{\boldsymbol{e}}' := \underset{(n+l) \times 1}{\begin{bmatrix} \tilde{\boldsymbol{e}} \\ \tilde{\boldsymbol{e}}_0 \end{bmatrix}}, \quad \text{and} \quad \Sigma'_0 := \underset{(n+l) \times (n+l)}{\begin{bmatrix} \Sigma_0 & 0 \\ 0 & \sigma_{0,0}^2 P_0^{-1} \end{bmatrix}}.$$

$$(16.27)$$

Let σ_0^2 represent the unknown variance component associated with $Q_0 = P_0^{-1}$ and let σ_1^2, σ_{12}, ..., σ_l^2 be associated with Σ_0 as defined in Section 16.2.1. Then we need only to replace the observation vector $\boldsymbol{y}$ in (16.19) with its extended form $\boldsymbol{y}'$, or with $\tilde{\boldsymbol{e}}'$, and modify W_0 as follows:

$$W_0 := \Sigma'^{-1}_0 - \Sigma'^{-1}_0 A'(A'^T \Sigma'^{-1}_0 A')^- A'^T \Sigma'^{-1}_0 \Rightarrow \qquad (16.28a)$$

$$W_0 = \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & \sigma_{0,0}^{-2} P_0 \end{bmatrix} -$$

$$- \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & \sigma_{0,0}^{-2} P_0 \end{bmatrix} \begin{bmatrix} A \\ K \end{bmatrix} (A^T \Sigma_0^{-1} A + \sigma_{0,0}^{-2} K^T P_0 K)^{-1} \begin{bmatrix} A^T & K^T \end{bmatrix} \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & \sigma_{0,0}^{-2} P_0 \end{bmatrix} =$$

$$= \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & \sigma_{0,0}^{-2} P_0 \end{bmatrix} - \left[\begin{array}{c|c} \Sigma_0^{-1} A (A^T \Sigma_0^{-1} A + \sigma_{0,0}^{-2} K^T P_0 K)^{-1} A^T \Sigma_0^{-1} & \\ \hline \sigma_{0,0}^{-2} P_0 K (A^T \Sigma_0^{-1} A + \sigma_{0,0}^{-2} K^T P_0 K)^{-1} A^T \Sigma_0^{-1} & \end{array} \right.$$

$$\left. \begin{array}{c|c} & \Sigma_0^{-1} A (A^T \Sigma_0^{-1} A + \sigma_{0,0}^{-2} K^T P_0 K)^{-1} \sigma_{0,0}^{-2} K^T P_0 \\ \hline & \sigma_{0,0}^{-4} P_0 K (A^T \Sigma_0^{-1} A + \sigma_{0,0}^{-2} K^T P_0 K)^{-1} K^T P_0 \end{array} \right]$$

$$(16.28b)$$

with

$$\Sigma_0 = \sigma_{1,0}^2 Q_1 + \sigma_{12,0} Q_2 + \cdots + \sigma_{l,0}^2 Q_k \quad \text{and} \quad k = l(l+1)/2. \qquad (16.28c)$$

Here the index k pertains to the last cofactor matrix associated with the observations. Thus, there are actually $k + 1$ components to estimate, including the one associated with the weight matrix P_0 for the parameter constraints. Also note that the trailing zeros in the subscripts of the variance components denote approximate values as usual.

Chapter 17

Prior Information

In this chapter we investigate the topic of prior information on the unknown parameters. More specifically, we decompose the parameter vector $\boldsymbol{\xi}$ into two parts, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, where we assume that prior information, in the form of *pseudo-observations* $\boldsymbol{b}_0$, is available only for $\boldsymbol{\xi}_1$. Furthermore, we associate a random error vector $\boldsymbol{e}_0$ with the prior information and assume that it is uncorrelated with the random error vector $\boldsymbol{e}$ associated with the observations $\boldsymbol{y}$, i.e., we assume that $C\{\boldsymbol{e}, \boldsymbol{e}_0\} = \boldsymbol{0}$. In practice, the prior information may come from a previous adjustment with its dispersion matrix provided by the covariance matrix of the estimated parameters from that adjustment. The data model with prior-information can be written as an extended Gauss-Markov Model (GMM) as follows:

$$\underset{n \times 1}{\boldsymbol{y}} = A_1 \boldsymbol{\xi}_1 + A_2 \boldsymbol{\xi}_2 + \boldsymbol{e}, \tag{17.1a}$$

$$\boldsymbol{\xi}_1 \in \mathbb{R}^{r \times 1}, \quad \boldsymbol{\xi}_2 \in \mathbb{R}^{(m-r) \times 1}, \quad \operatorname{rk} A_1 \leq r, \quad \operatorname{rk} A_2 = m - r \tag{17.1b}$$

$$\underset{r \times 1}{\boldsymbol{b}_0} = \boldsymbol{\xi}_1 + \boldsymbol{e}_0, \tag{17.1c}$$

$$\begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e}_0 \end{bmatrix} \sim (\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & Q_0 \end{bmatrix}) \tag{17.1d}$$

Here the full design matrix and parameter vector are denoted by

$$\underset{n \times m}{A} := \begin{bmatrix} A_1 & \big| & A_2 \end{bmatrix} \quad \text{and} \quad \underset{m \times 1}{\boldsymbol{\xi}} := \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}. \tag{17.2}$$

The variables in the model are described by

$\boldsymbol{y}$ is a given $n \times 1$ vector of observations.

$\boldsymbol{\xi}_1$ is an $r \times 1$ vector of unknown parameters.

$\boldsymbol{\xi}_2$ is an $(m - r) \times 1$ vector of unknown parameters.

A_1 is a given $n \times r$ coefficient (design) matrix.

A_2 is a given $n \times (m - r)$ coefficient (design) matrix.

$\boldsymbol{e}$ is an $n \times 1$ vector of unknown random errors associated with the observations $\boldsymbol{y}$.

$\boldsymbol{b}_0$ is a given $r \times 1$ vector of (random) pseudo-observations called prior information.

$\boldsymbol{e}_0$ is an $r \times 1$ vector of unknown random errors associated with the pseudo-observations $\boldsymbol{b}_0$.

P is a given $n \times n$ positive-definite weight matrix for the observations $\boldsymbol{y}$.

Q_0 is a given $r \times r$ positive-definite cofactor matrix for the pseudo-observations $\boldsymbol{b}_0$.

σ_0^2 is an unknown variance component.

Note that matrix A_2 is assumed to have full column rank, i.e., rk $A_2 = m - r$, while A_1 does not necessarily have full column rank. Typically, $\boldsymbol{b}_0$ is a vector of zeros due to linearization (though still a random vector). If $\boldsymbol{b}_0$ is not zero, then it contains the bias of the prior information with respect to the initial approximations for the parameters $\boldsymbol{\xi}$ (assuming linearization). Finally, we note that the model uses a single variance component σ_0^2, multiplying both cofactor matrices P^{-1} and $Q_0 =: P_0^{-1}$, where P and P_0 are called weight matrices.

17.1   Pseudo-observations

The extended GMM includes pseudo-observations, which are considered to be direct observations of the unknown parameters $\boldsymbol{\xi}_1$. Since the model uses only a single variance component, it indeed belongs to the class of Gauss-Markov models. The following expression summarizes the model in a more compact form than does (17.1):

$$
\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{b}_0 \end{bmatrix} \sim (\begin{bmatrix} A_1 & A_2 \\ I_r & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & P_0^{-1} \end{bmatrix}).
\tag{17.3}
$$

Because the model is a type of GMM, we can immediately write the LEast-Squares Solution (LESS) for the unknown parameters $\boldsymbol{\xi}$ and the associated dispersion matrix, in accordance with the *addition theory of normal equations*.

$$
\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \left(\begin{bmatrix} A_1^T & I_r \\ A_2^T & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P_0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ I_r & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} A_1^T P & P_0 \\ A_2^T P & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{b}_0 \end{bmatrix} =
$$

$$
=: \left[\begin{array}{c|c} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{array} \right]^{-1} \begin{bmatrix} \boldsymbol{c}_1 + P_0 \boldsymbol{b}_0 \\ \boldsymbol{c}_2 \end{bmatrix},
\tag{17.4a}
$$

$$D\left\{ \begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} \right\} = \sigma_0^2 \left[\begin{array}{c|c} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{array} \right]^{-1}, \tag{17.4b}$$

$$\text{with } N_{ij} := A_i^T P A_j, \ \ i, j \in \{1, 2\}. \tag{17.4c}$$

It is evident from the upper-left block of the dispersion matrix in (17.4b) that the magnitude of the variances of $\hat{\boldsymbol{\xi}}_1$ have been reduced due to the prior information on $\boldsymbol{\xi}_1$.

Now we want to find an equivalent estimator and dispersion matrix in terms of previous estimates made within a model without prior information. A solution of this form is more revealing of what is gained by adding the prior information to the model. For simplicity, we assume that the complete design matrix A has full column rank, though it does not have to in general. We start with the cofactor matrix $Q_{\hat{\boldsymbol{\xi}}}$ (inverted matrix on right side of (17.4b)), and rewrite it as follows:

$$Q_{\hat{\boldsymbol{\xi}}} := \sigma_0^{-2} D\left\{ \begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} \right\} = \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} + \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} = \tag{17.5a}$$

$$= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} + \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} \right) \left[\begin{array}{c|c} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{array} \right]^{-1} = \tag{17.5b}$$

$$= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} \left[\begin{array}{c|c} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{array} \right] \left[\begin{array}{c|c} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{array} \right]^{-1} -$$

$$- \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} \left[\begin{array}{c|c} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{array} \right]^{-1}. \tag{17.5c}$$

Now, introducing the first Schur compliment of the partitioned matrix of N as $S_1 := N_{11} - N_{12} N_{22}^{-1} N_{21}$, and then using the rules for inverting a partitioned matrix, we may write

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} = \left[\begin{array}{c|c} S_1^{-1} & -S_1^{-1} N_{12} N_{22}^{-1} \\ \hline -N_{22}^{-1} N_{21} S_1^{-1} & N_{22}^{-1} + N_{22}^{-1} N_{21} S_1^{-1} N_{12} N_{22}^{-1} \end{array} \right]. \tag{17.5d}$$

Now we continue manipulating the cofactor matrix from (17.5c) to result in

$$
Q_{\hat{\xi}} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} - \left[\begin{array}{c|c} S_1^{-1}P_0 & 0 \\ \hline -N_{22}^{-1}N_{21}S_1^{-1}P_0 & 0 \end{array} \right] \cdot
$$

$$
\cdot \left[\begin{array}{c|c} (S_1+P_0)^{-1} & -(S_1+P_0)^{-1}N_{12}N_{22}^{-1} \\ \hline -N_{22}^{-1}N_{21}(S_1+P_0)^{-1} & N_{22}^{-1} + N_{22}^{-1}N_{21}(S_1+P_0)^{-1}N_{12}N_{22}^{-1} \end{array} \right].
$$

$$(17.5e)$$

This result implies that the dispersion matrix of $\hat{\xi}$ can be written as

$$
D\left\{ \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} \right\} = \sigma_0^2 \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} -
$$

$$
- \sigma_0^2 \left[\begin{array}{c|c} S_1^{-1}P_0(S_1+P_0)^{-1} & -S_1^{-1}P_0(S_1+P_0)^{-1}N_{12}N_{22}^{-1} \\ \hline -N_{22}^{-1}N_{21}S_1^{-1}P_0(S_1+P_0)^{-1} & N_{22}^{-1}N_{21}S_1^{-1}P_0(S_1+P_0)^{-1}N_{12}N_{22}^{-1} \end{array} \right].
$$

$$(17.6)$$

Note that (17.6) is still symmetric since $S_1^{-1}P_0(S_1+P_0)^{-1}$ is symmetric as shown in the following:

$$
S_1^{-1}P_0(S_1+P_0)^{-1} =
$$
$$
= S_1^{-1}P_0\big[S_1(I_r+S_1^{-1}P_0)\big]^{-1} = S_1^{-1}P_0(I_r+S_1^{-1}P_0)^{-1}S_1^{-1} =
$$

(now applying (A.8a))

$$
= S_1^{-1}(I_r+P_0S_1^{-1})^{-1}P_0S_1^{-1} =
$$
$$
= \big[(I_r+P_0S_1^{-1})S_1\big]^{-1}P_0S_1^{-1} =
$$
$$
= (S_1+P_0)^{-1}P_0S_1^{-1}.
$$

An interesting observation from the dispersion $D\{\hat{\xi}\}$ given in (17.6) is that though prior information is only provided for ξ_1, we also gain an improvement in the dispersion of $\hat{\xi}_2$.

We are now ready to express the estimator for the parameter vector in terms of the estimator within the model that does not include prior information. For convenience, we write the latter using cup symbols. We then make use of the

cofactor matrix appearing in (17.6).

$$
\begin{bmatrix} \breve{\boldsymbol{\xi}}_1 \\ \breve{\boldsymbol{\xi}}_2 \end{bmatrix} := \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{bmatrix} =
$$

$$
= \left[\begin{array}{c|c} S_1^{-1} & -S_1^{-1} N_{12} N_{22}^{-1} \\ \hline -N_{22}^{-1} N_{21} S_1^{-1} & N_{22}^{-1} + N_{22}^{-1} N_{21} S_1^{-1} N_{12} N_{22}^{-1} \end{array} \right] \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{bmatrix} \Rightarrow
$$

(17.7a)

$$
\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} \breve{\boldsymbol{\xi}}_1 \\ \breve{\boldsymbol{\xi}}_2 \end{bmatrix} - \begin{bmatrix} -I_r \\ N_{22}^{-1} N_{21} \end{bmatrix} S_1^{-1} P_0 \big(S_1 + P_0 \big)^{-1} \big[-I_r, N_{12} N_{22}^{-1} \big] \cdot
$$

$$
\cdot \begin{bmatrix} \boldsymbol{c}_1 + P_0 \boldsymbol{b}_0 \\ \boldsymbol{c}_2 \end{bmatrix} + \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} \begin{bmatrix} P_0 \boldsymbol{b}_0 \\ \boldsymbol{0} \end{bmatrix}
$$

(17.7b)

Note that it is important not to ignore the prior information vector $\boldsymbol{b}_0$ even if it is numerically zero. This is because it is a random variable and thus its impact will not be zero in the dispersion matrix $D\{\hat{\boldsymbol{\xi}}\}$. Making use of (17.5d), and performing certain algebraic manipulations, we can further modify (17.7b) with the objective of reaching a vector of "parameter improvements."

$$
\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} - \begin{bmatrix} \breve{\boldsymbol{\xi}}_1 \\ \breve{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} -I_r \\ N_{22}^{-1} N_{21} \end{bmatrix} S_1^{-1} P_0 \big(I_r + S_1^{-1} P_0 \big)^{-1} S_1^{-1} \big[\boldsymbol{c}_1 + P_0 \boldsymbol{b}_0 - N_{12} N_{22}^{-1} \boldsymbol{c}_2 \big] +
$$

$$
+ \left(\begin{bmatrix} -I_r \\ N_{22}^{-1} N_{21} \end{bmatrix} S_1^{-1} \big[-I_r, N_{12} N_{22}^{-1} \big] + \begin{bmatrix} 0 & 0 \\ 0 & N_{22}^{-1} \end{bmatrix} \right) \begin{bmatrix} P_0 \boldsymbol{b}_0 \\ \boldsymbol{0} \end{bmatrix} =
$$

$$
= \begin{bmatrix} -I_r \\ N_{22}^{-1} N_{21} \end{bmatrix} S_1^{-1} \big(I_r + P_0 S_1^{-1} \big)^{-1} \big[-P_0 S_1^{-1} \big(N_{12} N_{22}^{-1} \boldsymbol{c}_2 - \boldsymbol{c}_1 \big) + P_0 S_1^{-1} P_0 \boldsymbol{b}_0 \big] +
$$

$$
+ \begin{bmatrix} -I_r \\ N_{22}^{-1} N_{21} \end{bmatrix} S_1^{-1} \left(\big[-I_r, N_{12} N_{22}^{-1} \big] \begin{bmatrix} P_0 \boldsymbol{b}_0 \\ \boldsymbol{0} \end{bmatrix} \right) =
$$

$$
= \begin{bmatrix} -I_r \\ N_{22}^{-1} N_{21} \end{bmatrix} S_1^{-1} \big(I_r + P_0 S_1^{-1} \big)^{-1} \big[-P_0 S_1^{-1} \big(N_{12} N_{22}^{-1} \boldsymbol{c}_2 - \boldsymbol{c}_1 \big) +
$$

$$
+ P_0 S_1^{-1} P_0 \boldsymbol{b}_0 - \big(I_r + P_0 S_1^{-1} \big) P_0 \boldsymbol{b}_0 \big] =
$$

$$= \begin{bmatrix} -I_r \\ N_{22}^{-1}N_{21} \end{bmatrix} S_1^{-1}\big(I_r + P_0 S_1^{-1}\big)^{-1}\big[P_0 S_1^{-1}\big(\boldsymbol{c}_1 - N_{12}N_{22}^{-1}\boldsymbol{c}_2\big) - P_0\boldsymbol{b}_0\big] =$$

$$= \begin{bmatrix} -I_r \\ N_{22}^{-1}N_{21} \end{bmatrix} S_1^{-1}\big(I_r + P_0 S_1^{-1}\big)^{-1}P_0\big(\breve{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0\big)$$

In summary, we can express the vector of parameter improvements as

$$\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} - \begin{bmatrix} \breve{\boldsymbol{\xi}}_1 \\ \breve{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} -I_r \\ N_{22}^{-1}N_{21} \end{bmatrix}\big(I_r + S_1^{-1}P_0\big)^{-1}S_1^{-1}P_0\big(\breve{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0\big) \Rightarrow \qquad (17.8\text{a})$$

$$\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} - \begin{bmatrix} \breve{\boldsymbol{\xi}}_1 \\ \breve{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} -I_r \\ N_{22}^{-1}N_{21} \end{bmatrix}\big(I_r + P_0^{-1}S_1\big)^{-1}\big(\breve{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0\big). \qquad (17.8\text{b})$$

Equation (17.8a) may be used if P_0 is not invertible, and equation (17.8b) may be used if P_0 is invertible. The vector $\breve{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0$ is the discrepancy vector between the prior information and what would have been estimated using the new data set without the prior information. Since the matrix $(I_r + P_0^{-1}S_1)$ has positive eigenvalues, multiplication by its inverse reduces the discrepancy vector.

If we had introduced a second variance component σ_1^2 associated with the new data set, this would only have had a second-order effect on the estimates and would have required the ratio σ_0^2/σ_1^2 in front of the prior information weight matrix P_0.

How does the prior information change the predicted random error (residual) vector $\tilde{\boldsymbol{e}}$? We want to express the change as an update to the residual vector $\breve{\boldsymbol{e}}$ and also as a function of $\breve{\boldsymbol{\xi}}$, which would be predicted, respectively, estimated within a GMM without prior information.

$$\tilde{\boldsymbol{e}} = \boldsymbol{y} - A_1\hat{\boldsymbol{\xi}}_1 - A_2\hat{\boldsymbol{\xi}}_2 =$$

$$= \big[\boldsymbol{y} - A_1\breve{\boldsymbol{\xi}}_1 - A_2\breve{\boldsymbol{\xi}}_2\big] - A_1\big(\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1\big) - A_2\big(\hat{\boldsymbol{\xi}}_2 - \breve{\boldsymbol{\xi}}_2\big) =$$

$$= \breve{\boldsymbol{e}} - A_1\big(\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1\big) - A_2\big(\hat{\boldsymbol{\xi}}_2 - \breve{\boldsymbol{\xi}}_2\big) =$$

$$= \breve{\boldsymbol{e}} + \big(A_1 - A_2 N_{22}^{-1}N_{21}\big)\big(I_r + P_0^{-1}S_1\big)^{-1}\big(\breve{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0\big) \qquad (17.9)$$

We note that the product $(I_r + P_0^{-1}S_1)^{-1}(\breve{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0)$ appears frequently in the above equations and so in practice it may be worth computing it once at the outset and then saving it for subsequent use.

17.2 Alternative Normal Equations

In this section we introduce an alternative system of normal equation to accommodate prior information. The resulting solution is identical to that presented in the preceding section, however this alternative form allows for a singular cofactor matrix Q_0.

The normal equations are written as

$$(N_{11} + P_0)\hat{\boldsymbol{\xi}}_1 + N_{12}\hat{\boldsymbol{\xi}}_2 = \boldsymbol{c}_1 + P_0\boldsymbol{b}_0, \qquad (17.10\text{a})$$

$$N_{21}\hat{\boldsymbol{\xi}}_1 + N_{22}\hat{\boldsymbol{\xi}}_2 = \boldsymbol{c}_2, \qquad (17.10\text{b})$$

$$\hat{\lambda} = P_0(\hat{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0). \qquad (17.10\text{c})$$

The preceding three equations can be combined in matrix form as follows:

$$\begin{bmatrix} N_{11} & N_{12} & I_r \\ N_{21} & N_{22} & 0 \\ I_r & 0 & -P_0^{-1} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{b}_0 \end{bmatrix}. \qquad (17.11)$$

Here, $\hat{\lambda}$ is an $r \times 1$ vector of estimated Lagrange multipliers. The *normal-equations matrix* on the left side of (17.11) is of size $(m + r) \times (m + r)$. We could use the Cholesky algorithm to reduce the upper 2×2 sub-matrix block and then proceed with Gaussian elimination.

The inverse of the normal-equations matrix yields the cofactor matrix of the estimates. However, we only need to concern ourselves with the upper 2×2 sub-matrix block of the inverse in order to find the dispersion of the parameter estimates $\hat{\boldsymbol{\xi}}_1$ and $\hat{\boldsymbol{\xi}}_2$. In the equation that follows, the other terms of no special interest have been replaced with the symbol X.

$$\begin{bmatrix} N_{11} & N_{12} & I_r \\ N_{21} & N_{22} & 0 \\ I_r & 0 & -P_0^{-1} \end{bmatrix}^{-1} = \left[\begin{array}{c|c} \left(\begin{bmatrix} N_{11} & N_{11} \\ N_{11} & N_{11} \end{bmatrix} + \begin{bmatrix} I_r \\ 0 \end{bmatrix} P_0 \begin{bmatrix} I_r & 0 \end{bmatrix} \right)^{-1} & X \\ \hline X & X \end{array} \right] =$$

$$= \left[\begin{array}{c|c} \sigma_0^{-2} D\{\hat{\boldsymbol{\xi}}\} & X \\ \hline X & X \end{array} \right] = \left[\begin{array}{c|c} \begin{bmatrix} N_{11} + P_0 & N_{11} \\ N_{11} & N_{11} \end{bmatrix}^{-1} & X \\ \hline X & X \end{array} \right] \qquad (17.12)$$

It is interesting to investigate the consequences of diminishing the weight of the prior information. Suppose the prior information weight matrix is defined as $P_0 :=$

εP_0 and we have the situation where $\varepsilon \to 0$. This means that the prior information looses its influence, resulting in $\hat{\boldsymbol{\xi}}_1 \to \breve{\boldsymbol{\xi}}_1$ with degrees of freedom (redundancy) $n +$ $r - m$. However, if $\varepsilon = 0$ then we have $\hat{\boldsymbol{\xi}}_1 = \breve{\boldsymbol{\xi}}_1$ with degree of freedom $n - m$. In other words, as ε approaches zero, the resulting estimate numerically approaches what would be obtained if prior information were not included in the model. However, the degrees of freedom of the model with prior information is larger than that of the model without prior information by a constant r, which is the number of parameters that we supposedly have prior information for. This has an unsatisfactory result on our estimated variance component $\hat{\sigma}_0^2$; it makes it look better than what it is. We might rather specify redundancy as a function of ε, but exactly how best to do that is still an open question.

Suppose we are given values for $\breve{\boldsymbol{\xi}}_1$ and $\breve{\boldsymbol{\xi}}_2$, together with the prior information $\boldsymbol{b}_0$ and associated weights P_0, and suppose we want to find the solution for $\hat{\boldsymbol{\xi}}_1$ and $\hat{\boldsymbol{\xi}}_2$. From row 1 of (17.8b) we can solve

$$(P_0 + S_1)(\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1) = P_0(\boldsymbol{b}_0 - \breve{\boldsymbol{\xi}}_1). \tag{17.13}$$

Then, by substitution of the first row of (7.13) into the second row, we can write

$$\hat{\boldsymbol{\xi}}_2 - \breve{\boldsymbol{\xi}}_2 = -N_{22}^{-1} N_{21}(\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1). \tag{17.14}$$

The update for the dispersion is then given by

$$D\{\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix}\} - D\{\begin{bmatrix} \breve{\boldsymbol{\xi}}_1 \\ \breve{\boldsymbol{\xi}}_2 \end{bmatrix}\} = -\sigma_0^2 \begin{bmatrix} N_{11} + P_0 & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1}, \tag{17.15}$$

which already was evident from (17.5c).

Suppose we are given only the prior information $\boldsymbol{b}_0$ and we would like to find the solutions for $\hat{\boldsymbol{\xi}}_1$ and $\hat{\boldsymbol{\xi}}_2$. The solution is developed by starting with (17.4a) and using the relationship $Q_0 = P_0^{-1}$ as follows:

$$\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{bmatrix}^{-1} \left(\begin{bmatrix} \boldsymbol{c}_1 - N_{11}\boldsymbol{b}_0 \\ \boldsymbol{c}_2 - N_{21}\boldsymbol{b}_0 \end{bmatrix} + \begin{bmatrix} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_0 \\ \boldsymbol{0} \end{bmatrix} \right),$$

which implies that

$$\begin{bmatrix} \hat{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} N_{11} + P_0 & N_{12} \\ \hline N_{21} & N_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_1^T P(\boldsymbol{y} - A_1\boldsymbol{b}_0) \\ A_2^T P(\boldsymbol{y} - A_1\boldsymbol{b}_0) \end{bmatrix} =$$

$$= \left(\left[\begin{array}{c|c} I_r + N_{11}Q_0 & N_{12} \\ \hline N_{21}Q_0 & N_{22} \end{array} \right] \left[\begin{array}{cc} P_0 & 0 \\ 0 & I_{m-r} \end{array} \right] \right)^{-1} \left[\begin{array}{c} A_1^T P(\boldsymbol{y} - A_1\boldsymbol{b}_0) \\ A_2^T P(\boldsymbol{y} - A_1\boldsymbol{b}_0) \end{array} \right] =$$

$$\left[\begin{array}{c} \hat{\boldsymbol{\xi}}_1 - \boldsymbol{b}_0 \\ \hat{\boldsymbol{\xi}}_2 \end{array} \right] = \left[\begin{array}{cc} Q_0 & 0 \\ 0 & I_{m-r} \end{array} \right] \left[\begin{array}{c|c} I_r + N_{11}Q_0 & N_{12} \\ \hline N_{21}Q_0 & N_{22} \end{array} \right]^{-1} \left[\begin{array}{c} A_1^T P(\boldsymbol{y} - A_1\boldsymbol{b}_0) \\ A_2^T P(\boldsymbol{y} - A_1\boldsymbol{b}_0) \end{array} \right]. \quad (17.16)$$

The first matrix on the right side of (17.16) is singular if Q_0 is singular, but this is of no consequence since we do not need to invert it. The second matrix on the right side is regular (non-singular) even if Q_0 is singular. The dispersion is given by

$$D\left\{ \left[\begin{array}{c} \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{array} \right] \right\} = \sigma_0^2 \left[\begin{array}{cc} Q_0 & 0 \\ 0 & I_{m-r} \end{array} \right] \left[\begin{array}{c|c} I_r + N_{11}Q_0 & N_{12} \\ \hline N_{21}Q_0 & N_{22} \end{array} \right]^{-1}. \quad (17.17)$$

In order to confirm the consistency between the current data and the prior information, we can test the validity of the null hypothesis

$$H_0 : E\{\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1\} = \boldsymbol{0}. \quad (17.18)$$

The test statistic T is defined as

$$T := \frac{(\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1)^T \cdot D\{\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1\}^{-1} \cdot (\hat{\boldsymbol{\xi}}_1 - \breve{\boldsymbol{\xi}}_1)}{r(\hat{\sigma}_0^2/\sigma_0^2)} \sim F(r, n - m). \quad (17.19)$$

17.3 Mixed Linear Model (Helmert's Knack)

The idea underlying the mixed linear model is that some of the parameters are random, while others are fixed. This is different than all models presented up to this point, where we have consistently defined the unknown parameters to be fixed (non-random). Here we introduce a non-random analogue to the vector of prior information $\boldsymbol{b}_0$, denoted by $\boldsymbol{\beta}_0$. Numerically, $\boldsymbol{b}_0$ and $\boldsymbol{\beta}_0$ are equivalent, but stochastically their equivalence is obtained only by addition of a *random zero-vector*, denoted by $\underaccent{\sim}{\boldsymbol{0}}$, as follows:

$$\boldsymbol{b}_0 = \boldsymbol{\beta}_0 + \underaccent{\sim}{\boldsymbol{0}} = \boldsymbol{\xi}_1 + \boldsymbol{e}_0, \quad \boldsymbol{e}_0 \sim (\boldsymbol{0}, \sigma_0^2 P_0^{-1} = \sigma_0^2 Q_0), \quad (17.20\text{a})$$

$$\boldsymbol{\beta}_0 = (\boldsymbol{\xi}_1 - \underaccent{\sim}{\boldsymbol{0}}) + \boldsymbol{e}_0 = \boldsymbol{x}_1 + \boldsymbol{e}_0, \quad \text{where } \boldsymbol{x}_1 := \boldsymbol{\xi}_1 - \underaccent{\sim}{\boldsymbol{0}}. \quad (17.20\text{b})$$

Equation (17.20b) is known as Helmert's knack. It is used to transform the non-random parameter vector $\boldsymbol{\xi}_1$ to a random parameter vector $\boldsymbol{x}_1$. Some explanation about the notation might be helpful. As usual, we use Greek letters for non-random variables and Latin letters for random variables. In this case we have also placed

a tilde beneath the zero to denote a random vector of zeros associated with the unknown parameters. The expectation and dispersion of the unknown, random parameters $\boldsymbol{x}_1$ are

$$E\{\boldsymbol{x}_1\} = E\{\boldsymbol{\beta}_0 - \boldsymbol{e}_0\} = \boldsymbol{\beta}_0 - E\{\boldsymbol{e}_0\} = \boldsymbol{\beta}_0, \tag{17.21a}$$

$$D\{\boldsymbol{x}_1\} = D\{\boldsymbol{\beta}_0 - \boldsymbol{e}_0\} = D\{\boldsymbol{e}_0\} = \sigma_0^2 Q_0. \tag{17.21b}$$

Since we have used the random vector $\underset{\sim}{\boldsymbol{0}}$ in the pseudo-observations, we need to modify the original observation equations given in (17.1a) by subtracting $A_1\underset{\sim}{\boldsymbol{0}}$ from both sides of the equation. This does not change the numerical values on the left side, but it does make it a different vector in terms of its stochastic properties. We denote the revised left-side vector as $\bar{\boldsymbol{y}}$.

$$\bar{\boldsymbol{y}} = \boldsymbol{y} - A_1\underset{\sim}{\boldsymbol{0}} =$$
$$= A_1\boldsymbol{\xi}_1 - A_1\underset{\sim}{\boldsymbol{0}} + A_2\boldsymbol{\xi}_2 + \boldsymbol{e} =$$
$$= A_1\big(\boldsymbol{\xi}_1 - \underset{\sim}{\boldsymbol{0}}\big) + A_2\boldsymbol{\xi}_2 + \boldsymbol{e} \Rightarrow$$
$$\bar{\boldsymbol{y}} = A_1\boldsymbol{x}_1 + A_2\boldsymbol{\xi}_2 + \boldsymbol{e} \tag{17.22}$$

Again we note that $\bar{\boldsymbol{y}}$ contains the same numerical values as $\boldsymbol{y}$, but now with dispersion matrix

$$D\{\bar{\boldsymbol{y}}\} = \sigma_0^2(A_1 Q_0 A_1^T + P^{-1}). \tag{17.23}$$

On the right side of (17.22), we have a random parameter-vector $\boldsymbol{x}_1$ and a non-random parameter-vector $\boldsymbol{\xi}_2$; the equation is linear in these unknowns. This is why we call the model a *mixed* linear model (MLM); it has a mix of fixed and random unknown parameters. We summarize the MLM in the box below.

$$\bar{\boldsymbol{y}} := \big(\boldsymbol{y} - A_1\underset{\sim}{\boldsymbol{0}}\big) = A_1\boldsymbol{x}_1 + A_2\boldsymbol{\xi}_2 + \boldsymbol{e}, \tag{17.24a}$$

$$\boldsymbol{x}_1 = \boldsymbol{\beta}_0 - \boldsymbol{e}_0, \quad \boldsymbol{x}_1 \sim (\boldsymbol{\beta}_0, \sigma_0^2 Q_0), \quad \mathrm{rk}\, A_2 = m - r, \tag{17.24b}$$

$$\begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e}_0 \end{bmatrix} \sim \big(\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & Q_0 \end{bmatrix}\big). \tag{17.24c}$$

In going from the extended GMM (17.1) to the MLM (17.24), we have changed from a model that has no a-priori information about the *non-random* parameters $\boldsymbol{\xi}_1$ to a model that has a-priori information about the *random* parameters $\boldsymbol{x}_1$. In either case, we know nothing a priori about the parameters in $\boldsymbol{\xi}_2$. We claim that the MLM is more flexible, in general, than the extended GMM. The following discussion supports this claim.

In the extended GMM, the class of linear estimators is represented by

$$\big[\hat{\boldsymbol{\xi}}_1^T,\ \hat{\boldsymbol{\xi}}_2^T\big]^T = L_1\boldsymbol{y} + L_2\boldsymbol{b}_0 + \boldsymbol{\gamma}_0, \tag{17.25}$$

where L_1 and L_2 are unknown matrices. In contrast, in the MLM the class of linear predictors/estimators is represented by

$$\left[\tilde{\boldsymbol{x}}_1^T,\ \hat{\boldsymbol{\xi}}_2^T\right]^T = L\bar{\boldsymbol{y}} + \boldsymbol{\gamma}, \tag{17.26}$$

where L is unknown and the vector $\boldsymbol{\beta}_0$, that $\boldsymbol{x}_1$ depends on, could be nonlinear. So we see that the linear class is larger for the MLM than for the extended GMM, which makes the MLM more flexible. However, it might be that the optimal estimate found in the MLM could also be found in the extended GMM; it depends on the linearity of $\boldsymbol{\beta}_0$.

Moritz (1970, 1972) used the MLM to introduce least-squares collocation. The collocation solution was linear for both $\bar{\boldsymbol{y}}$ and $\boldsymbol{\beta}_0$; so it could be described by the extended GMM. Schaffrin prefers the MLM to the extended GMM because it permits nonlinear forms of $\boldsymbol{\beta}_0$.
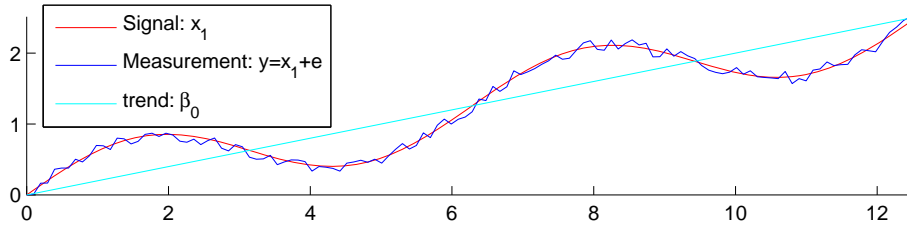
Figure 17.1: Sinusoidal signal containing random noise and linear trend

We now list some practical examples for the use of the MLM.

Example 1:

A typical application comes from signal theory. Here we are interested in a signal $\boldsymbol{x}_1$, which may include a linear or nonlinear component $\boldsymbol{\beta}_0$. Assuming no fixed parameters $\hat{\boldsymbol{\xi}}_2$ and $A_1 = I$, the observation equations become $\boldsymbol{y} = \boldsymbol{x}_1 + \boldsymbol{e}$. Figure 17.1 illustrates this example.

Example 2:

The MLM can be applied to deformation analysis, for example the monitoring of bridges or dams. In this case, we may have a-priori information about how we believe the structure should deform under load, but we are most interested in the actual deformation; that is the random signal to determine. We observe the signal plus noise; we must remove the noise (i.e., extract the signal from the noise).

Example 3:

Moritz applied the MLM to the problem of determining the gravity field of the earth. Here, the normal gravity field is the prior information, and the random signal to determine is is the earth's anomalous gravity field.

Moritz (1972) provides other examples of least-squares collocation in *Reports of the Department of Geodetic Science, Number 175*.

17.4   Solutions for the Mixed Linear Model

To obtain solutions for the unknown parameters of the MLM, we start by deriving the BLUUE for the non-random parameters $\boldsymbol{\xi}_2$. Substituting the equation for $\boldsymbol{x}_1$ from (17.24b) into the observation equation of (17.24) allows us to write the Mixed Linear Model (MLM) in an alternative form as

$$\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 = A_2\boldsymbol{\xi}_2 + (\boldsymbol{e} - A_1\boldsymbol{e}_0), \tag{17.27a}$$

$$(\boldsymbol{e} - A_1\boldsymbol{e}_0) \sim (\boldsymbol{0}, \sigma_0^2\big[P^{-1} + A_1Q_0A_1^T\big]). \tag{17.27b}$$

The MLM in (17.27) appears in the form of a GMM. The left side of (17.27a) is known and so are the characteristics of the combined error vector $\boldsymbol{e} - A_1\boldsymbol{e}_0$ on the right side. So, we can estimate $\boldsymbol{\xi}_2$ using least-squares principles via the following formula:

$$\hat{\boldsymbol{\xi}}_2 = \big[A_2^T(P^{-1} + A_1Q_0A_1^T)^{-1}A_2\big]^{-1}A_2^T(P^{-1} + A_1Q_0A_1^T)^{-1}(\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0). \tag{17.28}$$

The first inverted matrix in (17.28) is the cofactor matrix for $\hat{\boldsymbol{\xi}}_2$ so that the dispersion matrix of $\hat{\boldsymbol{\xi}}_2$ is given by

$$D\{\hat{\boldsymbol{\xi}}_2\} = \sigma_0^2\big[A_2^T(P^{-1} + A_1Q_0A_1^T)^{-1}A_2\big]^{-1}. \tag{17.29}$$

An alternative form of the dispersion matrix is obtained as follows: By use of (A.6a) we obtain

$$(P^{-1} + A_1Q_0A_1^T)^{-1} = P - PA_1(Q_0^{-1} + A_1^TPA_1)^{-1}A_1^TP, \tag{17.30a}$$

with

$$(Q_0^{-1} + A_1^TPA_1)^{-1} = (I_r + Q_0A_1^TPA_1)^{-1}Q_0. \tag{17.30b}$$

Upon substitution of (17.30a), together with (17.30b), into the dispersion formula (17.29) we get

$$D\{\hat{\boldsymbol{\xi}}_2\} = \sigma_0^2\big\{A_2^T\big[P - PA_1(I_r + Q_0A_1^TPA_1)^{-1}Q_0A_1^TP\big]A_2\big\}^{-1} =$$

$$= \sigma_0^2\big[N_{22} - N_{21}(I_r + Q_0N_{11})^{-1}Q_0N_{12}\big]^{-1} = \sigma_0^2\big[N_{22} - N_{21}Q_0^{-1} + N_{11})^{-1}N_{12}\big]^{-1} =$$

$$= \sigma_0^2N_{22}^{-1} + \sigma_0^2N_{22}^{-1}N_{21}\big[(Q_0^{-1} + N_{11})N_{12}N_{22}^{-1}N_{21}\big]^{-1}N_{12}N_{22}^{-1},$$

or

$$D\{\hat{\boldsymbol{\xi}}_2\} = \sigma_0^2N_{22}^{-1} + \sigma_0^2N_{22}^{-1}N_{21}(I_r + Q_0S_1)^{-1}Q_0N_{12}N_{22}^{-1}, \tag{17.31}$$

where

$$S_1 := N_{11} - N_{12}N_{22}^{-1}N_{21}. \tag{17.32}$$

Also, we have used the familiar relations

$$N_{ij} := A_i^T P A_j \text{ and } \bar{c}_i := A_i^T P \bar{y}, \tag{17.33}$$

where the symbol $\bar{c}_i$ is used below. To reach an alternative expression for ξ_2, we use (17.30a) through (17.31) to modify (17.28) as follows:

$$\hat{\xi}_2 = \left[N_{22}^{-1} + N_{22}^{-1} N_{21} (I_r + Q_0 S_1)^{-1} Q_0 N_{12} N_{22}^{-1} \right] \cdot \\ \cdot \left[A_2^T P - N_{21} (I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P \right] (\bar{y} - A_1 \beta_0). \tag{17.34}$$

For convenience, and for future reference, we also write

$$\hat{\xi}_2 = G_2 (\bar{y} - A_1 \beta_0), \tag{17.35a}$$

with

$$G_2 := \left[N_{22}^{-1} + N_{22}^{-1} N_{21} (I_r + Q_0 S_1)^{-1} Q_0 N_{12} N_{22}^{-1} \right] \cdot \\ \cdot \left[A_2^T P - N_{21} (I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P \right]. \tag{17.35b}$$

Expanding (17.35a) leads to

$$\hat{\xi}_2 = N_{22}^{-1} (\bar{c}_2 - N_{21} \beta_0) - \underline{N_{22}^{-1} N_{21}} \underline{(I_r + Q_0 N_{11})^{-1} Q_0 (\bar{c}_1 - N_{11} \beta_0)} + \\ + N_{22}^{-1} N_{21} (I_r + Q_0 S_1)^{-1} Q_0 N_{12} N_{22}^{-1} (\bar{c}_2 - N_{21} \beta_0) - \\ - \underline{N_{22}^{-1} N_{21}} (I_r + Q_0 S_1)^{-1} Q_0 N_{12} N_{22}^{-1} N_{21} \underline{\underline{(I_r + Q_0 N_{11})^{-1} Q_0 (\bar{c}_1 - N_{11} \beta_0)}}. \tag{17.36}$$

The single and double underlines in the second and fourth lines of the above equation are used to highlight similar terms. We may insert the identity matrix

$$(I_r + Q_0 S_1)^{-1} (I_r + Q_0 N_{11} - Q_0 N_{12} N_{22}^{-1} N_{21}) = I_r \tag{17.37}$$

between the underlined terms in the second line, which, after some algebraic manipulation, leads to

$$\boxed{\hat{\xi}_2 = N_{22}^{-1} (\bar{c}_2 - N_{21} \beta_0) - N_{22}^{-1} N_{21} (I_r + Q_0 S_1)^{-1} Q_0 (\bar{c}_1 - N_{12} N_{22}^{-1} \bar{c}_2 - S_1 \beta_0).} \tag{17.38}$$

After further algebraic manipulation, we can also write

$$\boxed{\hat{\xi}_2 = N_{22}^{-1} \bar{c}_2 - N_{22}^{-1} N_{21} (I_r + Q_0 S_1)^{-1} \left[Q_0 (\bar{c}_1 - N_{12} N_{22}^{-1} \bar{c}_2) + \beta_0 \right].} \tag{17.39}$$

In summary, we began with equation (17.27), which has the form of the GMM, and we applied least-squares criteria to reach a solution for ξ_2. We know that LESS within the (full-rank) GMM is equivalent to BLUUE. So, we claim that the various expressions of $\hat{\xi}_2$ above, beginning with (17.28), give the BLUUE within the mixed linear model for the non-random (fixed) parameter vector ξ_2.

From (17.27) we see that our solution will only lead to a prediction for $e - A_1 e_0$. But what we need a prediction for e_0 so that we can predict x_1. We can arrive at

LESS for $\tilde{\boldsymbol{e}}_0$ based on the following *Model of Condition Equations with Parameters* (see first set of Adjustment Notes):

$$\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 = A_2\boldsymbol{\xi}_2 + \begin{bmatrix} I_n, & -A_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e}_0 \end{bmatrix}, \tag{17.40a}$$

$$\begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e}_0 \end{bmatrix} \sim (\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & Q_0 \end{bmatrix}). \tag{17.40b}$$

This model leads to the following solution for the predicted random errors:

$$\begin{bmatrix} \tilde{\boldsymbol{e}} \\ \tilde{\boldsymbol{e}}_0 \end{bmatrix} = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q_0 \end{bmatrix} \begin{bmatrix} I_n \\ -A_1^T \end{bmatrix} \left(\begin{bmatrix} I_n, & -A_1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q_0 \end{bmatrix} \begin{bmatrix} I_n \\ -A_1^T \end{bmatrix} \right)^{-1} \cdot$$

$$\cdot (\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2) =$$

$$= \begin{bmatrix} P^{-1} \\ -Q_0 A_1^T \end{bmatrix} (P^{-1} + A_1 Q_0 A_1^T)^{-1} (\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2) \Rightarrow$$

$$\begin{bmatrix} \tilde{\boldsymbol{e}} \\ \tilde{\boldsymbol{e}}_0 \end{bmatrix} = \begin{bmatrix} P^{-1} \\ -Q_0 A_1^T \end{bmatrix} \begin{bmatrix} P - PA_1(I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P \end{bmatrix} (\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2). \tag{17.41}$$

The second row of (17.41) provides the following formula for $\tilde{\boldsymbol{e}}_0$:

$$\tilde{\boldsymbol{e}}_0 = -Q_0(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2) + (Q_0 N_{11} + I_r - I_r)(I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P \cdot$$

$$\cdot (\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2) =$$

$$= -Q_0(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2) + Q_0(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2) -$$

$$- (I_r + Q_0 N_{11})^{-1} Q_0(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2) =$$

$$= -(I_r + Q_0 N_{11})^{-1} Q_0(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2) \Rightarrow$$

$$\tilde{\boldsymbol{e}}_0 = -Q_0(I_r + N_{11}Q_0)^{-1}(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2). \tag{17.42}$$

By comparing the first and second rows of (17.41), we immediately see $\tilde{\boldsymbol{e}}_0$ as a function of $\tilde{\boldsymbol{e}}$:

$$\tilde{\boldsymbol{e}}_0 = -Q_0 A_1^T P \tilde{\boldsymbol{e}}. \tag{17.43}$$

Now, it is obvious from the MLM that we have $\tilde{\boldsymbol{x}}_1 = \boldsymbol{\beta}_0 - \tilde{\boldsymbol{e}}_0$, which upon substitution of (17.42) yields

$$\boxed{\tilde{\boldsymbol{x}}_1 = \boldsymbol{\beta}_0 + Q_0(I_r + N_{11}Q_0)^{-1}(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2),} \tag{17.44}$$

or, alternatively,

$$\tilde{\boldsymbol{x}}_1 = (I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P(\bar{\boldsymbol{y}}_1 - A_2 \hat{\boldsymbol{\xi}}_2) + \big[I_r - (I_r + Q_0 N_{11})^{-1} Q_0 N_{11}\big] \boldsymbol{\beta}_0 = \tag{17.45a}$$

$$= (I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P(\bar{\boldsymbol{y}}_1 - A_2 \hat{\boldsymbol{\xi}}_2) + (I_r + Q_0 N_{11})^{-1} \boldsymbol{\beta}_0. \tag{17.45b}$$

Here we used the general relationship $(I + A)^{-1} = I - (I + A)^{-1} A$ in the last step to reach (17.45b). Note that we have arrived at the prediction $\tilde{\boldsymbol{x}}_1$ strictly by least-squares principles. However, in this model we have the equivalence of LESS to the inhomBLIP (Best inhomogeneous Linear Predictor). The idea behind inhomBLIP is given in the following section.

17.5   Best Inhomogeneous Linear Predictor

The idea behind inhomBLIP is that in the class of linear predictors

$$\big\{ L(\bar{\boldsymbol{y}} - A_2 \hat{\boldsymbol{\xi}}_2) + \boldsymbol{\gamma} \mid L \text{ is an } r \times n \text{ matrix}, \boldsymbol{\gamma} \text{ is an } r \times 1 \text{ vector} \big\} \tag{17.46a}$$

the predictor

$$\tilde{\boldsymbol{x}}_1 = (I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P(\bar{\boldsymbol{y}}_1 - A_2 \hat{\boldsymbol{\xi}}_2) + (I_r + Q_0 N_{11})^{-1} \boldsymbol{\beta}_0 =$$
$$= L_1(\bar{\boldsymbol{y}}_1 - A_2 \hat{\boldsymbol{\xi}}_2) + \boldsymbol{\gamma}_1 \tag{17.46b}$$

has minimum mean square prediction error (MSPE). That is,

$$\boxed{\begin{aligned} \operatorname{tr} \operatorname{MSPE}\{\tilde{\boldsymbol{x}}_1\} = \operatorname{tr} E\{(\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1)(\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1)^T\} = \\ = \operatorname{tr} E\big\{ \big[L_1(\bar{\boldsymbol{y}}_1 - A_2 \hat{\boldsymbol{\xi}}_2) + \boldsymbol{\gamma}_1 - \boldsymbol{x}_1\big]\big[L_1(\bar{\boldsymbol{y}}_1 - A_2 \hat{\boldsymbol{\xi}}_2) + \boldsymbol{\gamma}_1 - \boldsymbol{x}_1\big]^T\big\} = \min_{L_1, \boldsymbol{\gamma}_1} \end{aligned}}$$

$$\tag{17.46c}$$

The variables L_1 and $\boldsymbol{\gamma}_1$ are defined as follows:

$$L_1 := (I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P = Q_0 A_1^T (P^{-1} + A_1 Q_0 A_1^T)^{-1}, \tag{17.47a}$$

$$\boldsymbol{\gamma}_1 := (I_r + Q_0 N_{11})^{-1} \boldsymbol{\beta}_0 = \boldsymbol{\beta}_0 - L_1 A_1 \boldsymbol{\beta}_0. \tag{17.47b}$$

The minimization of (17.46c) is not developed further here. However, we do note that the predictor $\tilde{\boldsymbol{x}}_1$ is automatically weakly unbiased in the sense that

$$E\{\tilde{\boldsymbol{x}}_1\} = (I_r + Q_0 N_{11})^{-1}[Q_0 A_1^T P \cdot E\{\bar{\boldsymbol{y}}_1 - A_2 \hat{\boldsymbol{\xi}}_2\} + E\{\boldsymbol{\beta}_0\}] = \tag{17.48a}$$

$$= (I_r + Q_0 N_{11})^{-1}[Q_0 A_1^T P A_1 \boldsymbol{\beta}_0 + \boldsymbol{\beta}_0] = \boldsymbol{\beta}_0 \tag{17.48b}$$

for the given vector $\boldsymbol{\beta}_0$. Note that (17.48b) does not necessarily hold for any arbitrary vector $\boldsymbol{\beta}_0$, but rather for the given $\boldsymbol{\beta}_0$, hence the term *weakly unbiased*.

Let us now consider in detail the mean-square prediction error MSPE of $\{\tilde{\boldsymbol{x}}_1\}$. Because of unbiasedness, we can write

$$\text{MSPE}\{\tilde{\boldsymbol{x}}_1\} = D\{\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1\}. \tag{17.49a}$$

Also, because the vector differences $\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1$ and $\boldsymbol{e}_0 - \tilde{\boldsymbol{e}}_0$ only differ by $\boldsymbol{\beta}_0$, we have

$$\text{MSPE}\{\tilde{\boldsymbol{x}}_1\} = D\{\boldsymbol{e}_0 - \tilde{\boldsymbol{e}}_0\} = D\{\boldsymbol{e}_0\} - C\{\boldsymbol{e}_0, \tilde{\boldsymbol{e}}_0\} - C\{\tilde{\boldsymbol{e}}_0, \boldsymbol{e}_0\} + D\{\tilde{\boldsymbol{e}}_0\}. \tag{17.49b}$$

Let us compute the last four terms of (17.49b) individually.

$$D\{\boldsymbol{e}_0\} = \sigma_0^2 Q_0 \tag{17.50a}$$

In computing $D\{\tilde{\boldsymbol{e}}_0\}$, we first write the dispersion for the term $\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2$. This term, as we have already seen, is equivalent to the prediction $\boldsymbol{e} - \widetilde{A_1\boldsymbol{e}_0}$. Also, equation (17.40b) implies no covariance between $\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0$ and $A_2\hat{\boldsymbol{\xi}}_2$, i.e.

$$D\{\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2\} = D\{\boldsymbol{e} - \widetilde{A_1\boldsymbol{e}_0}\} = D\{\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0\} - D\{A_2\hat{\boldsymbol{\xi}}_2\}. \tag{17.50b}$$

Now making use of (17.42), we can write

$$D\{\tilde{\boldsymbol{e}}_0\} = Q_0(I_r + N_{11}Q_0)^{-1}A_1^T P D\{\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2\} P A_1 Q_0(I_r + N_{11}Q_0)^{-1}. \tag{17.50c}$$

For the covariance terms, we have $C\{\boldsymbol{e}_0, \tilde{\boldsymbol{e}}_0\} = C\{\tilde{\boldsymbol{e}}_0, \boldsymbol{e}_0\}^T$, and with the help of (17.35b), we write the covariance $C\{\boldsymbol{e}_0, \tilde{\boldsymbol{e}}_0\}$ as follows:

$$C\{\boldsymbol{e}_0, \tilde{\boldsymbol{e}}_0\} = Q_0(I_r + N_{11}Q_0)^{-1}A_1 P \cdot C\{\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2, \boldsymbol{e}_0\} =$$

$$= Q_0(I_r + N_{11}Q_0)^{-1}A_1 P \cdot C\{(I_n - A_2 G_2)(\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0), \boldsymbol{e}_0\} =$$

$$= Q_0(I_r + N_{11}Q_0)^{-1}A_1 P(I_n - A_2 G_2) \cdot C\{(A_2\boldsymbol{\xi}_2 + \boldsymbol{e} - A_1\boldsymbol{e}_0), \boldsymbol{e}_0\} =$$

$$= -\left[Q_0(I_r + N_{11}Q_0)^{-1}A_1 P\right](I_n - A_2 G_2)A_1(\sigma_0^2 Q_0) = C\{\tilde{\boldsymbol{e}}_0, \boldsymbol{e}_0\}^T \tag{17.50d}$$

To recap, equation (17.49b) is comprised of equations (17.50a) through (17.50d). The way we would actually form the dispersion matrix is as follows:

$$D\left\{\begin{bmatrix} \tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix}\right\} = \left[\begin{array}{c|c} I_r + Q_0 N_{11} & Q_0 N_{12} \\ \hline N_{21} & N_{22} \end{array}\right]^{-1} \left[\begin{array}{cc} Q_0 & 0 \\ 0 & I_{m-r} \end{array}\right] =$$

$$= \left[\begin{array}{c|c} \text{MSPE}\{\tilde{\boldsymbol{x}}_1\} & C\{\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1, \hat{\boldsymbol{\xi}}_2\} \\ \hline C\{\hat{\boldsymbol{\xi}}_2, \tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1\} & D\{\hat{\boldsymbol{\xi}}_2\} \end{array}\right]. \tag{17.51}$$

Here we stress that we are not interested in the dispersion $D\{\tilde{\boldsymbol{x}}_1\}$, since this is an indicator of variation between $\tilde{\boldsymbol{x}}_1$ and $E\{\tilde{\boldsymbol{x}}_1\}$. Rather we are interested in the variation between $\tilde{\boldsymbol{x}}_1$ and the true variable $\boldsymbol{x}_1$, a concept that the following formula makes clear:

$$\text{MSPE}\{\tilde{\boldsymbol{x}}_1\} = E\{(\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1)(\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1)^T\} = D\{\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1\}, \tag{17.52}$$

since $E\{\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1\} = \boldsymbol{0}$.

17.6 Alternative Normal Equations for the Mixed Linear Model

In the previous section we showed different, but equivalent, expressions for the predicted parameter vector $\tilde{\boldsymbol{x}}_1$. All of these expressions depended on the estimate $\hat{\boldsymbol{\xi}}_2$ for the fixed parameters. Our goal in this section is to find a system of normal equations that will permit the random parameters $\tilde{\boldsymbol{x}}_1$ to be predicted without the need to compute the fixed parameters $\hat{\boldsymbol{\xi}}_2$. With reference to (17.28), we begin with the following orthogonality relations, which are analogous to $A^T P \tilde{\boldsymbol{e}} = \boldsymbol{0}$ in the GMM:

$$A_2^T (P^{-1} + A_1 I_r Q_0 A_1^T)^{-1}(\tilde{\boldsymbol{e}} - A_1 \tilde{\boldsymbol{e}}_0) = \tag{17.53a}$$

$$= A_2^T \big[P - P A_1 (I_r + Q_0 A_1^T P A_1)^{-1} Q_0 A_1^T P \big] (\tilde{\boldsymbol{e}} - A_1 \tilde{\boldsymbol{e}}_0) = \tag{17.53b}$$

$$= A_2^T P \tilde{\boldsymbol{e}} - N_{21} \tilde{\boldsymbol{e}}_0 - N_{21} (I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P \check{\boldsymbol{e}} + $$
$$+ N_{21}(I_r + Q_0 N_{11})^{-1} Q_0 N_{11} \tilde{\boldsymbol{e}}_0 = \boldsymbol{0}. \tag{17.53c}$$

Now we make use of the relations

$$\tilde{\boldsymbol{e}} = (\bar{\boldsymbol{y}} - A_1 \tilde{\boldsymbol{x}}_1 - A_2 \hat{\boldsymbol{\xi}}_2) \tag{17.54a}$$

and

$$\tilde{\boldsymbol{e}}_0 = \boldsymbol{\beta}_0 - \tilde{\boldsymbol{x}}_1 \tag{17.54b}$$

in order to write

$$\bar{\boldsymbol{c}}_2 = N_{21} \tilde{\boldsymbol{x}}_1 + N_{22} \hat{\boldsymbol{\xi}}_2 + N_{21}(\boldsymbol{\beta}_0 - \tilde{\boldsymbol{x}}_1) + N_{21}(I_r + Q_0 N_{11})^{-1} Q_0 A_1^T P(\tilde{\boldsymbol{e}} - A_1 \tilde{\boldsymbol{e}}_0) =$$
$$= N_{21} \boldsymbol{\beta}_0 + N_{22} \hat{\boldsymbol{\xi}}_2 + A_2^T P \tilde{\boldsymbol{e}} - N_{21} \tilde{\boldsymbol{e}}_0. \tag{17.55}$$

Also, multiplying the residuals in (17.54) by $A_1^T P$ leads to

$$\bar{\boldsymbol{c}}_1 = N_{11} \boldsymbol{\beta}_0 + N_{12} \hat{\boldsymbol{\xi}}_2 + A_1^T P \tilde{\boldsymbol{e}} - N_{11} \tilde{\boldsymbol{e}}_0. \tag{17.56}$$

Now we introduce a new symbol

$$\hat{\boldsymbol{\nu}} := A_1^T P \tilde{\boldsymbol{e}} \tag{17.57}$$

and note that $A_2^T P \tilde{\boldsymbol{e}} = \boldsymbol{0}$. Combining equations (17.55) through (17.57) into a single systems of equations yields the normal equations

$$\begin{bmatrix} N_{11} & N_{12} & I_r \\ N_{21} & N_{22} & 0 \\ I_r & 0 & -Q_0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{\xi}}_2 \\ \hat{\boldsymbol{\nu}} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{c}}_1 \\ \bar{\boldsymbol{c}}_2 \\ \boldsymbol{\beta}_0 \end{bmatrix}. \tag{17.58}$$

The solution to (17.58) yields both $\tilde{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{\xi}}_2$; it also allows us to invert the normal-equations matrix when Q_0 is singular. If Q_0 is *regular* (non-singular), we may reduce the size of the system as follows:

$$\begin{bmatrix} P_0 + N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{c}}_1 + P_0\boldsymbol{\beta}_0 \\ \bar{\boldsymbol{c}}_2 \end{bmatrix}. \tag{17.59}$$

Consistent with previous claims, the solution to (17.59) yields inhomBLIP for $\tilde{\boldsymbol{x}}_1$ and BLUUE for $\hat{\boldsymbol{\xi}}_2$. It also leads to $\hat{\boldsymbol{\nu}} = P_0(\tilde{\boldsymbol{x}}_1 - \boldsymbol{\beta}_0)$.

After inverting the matrix on the left side of (17.59) (see Appendix A for inversion formula), we can write the prediction for $\boldsymbol{x}_1$ as follows:

$$\begin{aligned} \tilde{\boldsymbol{x}}_1 = (P_0 + N_{11})^{-1}(\bar{\boldsymbol{c}}_1 + P_0\boldsymbol{\beta}_0) + (P_0 + N_{11})^{-1}\cdot \\ \cdot N_{12}\big[N_{22} - N_{21}(P_0 + N_{11})^{-1}N_{12}\big]^{-1}\cdot \\ \cdot \big[N_{21}(P_0 + N_{11})^{-1}(\bar{\boldsymbol{c}}_1 + P_0\boldsymbol{\beta}_0) - \bar{\boldsymbol{c}}_2\big]. \end{aligned} \tag{17.60a}$$

Likewise, the estimation for $\boldsymbol{\xi}_2$ is given by

$$\hat{\boldsymbol{\xi}}_2 = -\big[N_{22} - N_{21}(P_0 + N_{11})^{-1}N_{12}\big]^{-1}\big[N_{21}(P_0 + N_{11})^{-1}(\bar{\boldsymbol{c}}_1 + P_0\boldsymbol{\beta}_0) - \bar{\boldsymbol{c}}_2\big].$$

$$\tag{17.60b}$$

Note that (17.60b) is equivalent to (17.28), which can be seen by confirming the following two equivalences:

$$\big[A_2^T(P^{-1} + A_1Q_0A_1^T)^{-1}A_2\big]^{-1} = \big[N_{22} - N_{21}(P_0 + N_{11})^{-1}N_{12}\big]^{-1}, \tag{17.61a}$$

$$N_{21}(P_0 + N_{11})^{-1}(\bar{\boldsymbol{c}}_1 + P_0\boldsymbol{\beta}_0) = \bar{\boldsymbol{c}}_2 + A_2^T(P^{-1} + A_1Q_0A_1^T)^{-1}(A_1\boldsymbol{\beta}_0 - \bar{\boldsymbol{y}}). \tag{17.61b}$$

Combining (17.60a) and (17.60b) yields the following expression for the predicted random effects vector $\tilde{\boldsymbol{x}}_1$ as a function of the estimated fixed parameters $\hat{\boldsymbol{\xi}}_2$:

$$\tilde{\boldsymbol{x}}_1 = \boldsymbol{\beta}_0 + (P_0 + N_{11})^{-1}(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2), \tag{17.62a}$$

which agrees with (17.44).

Recall that $\bar{\boldsymbol{y}}$, $\bar{\boldsymbol{c}}_1$, and $\bar{\boldsymbol{c}}_2$ are numerically equivalent to $\boldsymbol{y}$, $\boldsymbol{c}_1$, and $\boldsymbol{c}_2$, respectively, but they have different stochastic properties due to the randomness of $\boldsymbol{x}_1$ (see (17.33) and (17.22)). If we factor out the term $A_1^T P$ from second parenthetical expression in (17.62a), we get

$$\tilde{\boldsymbol{x}}_1 = \boldsymbol{\beta}_0 + \big(P_0 + N_{11}\big)^{-1}A_1^T P(\bar{\boldsymbol{y}} - A_1\boldsymbol{\beta}_0 - A_2\hat{\boldsymbol{\xi}}_2), \tag{17.62b}$$

which can be re-written as

$$\tilde{\boldsymbol{x}}_1 = \boldsymbol{\beta}_0 + C\{\boldsymbol{x}_1, \boldsymbol{y}\}[D\{\boldsymbol{y}\}]^{-1}(\bar{\boldsymbol{c}}_1 - N_{11}\boldsymbol{\beta}_0 - N_{12}\hat{\boldsymbol{\xi}}_2), \tag{17.63}$$

since

$$C\{\boldsymbol{x}_1, \boldsymbol{y}\} = \sigma_0^2 P_0^{-1} A_1^T \tag{17.64a}$$

and

$$D\{\boldsymbol{y}\} = \sigma_0^2 (A_1 P_0^{-1} A_1^T + P^{-1}). \tag{17.64b}$$

We conclude this chapter by commenting that LESS from the extended GMM yields the same numerical results as LESS from the MLM, but the interpretation is completely different. In the mixed linear model, $\tilde{\boldsymbol{x}}_1$ is predicted, while $\hat{\boldsymbol{\xi}}_1$ is estimated within the extended GMM. Therefore, we are not interested in the dispersion of $\tilde{\boldsymbol{x}}_1$ itself but rather its MSPE.

$$D\{\begin{bmatrix} \tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix}\} = \sigma_0^2 \begin{bmatrix} P_0 + N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} \neq D\{\begin{bmatrix} \tilde{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix}\}, \tag{17.65a}$$

where

$$D\{\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1\} = \text{MSPE}\{\tilde{\boldsymbol{x}}_1\}. \tag{17.65b}$$

Finally, we state that the estimated variance component as shown below is the Best Invariant Quadratic Uniformly Unbiased Estimate within the MLM.

$$\boxed{\hat{\sigma}_0^2 = \frac{(\tilde{\boldsymbol{e}} - A_1 \tilde{\boldsymbol{e}}_0)^T (P^{-1} + A_1 Q_0 A_1^T)(\tilde{\boldsymbol{e}} - A_1 \tilde{\boldsymbol{e}}_0)}{(n - m + r)}} \tag{17.66a}$$

Or, alternatively, using (17.40b), we can write

$$\boxed{\hat{\sigma}_0^2 = \frac{\tilde{\boldsymbol{e}}^T (I_n + Q_0 N_{11})^T (P^{-1} + A_1 Q_0 A_1^T)(I_n + Q_0 N_{11})\tilde{\boldsymbol{e}}}{(n - m + r)}.} \tag{17.66b}$$

Here, we have assumed that rk $A = m$. Also, recall that for the MLM, r is the dimension of $\boldsymbol{x}_1$.

Chapter 18

The Dynamic Linear Model and Kalman Filtering

The Dynamic Linear Model (DLM) is a linearized model that consists of an *initial value problem* (IVP) and *observed variables*. It can be viewed as a model of *observation equations with differential constraints*. The constraints are not imposed on the parameters but rather on the parameter changes (i.e., changes that occur in time).

After linearizing and discretizing the original differential equations, we arrive at the following (differential) *state equation* at epoch 1 as a function of the *state variables* $\boldsymbol{x}_0$ at epoch 0:

$$\boldsymbol{x}_1 = \Phi_0 \boldsymbol{x}_0 + \boldsymbol{u}_1. \tag{18.1}$$

The symbols are defined as follows:

$\boldsymbol{x}_i$ is an $m \times 1$ unknown *state vector* at epoch $i = 0, 1$.

Φ_0 is an $m \times m$ given *state transition matrix*.

$\boldsymbol{u}_1$ is an $m \times 1$ random noise vector.

From (18.1) we can write the following *stochastic constraints* for $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$:

$$\begin{bmatrix} I_m, & -\Phi_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_0 \end{bmatrix} = \boldsymbol{u}_1. \tag{18.2}$$

We mentioned already that the DLM is an initial value problem. The initial conditions are expressed by

$$\tilde{\boldsymbol{x}}_0 = \boldsymbol{x}_0 + \boldsymbol{e}_0^0. \tag{18.3}$$

Here the subscript 0 denotes epoch 0, while the superscript 0 denotes the initial condition. We note that the subscript for $\boldsymbol{u}_1$ is sometimes shown as 0 rather than 1

in the literature. This is merely a convention, as the variable $\boldsymbol{u}_1$ represents the noise (random error) of the difference of the states $\boldsymbol{x}_1$ and $\Phi_0 \boldsymbol{x}_0$, between epochs 1 and 0, respectively. Also, for any epoch i, the state transition matrix Φ_i is unique; that is, it changes from epoch to epoch. Our knowledge of the initial state vector $\tilde{\boldsymbol{x}}_0$ can be improved by using a *backward filter*, but its use is not possible in real-time applications.

Equations (18.2) and (18.3) constitute the IVP in discrete form. Their stochastic properties (expectation and dispersion) are written as

$$
\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{e}_0^0 \end{bmatrix} \sim (\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Sigma_0^0 \end{bmatrix}). \tag{18.4}
$$

At this stage we have $2m$ unknowns and $2m$ equations (owing to the unknown $m \times 1$ vectors $\boldsymbol{x}$ and $\boldsymbol{u}$). Since there is no redundancy in the model, we cannot determine the unknowns in a least-squares sense. The redundancy enters the model via the following observation equations:

$$
\boldsymbol{y}_1 = A_1 \boldsymbol{x}_1 + \boldsymbol{e}_1, \quad \boldsymbol{y}_1 \in \mathbb{R}^n, \quad A_1 \in \mathbb{R}^{n \times m}, \tag{18.5a}
$$

with the stochastic model

$$
\boldsymbol{e}_1 \sim (\boldsymbol{0}, \Sigma_1), \quad C\{\boldsymbol{e}_1, \boldsymbol{u}_1\} = \boldsymbol{0}, \quad C\{\boldsymbol{e}_1, \boldsymbol{e}_0^0\} = \boldsymbol{0}. \tag{18.5b}
$$

Thus we see that the DLM is comprised of three components: observation equations (18.5a), state equations (18.1), and initial conditions (18.3). We may combine all three parts of the model into one succinct statement as follows:

$$
\left. \begin{aligned} \boldsymbol{y}_1 &= A_1 \boldsymbol{x}_1 + \boldsymbol{e}_1 \\ \boldsymbol{x}_1 &= \Phi_0 \boldsymbol{x}_0 + \boldsymbol{u}_1 \\ \tilde{\boldsymbol{x}}_0 &= \boldsymbol{x}_0 + \boldsymbol{e}_0^0 \end{aligned} \right\} \quad \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{u}_1 \\ \boldsymbol{e}_0^0 \end{bmatrix} \sim (\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Theta_1 & 0 \\ 0 & 0 & \Sigma_0^0 \end{bmatrix}). \tag{18.6}
$$

Notice the absence of a (common) variance component, σ_0^2, in the model. We can consider it to be embedded within the covariance matrices Σ_1, Θ_1, and Σ_0^0.

Our goal is to predict the unknown state vector $\boldsymbol{x}_1$ and determine its mean squared error (MSE) matrix. The relations between the predicted variables (with tildes) and true variables (without tildes) are described by

$$
\tilde{\boldsymbol{x}}_1 = \boldsymbol{x}_1 + \boldsymbol{e}_1^0, \tag{18.7a}
$$

and

$$
D\{\boldsymbol{e}_1^0 = \tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1\} = \mathrm{MSPE}\{\tilde{\boldsymbol{x}}_1\} = \Sigma_1^0. \tag{18.7b}
$$

We may also wish to use "backward filtering" to compute the prediction $\tilde{\tilde{\boldsymbol{x}}}_0$ for the initial state vector $\boldsymbol{x}_0$. However, as mentioned previously, this is not feasible, or even possible, in real-time problems.

We now introduce a new prediction variable $\breve{\boldsymbol{x}}_1$ by combining the state equation and initial condition. This variable represents our prior knowledge (*prior information*) about the state vector

$$\boldsymbol{x}_1 = \Phi_0 \boldsymbol{x}_0 + \boldsymbol{u}_1 = \Phi_0(\tilde{\boldsymbol{x}}_0 - \boldsymbol{e}_0^0) + \boldsymbol{u}_1, \tag{18.8a}$$

which leads to the predictor

$$\breve{\boldsymbol{x}}_1 := \Phi_0 \tilde{\boldsymbol{x}}_0 = \boldsymbol{x}_1 - (\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0). \tag{18.8b}$$

We call the term in parenthesis in (18.8b) the *combined error*. Note that

$$E\{\breve{\boldsymbol{x}}_1\} = E\{\boldsymbol{x}_1\}, \tag{18.9a}$$

since

$$E\{\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0\} = \boldsymbol{0}. \tag{18.9b}$$

We note that the "prior information" (18.8a) and (18.8b) in the DLM is more complicated than that in the Mixed Linear Model (MLM). Here, we must determine the predicted state vector $\tilde{\boldsymbol{x}}_1$ (which is different than $\breve{\boldsymbol{x}}_1$) based on the new observations. The variable $\breve{\boldsymbol{x}}_1$ is the best prediction based on the state equation and the initial condition only. We essentially blend the prior knowledge $\breve{\boldsymbol{x}}_1$ with the observations $\boldsymbol{y}_1$. With this *fusion* of information we are able to determine the prediction $\tilde{\boldsymbol{x}}_1$. Note that all of the redundancy in the model comes from the observation equations. The initial value problem is only uniquely solvable.

This fusion process is called *Kalman filtering*. It can be done in real time, in which case the number of state parameters may be restricted by the speed of the computer processor. The key is to have good information about the state equation, not only the state transition matrix Φ_0 but also the associated covariance matrix, Θ_1, of the state equation. The information contained in matrices Φ_0 and Θ_1 describes how we think the dynamic system behaves. Our knowledge of the system is introduced as a differential equation, which is *linearized* and *discretized* to form the state equation. This work must be done before the adjustment stage.

With the introduction of (18.8b), we may write an equivalent version of the DLM as follows:

$$\left. \begin{array}{l} \boldsymbol{y}_1 = A_1 \boldsymbol{x}_1 + \boldsymbol{e}_1 \\[2mm] \breve{\boldsymbol{x}}_1 = \boldsymbol{x}_1 - (\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0) \end{array} \right\} \quad \begin{bmatrix} \boldsymbol{e}_1 \\[2mm] -(\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0) \end{bmatrix} \sim (\begin{bmatrix} \boldsymbol{0} \\[2mm] \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \Sigma_1 & 0 \\[2mm] 0 & \Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \end{bmatrix})$$

$$\tag{18.10}$$

The model (18.10) essentially combines observation equations (in $\boldsymbol{y}_1$) with pseudo-observation equations (in $\breve{\boldsymbol{x}}_1$). But here we are dealing with *random effects*, so the

DLM is not an extended GMM but rather is essentially an *extended random effects model* (REM).

The LEast-Squares Solution (LESS) within the DLM is equivalent to the inhomBLIP of $\boldsymbol{x}_1$. Based on the model (18.10), we can write the least-squares normal equations directly as follows:

$$
\begin{bmatrix} A_1^T & | & I_m \end{bmatrix}
\left[\begin{array}{c|c} \Sigma_1^{-1} & 0 \\ \hline 0 & \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \end{array} \right]
\begin{bmatrix} A_1 \\ I_m \end{bmatrix} \tilde{\boldsymbol{x}}_1 =
$$

$$
= \begin{bmatrix} A_1^T \Sigma_1^{-1} & | & \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \end{bmatrix}
\begin{bmatrix} \boldsymbol{y}_1 \\ \breve{\boldsymbol{x}}_1 \end{bmatrix} .
$$

(18.11)

Solving the normal equations results in

$$
\tilde{\boldsymbol{x}}_1 = \left[A_1^T \Sigma_1^{-1} A_1 + \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \right]^{-1} \left[A_1^T \Sigma_1^{-1} \boldsymbol{y}_1 + \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \breve{\boldsymbol{x}}_1 \right].
$$

(18.12)

Then, the following steps lead to the isolation of $\breve{\boldsymbol{x}}_1$:

$$
\tilde{\boldsymbol{x}}_1 = \left[A_1^T \Sigma_1^{-1} A_1 + \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \right]^{-1} \cdot
$$
$$
\cdot \left[A_1^T \Sigma_1^{-1} \boldsymbol{y}_1 + \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \breve{\boldsymbol{x}}_1 + A_1^T \Sigma_1^{-1} A_1 \breve{\boldsymbol{x}}_1 - A_1^T \Sigma_1^{-1} A_1 \breve{\boldsymbol{x}}_1 \right] \Rightarrow
$$

(18.13a)

$$
\boxed{ \tilde{\boldsymbol{x}}_1 = \breve{\boldsymbol{x}}_1 + \left[A_1^T \Sigma_1^{-1} A_1 + \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \right]^{-1} A_1^T \Sigma_1^{-1} \left(\boldsymbol{y}_1 - A_1 \breve{\boldsymbol{x}}_1 \right), }
$$

(18.13b)

or

$$
\boxed{ \tilde{\boldsymbol{x}}_1 = \breve{\boldsymbol{x}}_1 + K_1 \boldsymbol{z}_1. }
$$

(18.13c)

Here, the $m \times n$ matrix

$$
\boxed{ K_1 := \left[A_1^T \Sigma_1^{-1} A_1 + \left(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T \right)^{-1} \right]^{-1} A_1^T \Sigma_1^{-1} }
$$

(18.13d)

is called *Kalman gain matrix*, and the $n \times 1$ vector

$$
\boxed{ \boldsymbol{z}_1 := \left(\boldsymbol{y}_1 - A_1 \breve{\boldsymbol{x}}_1 \right) }
$$

(18.13e)

is called the *innovation*.

The form of the Kalman gain matrix in (18.13d) is useful for the case where the dimension, m, of the state vector is smaller than the number of observations n, since the size of matrix to invert is $m \times m$.

We may write alternative forms of the solution as follows:

$$\tilde{\boldsymbol{x}}_1 - \breve{\boldsymbol{x}}_1 = \left[I_m + (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \Sigma_1^{-1} A_1 \right]^{-1} \cdot$$
$$\cdot (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \Sigma_1^{-1} (\boldsymbol{y}_1 - A_1 \breve{\boldsymbol{x}}_1) = \tag{18.14a}$$

$$= (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \left[\Sigma_1 + A_1 (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \right]^{-1} (\boldsymbol{y}_1 - A_1 \breve{\boldsymbol{x}}_1). \tag{18.14b}$$

In (18.14a) we have used the the relations (A.9a) and (A.9c), and in (18.14b) we have used the relations (A.8a) and (A.8e). Both equations (18.14a) and (18.14b) are in the form of an *update*. However, equation (18.14a) requires the inversion of an $m \times m$ matrix, whereas equation (18.14b) requires the inversion of an $n \times n$ matrix. Oftentimes, in real-time applications, the number of observations n at a given epoch is small (perhaps only 1) compared to the number of state parameters m. In such cases, equation (18.14b) would be preferred over equation (18.14a).

We note that in the technical literature $\tilde{\boldsymbol{x}}_1$ is called the *filtered state*, while $\breve{\boldsymbol{x}}_1$ is called the *predicted state*. However, in the statistical literature, $\tilde{\boldsymbol{x}}_1$ is referred to as the *best prediction*. It is this best prediction $\tilde{\boldsymbol{x}}_1$ that we are interested in.

We summarize the various forms of the Kalman gain matrix appearing in the above formulas as follows:

$$K_1 = \left[A_1^T \Sigma_1^{-1} A_1 + (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)^{-1} \right]^{-1} A_1^T \Sigma_1^{-1} = \tag{18.15a}$$

$$= \left[I_m + (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \Sigma_1^{-1} A_1 \right]^{-1} (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \Sigma_1^{-1} = \tag{18.15b}$$

$$= (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \left[\Sigma_1 + A_1 (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T \right]^{-1}. \tag{18.15c}$$

By combining the two equations in (18.10), we can alternatively express the innovation vector as

$$\boldsymbol{z}_1 := (\boldsymbol{y}_1 - A_1 \breve{\boldsymbol{x}}_1) = \boldsymbol{e}_1 + A_1 (\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0) = \left[I_n \mid A_1 \mid -A_1 \Phi_0 \right] \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{u}_1 \\ \boldsymbol{e}_0^0 \end{bmatrix}. \tag{18.16}$$

The dispersion of the innovation vector is readily apparent from (18.16); by applying the law of error propagation we obtain

$$D\{\boldsymbol{z}_1\} = \left[I_n \mid A_1 \mid -A_1 \Phi_0 \right] \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Theta_1 & 0 \\ 0 & 0 & \Sigma_0^0 \end{bmatrix} \begin{bmatrix} I_n \\ A_1^T \\ -\Phi_0^T A_1^T \end{bmatrix} =$$
$$= \Sigma_1 + A_1 (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T. \tag{18.17}$$

We may express the stochastic properties of the innovation $\boldsymbol{z}_1$ more concisely as

$$\boldsymbol{z}_1 \sim (\boldsymbol{0}, D\{\boldsymbol{z}_1\}) \quad \text{and} \quad C\{\boldsymbol{z}_i, \boldsymbol{z}_j\} = \boldsymbol{0} \ \text{ for } i \neq j. \tag{18.18}$$

The statement of zero correlation means that the innovative sequence (from epoch to epoch) is uncorrelated. The expectation $E\{\boldsymbol{z}_i\} = \boldsymbol{0}$ should be tested for. If, through statistical testing, the expectation is found to be non-zero, this means that the state equations are inconsistent with the observation equations, implying that the state equations might need to be modified.

The familiar *model of condition equations* (see Chapter 4), along with the least-squares solution (LESS) for the residual vector within that model, is given by

$$\boldsymbol{w} := B\boldsymbol{e}, \tag{18.19}$$

and

$$\tilde{\boldsymbol{e}} = P^{-1}B^T(BP^{-1}B^T)^{-1}\boldsymbol{w}. \tag{18.20}$$

Comparing (18.16) with the first part of (18.20), we see that the innovation vector is in the form of the model of condition equations. Thus, in accordance with the LESS within that model, we can immediately write the solution to the vector of predicted errors (residuals) as

$$
\begin{bmatrix} \tilde{\boldsymbol{e}}_1 \\ \tilde{\boldsymbol{u}}_1 \\ \tilde{\boldsymbol{e}}_0^0 \end{bmatrix} = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Theta_1 & 0 \\ 0 & 0 & \Sigma_0^0 \end{bmatrix} \begin{bmatrix} I_n \\ A_1^T \\ -\Phi_0^T A_1^T \end{bmatrix} \cdot
$$
$$
\cdot \left(\begin{bmatrix} I_n & | & A_1 & | & -A_1 & | & \Phi_0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Theta_1 & 0 \\ 0 & 0 & \Sigma_0^0 \end{bmatrix} \begin{bmatrix} I_n \\ A_1^T \\ -\Phi_0^T A_1^T \end{bmatrix} \right)^{-1} \boldsymbol{z}_1, \tag{18.21a}
$$

or

$$
\begin{bmatrix} \tilde{\boldsymbol{e}}_1 \\ \tilde{\boldsymbol{u}}_1 \\ \tilde{\boldsymbol{e}}_0^0 \end{bmatrix} = \begin{bmatrix} \Sigma_1 \\ \Theta_1 A_1^T \\ -\Sigma_0^0 \Phi_0^T A_1^T \end{bmatrix} \left[\Sigma_1 + A_1\left(\Theta_1 + \Phi_0\Sigma_0^0\Phi_0^T\right)A_1^T \right]^{-1} \boldsymbol{z}_1. \tag{18.21b}
$$

If we substitute the predicted errors from (18.21b) into the second equation of (18.10), we arrive at

$$\tilde{\boldsymbol{x}}_1 = \breve{\boldsymbol{x}}_1 + \left(\tilde{\boldsymbol{u}}_1 - \Phi_0\tilde{\boldsymbol{e}}_0^0 \right), \tag{18.22}$$

which leads to the same update formula found in (18.14b).

We mentioned earlier that backwards filtering can be used to obtain a better prediction of the initial state vector $\tilde{\boldsymbol{x}}_0$, though this is usually not feasible in real-time applications. Substituting the predicted random error vector $\tilde{\boldsymbol{e}}_0^0$ of (18.21b)

into the third equation of (18.6), and making use of (18.13e) for $\boldsymbol{z}_1$, allows us to write the backwards filter in the form of an update to $\tilde{\boldsymbol{x}}_0$ as follows:

$$\tilde{\tilde{\boldsymbol{x}}}_0 = \tilde{\boldsymbol{x}}_0 + \tilde{\boldsymbol{e}}_0^0 = \tilde{\boldsymbol{x}}_0 - \Sigma_0^0 \Phi_0^T A_1^T \left[\Sigma_1 + A_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T\right]^{-1}(\boldsymbol{y}_1 - A_1 \Phi_0 \tilde{\boldsymbol{x}}_0). \tag{18.23}$$

In order to form the model for the next interval, we need the covariance matrix Σ_1^0. This matrix is defined as

$$\Sigma_1^0 := \mathrm{MSPE}\{\tilde{\boldsymbol{x}}_1\} = D\{\tilde{\boldsymbol{x}}_1 - \boldsymbol{x}_1\} = D\{(\tilde{\boldsymbol{x}}_1 - \breve{\boldsymbol{x}}_1) - (\boldsymbol{x}_1 - \breve{\boldsymbol{x}}_1)\}. \tag{18.24}$$

Referring to (18.8b), we may write a vector difference depending on unknown vectors $\boldsymbol{x}_1$, $\boldsymbol{u}_1$, and $\boldsymbol{e}_0^0$ as

$$\boldsymbol{x}_1 - \breve{\boldsymbol{x}}_1 = \boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0, \tag{18.25a}$$

which implies the following vector difference based on corresponding predicted variables:

$$\tilde{\boldsymbol{x}}_1 - \breve{\boldsymbol{x}}_1 = \tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0. \tag{18.25b}$$

So, with help of (18.25b), we may replace the differences in (18.24) with linear combinations of the residual vectors as in the following:

$$\begin{aligned} D\{(\tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0) - (\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0)\} = \\ D\{\tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0\} - C\{(\tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0), (\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0)\} - \\ - C\{(\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0), (\tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0)\} + D\{\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0\}. \end{aligned} \tag{18.26}$$

We now determine each of the four terms on the right side of (18.26) before combining them into a single equation. Comparing (18.14b) and (18.22) we see that

$$\begin{aligned} D\{\tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0\} = (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T \left[\Sigma_1 + A_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T\right]^{-1} D\{\boldsymbol{z}_1\} \cdot \\ \cdot \left[\Sigma_1 + A_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T\right]^{-1} A_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T), \end{aligned} \tag{18.27a}$$

which, upon substitution of (18.17), leads to

$$\begin{aligned} D\{\tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0\} = (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T \left[\Sigma_1 + A_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T\right]^{-1} \cdot \\ \cdot A_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T). \end{aligned} \tag{18.27b}$$

From the given model (18.10) we can write

$$D\{\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0\} = \Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T. \tag{18.27c}$$

Using (18.27c), we can rewrite (18.27b) as

$$\begin{aligned} D\{\tilde{\boldsymbol{u}}_1 - \Phi_0 \tilde{\boldsymbol{e}}_0^0\} = (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T \left[\Sigma_1 + A_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)A_1^T\right]^{-1} \cdot \\ \cdot A_1 D\{\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0\} = K_1 A_1 D\{\boldsymbol{u}_1 - \Phi_0 \boldsymbol{e}_0^0\}, \end{aligned} \tag{18.27d}$$

which leads to the following covariance terms:

$$C\{\tilde{\boldsymbol{u}}_1 - \Phi_0\tilde{\boldsymbol{e}}_0^0, \boldsymbol{u}_1 - \Phi_0\boldsymbol{e}_0^0\} = K_1 A_1 D\{\boldsymbol{u}_1 - \Phi_0\boldsymbol{e}_0^0\} = \tag{18.27e}$$

(and, due to symmetry,)

$$= C\{\boldsymbol{u}_1 - \Phi_0\boldsymbol{e}_0^0, \tilde{\boldsymbol{u}}_1 - \Phi_0\tilde{\boldsymbol{e}}_0^0\} = D\{\tilde{\boldsymbol{u}}_1 - \Phi_0\tilde{\boldsymbol{e}}_0^0\}. \tag{18.27f}$$

Summing the individual components (18.27c) through (18.27f) yields

$$\boxed{D\{(\tilde{\boldsymbol{u}}_1 - \Phi_0\tilde{\boldsymbol{e}}_0^0) - (\boldsymbol{u}_1 - \Phi_0\boldsymbol{e}_0^0)\} = (I_m - K_1 A_1)(\Theta_1 + \Phi_0\Sigma_0^0\Phi_0^T) =: \Sigma_1^0.}$$

$$\tag{18.28}$$

With the covariance matrix Σ_1^0, we are ready to process the data at epoch 2, and we can continue in a like manner with any epochs that follow.

Notice that the variance component σ_0^2 has not been included in the dispersion formulas. This is because we try to avoid extra computations in real-time applications. However, we may wish to test our hypothesis that the innovation vector $\boldsymbol{z}_1$ is zero. To do so, we form the test statistic

$$\Omega := \boldsymbol{z}_1^T \big(D\{\boldsymbol{z}_1\}\big)^{-1}\boldsymbol{z}_1, \tag{18.29}$$

which has redundancy n. Our hypothesis test (at each epoch) is

$$H_0 : E\{\boldsymbol{z}_1\} = \boldsymbol{0} \;\text{ versus }\; H_a : E\{\boldsymbol{z}_1\} \neq \boldsymbol{0}. \tag{18.30}$$

The distribution of the test statistic is

$$\Omega \sim \chi_n^2 \;\text{ under }\; H_0. \tag{18.31}$$

For some chosen level of significance α, we reject the null hypothesis H_0 if $\Omega > \chi_\alpha^2$. Note that we could also test the expectations of $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{u}}_1$, and $\tilde{\boldsymbol{e}}_0^0$ separately if the null hypothesis in (18.30) is rejected.

Appendix A

Useful Matrix Relations and Identities

Basic relationships
Product of transposes:

$$A^T B^T = (BA)^T \qquad\qquad (A.1)$$

Transpose of inverse:

$$(A^T)^{-1} = (A^{-1})^T \qquad\qquad (A.2)$$

Product of inverses:

$$A^{-1} B^{-1} = (BA)^{-1} \qquad\qquad (A.3)$$

Rank of triple product: Given: $A(m \times n)$, $B(m \times m)$, $C(n \times n)$:

$$B, C \text{ nonsingular} \Rightarrow \text{rk}(BAC) = \text{rk}(A) \;\; \text{or} \;\; \text{rk}(BA) = \text{rk}(A) \;\; \text{if} \;\; C = I \qquad (A.4)$$

Trace invariant with respect to a cyclic permutation of factors   If the product ABC is square, then the following trace operations are equivalent:

$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB). \qquad\qquad (A.5)$$

Sherman-Morrison-Woodbury-Schur formula

$$(T - UW^{-1}V)^{-1} = T^{-1} + T^{-1}U(W - VT^{-1}U)^{-1}VT^{-1} \qquad (A.6a)$$

Multiplying on the right by U and rearranging leads to the so-called push-through identity

$$T^{-1}U(W - VT^{-1}U)^{-1}W = (T - UW^{-1}V)^{-1}U. \tag{A.6b}$$

The origin of the phrase "push-through" is illustrated by the special case where $T = tI$ and $W = wI$, leading to

$$U(tI - (1/w)VU)^{-1} = (tI - (1/w)UV)^{-1}U. \tag{A.6c}$$

As a consequence of (A.6a), we also have:

$$(I \pm UW^{-1}V)^{-1} = I \mp U(W \pm VU)^{-1}V, \tag{A.7a}$$

$$(I \pm UV)^{-1} = I \mp U(I \pm VU)^{-1}V, \tag{A.7b}$$

$$(I \pm W^{-1}V)^{-1} = I \mp (W \pm V)^{-1}V, \tag{A.7c}$$

$$(I \pm V)^{-1} = I \mp (I \pm V)^{-1}V, \tag{A.7d}$$

$$(I \pm W^{-1})^{-1} = I \mp (W \pm I)^{-1}. \tag{A.7e}$$

Useful Matrix Equalities   Equations (39–43) of "Useful Matrix Equalities" (handout from Prof. Schaffrin, possibly originating from Urho A. Uotila).

$$DC(A + BDC)^{-1} = (D^{-1} + CA^{-1}B)^{-1}CA^{-1} = \tag{A.8a}$$

$$= D(I + CA^{-1}BD)^{-1}CA^{-1} = \tag{A.8b}$$

$$= DC(I + A^{-1}BDC)^{-1}A^{-1} = \tag{A.8c}$$

$$= DCA^{-1}(I + BDCA^{-1})^{-1} = \tag{A.8d}$$

$$= (I + DCA^{-1}B)^{-1}DCA^{-1} \tag{A.8e}$$

We may expand the above UME's by setting, in turn, each matrix equal to the identity matrix, thus generating four new sets of identities, as follows:

Let $A = I$:

$$DC(I + BDC)^{-1} = (D^{-1} + CB)^{-1}C = \tag{A.9a}$$

$$= D(I + CBD)^{-1}C = \tag{A.9b}$$

$$= (I + DCB)^{-1}DC. \tag{A.9c}$$

Let $B = I$:

$$DC(A + DC)^{-1} = (D^{-1} + CA^{-1})^{-1}CA^{-1} = \tag{A.10a}$$

$$= D(I + CA^{-1}D)^{-1}CA^{-1} = \tag{A.10b}$$

$$= DC(I + A^{-1}DC)^{-1}A^{-1} = \tag{A.10c}$$

$$= DCA^{-1}(I + DCA^{-1})^{-1} = \qquad \text{(A.10d)}$$
$$= (I + DCA^{-1})^{-1}DCA^{-1}. \qquad \text{(A.10e)}$$

Let $C = I$:

$$D(A + BD)^{-1} = (D^{-1} + A^{-1}B)^{-1}A^{-1} = \qquad \text{(A.11a)}$$
$$= D(I + A^{-1}BD)^{-1}A^{-1} = \qquad \text{(A.11b)}$$
$$= DA^{-1}(I + BDA^{-1})^{-1} = \qquad \text{(A.11c)}$$
$$= (I + DA^{-1}B)^{-1}DA^{-1}. \qquad \text{(A.11d)}$$

Let $D = I$:

$$C(A + BC)^{-1} = (I + CA^{-1}B)^{-1}CA^{-1} = \qquad \text{(A.12a)}$$
$$= C(I + A^{-1}BC)^{-1}A^{-1} = \qquad \text{(A.12b)}$$
$$= CA^{-1}(I + BCA^{-1})^{-1}. \qquad \text{(A.12c)}$$

Suppose the matrices A and B in (A.8) are identity matrices, then we have

$$DC(I + DC)^{-1} = (D^{-1} + C)^{-1}C = \qquad \text{(A.13a)}$$
$$= D(I + CD)^{-1}C = \qquad \text{(A.13b)}$$
$$= (I + DC)^{-1}DC. \qquad \text{(A.13c)}$$

Inverse of the partitioned normal equation matrix Assume the matrix N is of full rank and is partitioned as follows:

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \qquad \text{(A.14)}$$

The following steps lead to the inverse of N expressed in terms of the partitioned blocks:

$$\left[\begin{array}{cc|cc} N_{11} & N_{12} & I & 0 \\ N_{21} & N_{22} & 0 & I \end{array}\right] \rightarrow \left[\begin{array}{c|c|c|c} I & N_{11}^{-1}N_{12} & N_{11}^{-1} & 0 \\ \hline N_{21} & N_{22} & 0 & I \end{array}\right] \rightarrow$$

$$\left[\begin{array}{c|c|c|c} I & N_{11}^{-1}N_{12} & N_{11}^{-1} & 0 \\ \hline 0 & N_{22} - N_{21}N_{11}^{-1}N_{12} & -N_{21}N_{11}^{-1} & I \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c|c} I & N_{11}^{-1}N_{12} & N_{11}^{-1} & 0 \\ 0 & I & -\left(N_{22} - N_{21}N_{11}^{-1}N_{12}\right)^{-1}N_{21}N_{11}^{-1} & \left(N_{22} - N_{21}N_{11}^{-1}N_{12}\right)^{-1} \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c|c} I & 0 & N_{11}^{-1} + N_{11}^{-1}N_{12} \cdot W \cdot N_{21}N_{11}^{-1} & -N_{11}^{-1}N_{12} \cdot W \\ 0 & I & -W \cdot N_{21}N_{11}^{-1} & W \end{array}\right],$$

with $W := (N_{22} - N_{21}N_{11}^{-1}N_{12})^{-1}$. Finally we may write

$$\left[\begin{array}{cc} N_{11} & N_{12} \\ N_{21} & N_{22} \end{array}\right]^{-1} = \left[\begin{array}{c|c} N_{11}^{-1} + N_{11}^{-1}N_{12} \cdot W \cdot N_{21}N_{11}^{-1} & -N_{11}^{-1}N_{12} \cdot W \\ \hline -W \cdot N_{21}N_{11}^{-1} & W \end{array}\right]. \quad \text{(A.15)}$$

Note that other equivalent representations of this inverse exist. Taking directly from the Useful Matrix Equalities handout mentioned above, we write some additional expressions for the inverse.

$$\left[\begin{array}{cc} N_{11} & N_{12} \\ N_{21} & N_{22} \end{array}\right]^{-1} = \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array}\right] \quad \text{(A.16)}$$

$$Q_{11} = \left(N_{11} - N_{12}N_{22}^{-1}N_{21}\right)^{-1} = \quad \text{(A.17a)}$$
$$= N_{11}^{-1} + N_{11}^{-1}N_{12}\left(N_{22} - N_{21}N_{11}^{-1}N_{12}\right)^{-1}N_{21}N_{11}^{-1} = \quad \text{(A.17b)}$$
$$= N_{11}^{-1} + N_{11}^{-1}N_{12}Q_{22}N_{21}N_{11}^{-1} \quad \text{(A.17c)}$$

$$Q_{22} = \left(N_{22} - N_{21}N_{11}^{-1}N_{12}\right)^{-1} = \quad \text{(A.18a)}$$

$$= N_{22}^{-1} + N_{22}^{-1} N_{21} \left(N_{11} - N_{12} N_{22}^{-1} N_{21} \right)^{-1} N_{12} N_{22}^{-1} = \tag{A.18b}$$
$$= N_{22}^{-1} + N_{22}^{-1} N_{21} Q_{11} N_{12} N_{22}^{-1} \tag{A.18c}$$

$$Q_{12} = -\left(N_{11} - N_{12} N_{22}^{-1} N_{21} \right)^{-1} N_{12} N_{22}^{-1} = -Q_{11} N_{12} N_{22}^{-1} = \tag{A.19a}$$
$$= -N_{11}^{-1} N_{12} \left(N_{22} - N_{21} N_{11}^{-1} N_{12} \right)^{-1} = -N_{11}^{-1} N_{12} Q_{22} \tag{A.19b}$$

$$Q_{21} = -N_{22}^{-1} N_{21} \left(N_{11} - N_{12} N_{22}^{-1} N_{21} \right)^{-1} = -N_{22}^{-1} N_{21} Q_{11} = \tag{A.20a}$$
$$= -\left(N_{22} - N_{21} N_{11}^{-1} N_{12} \right)^{-1} N_{21} N_{11}^{-1} = -Q_{22} N_{21} N_{11}^{-1} \tag{A.20b}$$

In the case that $N_{22} = 0$, we have:

$$Q_{22} = -\left(N_{21} N_{11}^{-1} N_{12} \right)^{-1} \tag{A.21a}$$
$$Q_{11} = N_{11}^{-1} + N_{11}^{-1} N_{12} Q_{22} N_{21} N_{11}^{-1} \tag{A.21b}$$
$$Q_{12} = -N_{11}^{-1} N_{12} Q_{22} \tag{A.21c}$$
$$Q_{21} = -Q_{22} N_{21} N_{11}^{-1} \tag{A.21d}$$

Schur Complement: The parenthetical term $\left(N_{22} - N_{21} N_{11}^{-1} N_{12} \right)$ shown above is called the Schur Complement of N_{11}. In general, given the partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{A.22a}$$

if matrix D is invertible, the Schur complement of D is

$$S_1 = A - BD^{-1}C. \tag{A.22b}$$

Likewise, if matrix A is invertible, the Schur complement of A is

$$S_2 = D - CA^{-1}B. \tag{A.22c}$$

Now, let $S = S_2 = D - CA^{-1}B$ and assume M is nonsingular. Carlson (1986) shows that

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}, \tag{A.23a}$$

which can be used to write the inverse of M first as a triple product of matrices and then in the form of (A.15) as follows:

$$M^{-1} = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} =$$

$$= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \qquad \text{(A.23b)}$$

Carlson (ibid.) credits this formulation to Banachiewicz.

Vector and matrix norms   For a real number $p \geq 1$, the p-norm, or l_p-norm, of the $n \times 1$ vector $\boldsymbol{x}$ is defined by

$$\|\boldsymbol{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}. \qquad \text{(A.24)}$$

Particular cases

1. $p = 1$, 1-norm or l_1-norm:

$$\|\boldsymbol{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n| \qquad \text{(A.25a)}$$

2. $p = 2$, 2-norm or l_2-norm (Euclidean distance/norm):

$$\boxed{\|\boldsymbol{x}\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}} \qquad \text{(A.25b)}$$

3. $p = \infty$, ∞-norm or l_∞-norm ("infinity norm"):

$$\|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\} \qquad \text{(A.25c)}$$

In a similar way, entry-wise matrix norms for a $n \times m$ matrix A are defined by

$$\|A\|_p = \|\text{vec}\,A\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^p\right)^{1/p}, \qquad \text{(A.26)}$$

where vec is the operator that turns a matrix into a vector by stacking its columns on top of each other from the first to the last.

Particular cases

1. $p = 2$, "Frobenius norm," also called l_2-norm, Hilbert-Schmidt norm, Schur norm, and Euclidean norm: (Lütkepohl, 1996, pg. 103):

$$\boxed{\|A\|_2 = \|A\|_F = \sqrt{\text{tr}(A^T A)}} \qquad (A.27a)$$

2. $p = \infty$, Max norm:

$$\|A\|_\infty = \|A\|_{\text{max}} = \max_{i,j}[\![a_{ij}]\!] \qquad (A.27b)$$

Determinants and inverses of 2×2 and 3×3 matrices

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is defined by

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \qquad (A.28a)$$

The inverse of A can be found by

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \qquad (A.28b)$$

Writing a 3×3 matrix A as

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \qquad (A.29a)$$

the determinant of A is found by

$$\det A = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} =$$

$$= +a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} =$$

$$= -d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix} = \qquad (A.29b)$$

$$= +g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}.$$

The inverse of A is found by

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^{T} = \frac{1}{|A|} \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} c & b \\ i & h \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ i & g \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} b & a \\ h & g \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix} =$$

$$= \frac{1}{|A|} \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}. \tag{A.29c}$$

Kronecker product   The Kronecker-Zehfuss product of matrices is often simply called the Kronecker product. Its definition and several computational rules associated with it are given below.

Definition: let $G = [g_{ij}]$ be a $p \times q$ matrix and $H = [h_{ij}]$ be an $r \times s$ matrix, then

$$G \otimes H := \begin{bmatrix} g_{ij} \cdot H \end{bmatrix} \tag{A.30}$$

gives the Kronecker-Zehfuss product $G \otimes H$, which is of size $pr \times qs$.

Kronecker-Zehfuss computational rules:

(1) $\operatorname{vec} ABC^{T} = (C \otimes A) \operatorname{vec} B$ $\hspace{4em}$ (A.31)

(2) $\operatorname{tr} ABC^{T}D^{T} = \operatorname{tr} D^{T}ABC^{T} = (\operatorname{vec} D)^{T}(C \otimes A) \operatorname{vec} B$ $\hspace{2em}$ (A.32)

(3) $(G \otimes H)^{T} = G^{T} \otimes H^{T}$ $\hspace{4em}$ (A.33)

(4) $(G \otimes H)^{-1} = G^{-1} \otimes H^{-1}$ $\hspace{4em}$ (A.34)

(5) $\alpha(G \otimes H) = \alpha G \otimes H = G \otimes \alpha H$ for $\alpha \in \mathbb{R}$ $\hspace{2em}$ (A.35)

(6) $(F + G) \otimes H = (F \otimes H) + (G \otimes H)$ $\hspace{3em}$ (A.36)

(7) $G \otimes (H + J) = (G \otimes H) + (G \otimes J)$ $\hspace{3em}$ (A.37)

(8) $(A \otimes B)(G \otimes H) = AG \otimes BH$ $\hspace{4em}$ (A.38)

(9) $(H \otimes G) = K(G \otimes H)K$ for "commutation matrices" of suitable size

$\hspace{32em}$ (A.39)

(10) K^{T} is also a commutation matrix with $KK^{T} = I = K^{T}K \Rightarrow$

$\quad\quad K(H \otimes G) = (G \otimes H)K$ $\hspace{4em}$ (A.40)

$\quad\quad$(Note that K is a generic symbol; the two K matrices could be different.)

$\quad\quad$specially: $K(H \otimes g) = g \otimes H$ for any vector g

(11) $K \otimes K$ is also a commutation matrix $\Rightarrow \operatorname{vec}(G^{T}) = \operatorname{vec}(KK^{T}G^{T}) =$

$\quad\quad (G \otimes I)K \operatorname{vec} I = K(I \otimes G) \operatorname{vec} I = K \operatorname{vec} G$. Hence, K is called a "vec-

permutation matrix." $\hspace{10cm}$ (A.41)

(12) Let $\boldsymbol{\lambda}_G$ and $\boldsymbol{\lambda}_H$ be vectors with the respective eigenvalues of the matrices G and H; then the vector $(\boldsymbol{\lambda}_G \otimes \boldsymbol{\lambda}_H)$ contains exactly the eigenvalues of the matrix $(G \otimes H)$. $\hspace{4cm}$ (A.42)

(13) $\operatorname{tr}(G \otimes H) = \operatorname{tr} G \operatorname{tr} H$ $\hspace{7cm}$ (A.43)

(14) G and H positive (semi)definite $\Rightarrow G \otimes H$ positive (semi)definite $\hspace{0.5cm}$ (A.44)

The four fundamental matrix subspaces Let A be a matrix of size $n \times m$ with $\operatorname{rk} A =: q$. The four fundamental matrix subspaces are

The *column space* of A (also range of A) is denoted by $\mathcal{R}(A)$.

The *nullspace* of A (also the kernel of A) is denoted by $\mathcal{N}(A)$.

The *row space* of A, which is $\mathcal{R}(A^T)$.

The *left nullspace* of A, which is $\mathcal{N}(A^T)$.

The subspaces are elements of larger spaces, the sizes of which are determined by the dimension of A.

$$\mathcal{N}(A) \subset \mathbb{R}^m, \quad \mathcal{R}(A^T) \subset \mathbb{R}^m$$

$$\mathcal{N}(A^T) \subset \mathbb{R}^n, \quad \mathcal{R}(A) \subset \mathbb{R}^n$$

The dimensions of the subspaces are a function of the rank of A, which we denote by q.

$$\dim \mathcal{R}(A) = q$$

$$\dim \mathcal{N}(A) = m - q \text{ (also called the nullity of } A)$$

$$\dim \mathcal{R}(A^T) = q$$

$$\dim \mathcal{N}(A^T) = n - q$$

In summary, we may write

$$\mathcal{R}(A) = \text{column space of } A; \text{ dimension } q \tag{A.45a}$$
$$\mathcal{N}(A) = \text{nullspace of } A; \text{ dimension } m - q \tag{A.45b}$$
$$\mathcal{R}(A^T) = \text{row space of } A; \text{ dimension } q \tag{A.45c}$$
$$\mathcal{N}(A^T) = \text{left nullspace of } A; \text{ dimension } n - q \tag{A.45d}$$

Derivative of a quadratic form While some authors write the derivative of a quadratic form (a scalar-valued vector function) with respect to a column vector as a row vector, we write such a derivative as a column vector. This is in agreement with the following authors: Grafarend and Schaffrin (1993); Harville (2000, pg. 295); Koch (1999, pg. 69); Lütkepohl (1996, pg. 175); Strang and Borre (1997, pg. 300). For example, given $\boldsymbol{x} \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$, we have

$$\Phi(\boldsymbol{x}) = \boldsymbol{x}^T Q \boldsymbol{x} \Rightarrow \frac{\partial \Phi}{\partial \boldsymbol{x}} = 2Q\boldsymbol{x}. \tag{A.46}$$

Derivatives of the trace   (for additional formulas see Lütkepohl (1996, pp. 177–179))

$$X(m \times n), \; A(n \times m): \; \frac{\partial \operatorname{tr}(AX)}{\partial X} = \frac{\partial \operatorname{tr}(XA)}{\partial X} = A^T \tag{A.47a}$$

$$X(m \times n), \; A(m \times n): \; \frac{\partial \operatorname{tr}(X^T A)}{\partial X} = \frac{\partial \operatorname{tr}(AX^T)}{\partial X} = A \tag{A.47b}$$

$$X(m \times n): \; \frac{\partial \operatorname{tr}(X^T X)}{\partial X} = \frac{\partial \operatorname{tr}(XX^T)}{\partial X} = 2X \tag{A.47c}$$

$$X(m \times n), \; A(m \times m): \; \frac{\partial \operatorname{tr}(X^T AX)}{\partial X} = (A + A^T)X \tag{A.47d}$$

$$X(m \times n), \; A(m \times m) \text{ symmetric:} \; \frac{\partial \operatorname{tr}(X^T AX)}{\partial X} = 2AX \tag{A.47e}$$

$$X(m \times n), \; A(n \times n): \; \frac{\partial \operatorname{tr}(XAX^T)}{\partial X} = X(A + A^T) \tag{A.47f}$$

$$X(m \times n), \; A(n \times n) \text{ symmetric:} \; \frac{\partial \operatorname{tr}(XAX^T)}{\partial X} = 2XA \tag{A.47g}$$

$$X, A(m \times m): \; \frac{\partial \operatorname{tr}(XAX)}{\partial X} = X^T A^T + A^T X^T \tag{A.47h}$$

$$X(m \times n), \; A(p \times m): \; \frac{\partial \operatorname{tr}(AXX^T A^T)}{\partial X} = 2A^T AX \tag{A.47i}$$

See Lütkepohl (1996) for many more useful matrix relationships.

Appendix B

Linearization

A truncated Taylor series is frequently used to *linearize* a nonlinear function. Readers will remember from introductory calculus the Taylor series for a function of one variable. As a review, we present both Taylor's theorem and series, as well as quadratic and linear approximations to functions based on truncations of the series. Then we show the extension of the liner approximation to a multivariate function using matrices.

B.1 Taylor's Theorem and Series for a Function of a Single Variable

If the function f and its first n derivatives f', f'', ..., $f^{(n)}$ are continuous on the interval $[a, b]$ and if $f^{(n)}$ is differentiable on (a, b), then there exists a number c_{n+1} between a and b such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2}(b - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(b - a)^n +$$
$$+ \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(b - a)^{n+1}. \quad \text{(B.1)}$$

Taylor series   The Taylor series itself, for f about $x = a$, is given by

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \ldots \quad \text{(B.2)}$$

Quadratic approximation   A quadratic approximation of $f(x)$ near $x = a$ is

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2, \tag{B.3a}$$

with an error $e_2(x)$ that satisfies

$$\left| e_2(x) \right| \leq \frac{\left| \max f'''(c) \right|}{6} |x - a|^3 , \quad c \text{ between } a \text{ and } x. \tag{B.3b}$$

Linear approximation   Likewise, a linear approximation of $f(x)$ near $x = a$ is

$$f(x) \approx f(a) + f'(a)(x - a), \tag{B.4a}$$

with an error $e_1(x)$ that satisfies

$$\left| e_1(x) \right| \leq \frac{\left| \max f''(c) \right|}{2} (x - a)^2, \quad c \text{ between } a \text{ and } x. \tag{B.4b}$$

B.2   Linearization: A Truncated Taylor's Series for the Multivariate Case

Let $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{\Xi})$ represent an $n \times 1$ set of nonlinear functions of the independent $m \times 1$ vector $\boldsymbol{\Xi}$. Assume that the functions $\boldsymbol{f}$ are continuous over the interval $[\boldsymbol{\Xi}, \boldsymbol{\Xi}_0]$ and that their first derivatives exist over the interval $(\boldsymbol{\Xi}, \boldsymbol{\Xi}_0)$. Then, a linear approximation of $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{\Xi})$ near $\boldsymbol{\Xi} = \boldsymbol{\Xi}_0$ is given by

$$\boldsymbol{y} \approx \boldsymbol{f}(\boldsymbol{\Xi}_0) + \left. \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\Xi}^T} \right|_{\boldsymbol{\Xi}_0} \cdot (\boldsymbol{\Xi} - \boldsymbol{\Xi}_0), \tag{B.5a}$$

which, after introduction of the incremental vector $\boldsymbol{\xi} := \boldsymbol{\Xi} - \boldsymbol{\Xi}_0$ and the $n \times m$ matrix $A := \partial \boldsymbol{f} / \partial \boldsymbol{\Xi}^T$, can be rewritten as

$$\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{\Xi}_0) \approx A\boldsymbol{\xi}. \tag{B.5b}$$

More detailed representations of $\boldsymbol{f}(\boldsymbol{\Xi}_0)$ and A are as follows:

$$\underset{n \times 1}{\boldsymbol{f}(\boldsymbol{\Xi}_0)} = \begin{bmatrix} f_1(\Xi_1^0, \ldots, \Xi_m^0) \\ \vdots \\ f_n(\Xi_1^0, \ldots, \Xi_m^0) \end{bmatrix}, \quad \underset{n \times m}{A} = \begin{bmatrix} \left. \frac{\partial f_1}{\partial \Xi_1} \right|_{\Xi_1^0} & \cdots & \left. \frac{\partial f_1}{\partial \Xi_m} \right|_{\Xi_m^0} \\ \vdots & & \vdots \\ \left. \frac{\partial f_n}{\partial \Xi_1} \right|_{\Xi_1^0} & \cdots & \left. \frac{\partial f_n}{\partial \Xi_m} \right|_{\Xi_m^0} \end{bmatrix}. \tag{B.6}$$

B.2.1 Example

Distances y_1, y_2, and y_3 in the horizontal plane are given from three points with known horizontal coordinates to one new point with unknown horizontal coordinates (u, v). Using (u_1, v_1) as the coordinates of the first known point, etc., and (u_0, v_0) as an *approximation* for the unknown coordinates (u, v), linearize the distance functions $y_1 = f_1(u, v) = \sqrt{(u_1 - u)^2 + (v_1 - v)^2}$, etc.

Solution

$$
\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\boldsymbol{y}} - \underbrace{\begin{bmatrix} \sqrt{(u_1 - u_0)^2 + (v_1 - v_0)^2} \\ \sqrt{(u_2 - u_0)^2 + (v_2 - v_0)^2} \\ \sqrt{(u_3 - u_0)^2 + (v_3 - v_0)^2} \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{\Xi} = \boldsymbol{\Xi}_0)} \approx
$$

$$
\approx \underbrace{\begin{bmatrix} \dfrac{(u_0 - u_1)}{\sqrt{(u_1 - u_0)^2 + (v_1 - v_0)^2}} & \dfrac{(v_0 - v_1)}{\sqrt{(u_1 - u_0)^2 + (v_1 - v_0)^2}} \\ \dfrac{(u_0 - u_2)}{\sqrt{(u_2 - u_0)^2 + (v_2 - v_0)^2}} & \dfrac{(v_0 - v_2)}{\sqrt{(u_2 - u_0)^2 + (v_2 - v_0)^2}} \\ \dfrac{(u_0 - u_3)}{\sqrt{(u_3 - u_0)^2 + (v_3 - v_0)^2}} & \dfrac{(v_0 - v_3)}{\sqrt{(u_3 - u_0)^2 + (v_3 - v_0)^2}} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}}_{\boldsymbol{\xi}}. \quad (B.7)
$$

Note that we have not included a random error vector $\boldsymbol{e}$, as the focus here is on linearization, not modeling of random errors — we did not say that $\boldsymbol{y}$ is a vector of observations; we simply said that it contains three given distances.

Appendix C

Statistical Tables

C.1 Values of the Standard Normal Cumulative Distribution Function

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du = P[Z \leq z]$$
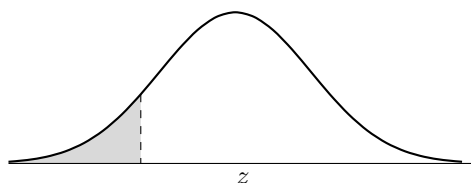
Table C.1: Probabilities $P[Z \leq z]$ computed by the MATLAB function `normcdf(z)` over the domain $[-3.09, 3.09]$ at an interval of 0.01

z	0	1	2	3	4	5	6	7	8	9
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233

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z	0	1	2	3	4	5	6	7	8	9
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

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C.2 Percentiles (Critical Values) of the *t*-Distribution

t_p
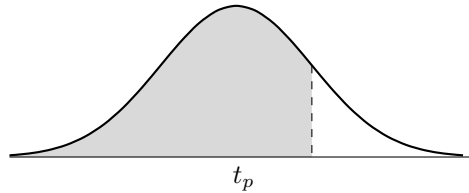
Table C.2: Percentiles (critical values) of the *t*-distribution computed by the MATLAB function `tinv(p,ν)` for percentile p and degrees of freedom ν

ν	$t_{0.55}$	$t_{0.60}$	$t_{0.65}$	$t_{0.70}$	$t_{0.75}$	$t_{0.80}$	$t_{0.85}$
1	0.1584	0.3249	0.5095	0.7265	1.0000	1.376	1.963
2	0.1421	0.2887	0.4447	0.6172	0.8165	1.061	1.386
3	0.1366	0.2767	0.4242	0.5844	0.7649	0.9785	1.250
4	0.1338	0.2707	0.4142	0.5686	0.7407	0.9410	1.190
5	0.1322	0.2672	0.4082	0.5594	0.7267	0.9195	1.156
6	0.1311	0.2648	0.4043	0.5534	0.7176	0.9057	1.134
7	0.1303	0.2632	0.4015	0.5491	0.7111	0.8960	1.119
8	0.1297	0.2619	0.3995	0.5459	0.7064	0.8889	1.108
9	0.1293	0.2610	0.3979	0.5435	0.7027	0.8834	1.100
10	0.1289	0.2602	0.3966	0.5415	0.6998	0.8791	1.093
11	0.1286	0.2596	0.3956	0.5399	0.6974	0.8755	1.088
12	0.1283	0.2590	0.3947	0.5386	0.6955	0.8726	1.083
13	0.1281	0.2586	0.3940	0.5375	0.6938	0.8702	1.079
14	0.1280	0.2582	0.3933	0.5366	0.6924	0.8681	1.076
15	0.1278	0.2579	0.3928	0.5357	0.6912	0.8662	1.074
16	0.1277	0.2576	0.3923	0.5350	0.6901	0.8647	1.071
17	0.1276	0.2573	0.3919	0.5344	0.6892	0.8633	1.069
18	0.1274	0.2571	0.3915	0.5338	0.6884	0.8620	1.067
19	0.1274	0.2569	0.3912	0.5333	0.6876	0.8610	1.066
20	0.1273	0.2567	0.3909	0.5329	0.6870	0.8600	1.064
21	0.1272	0.2566	0.3906	0.5325	0.6864	0.8591	1.063
22	0.1271	0.2564	0.3904	0.5321	0.6858	0.8583	1.061
23	0.1271	0.2563	0.3902	0.5317	0.6853	0.8575	1.060
24	0.1270	0.2562	0.3900	0.5314	0.6848	0.8569	1.059
25	0.1269	0.2561	0.3898	0.5312	0.6844	0.8562	1.058
26	0.1269	0.2560	0.3896	0.5309	0.6840	0.8557	1.058
27	0.1268	0.2559	0.3894	0.5306	0.6837	0.8551	1.057
28	0.1268	0.2558	0.3893	0.5304	0.6834	0.8546	1.056
29	0.1268	0.2557	0.3892	0.5302	0.6830	0.8542	1.055
30	0.1267	0.2556	0.3890	0.5300	0.6828	0.8538	1.055
40	0.1265	0.2550	0.3881	0.5286	0.6807	0.8507	1.050
60	0.1262	0.2545	0.3872	0.5272	0.6786	0.8477	1.045
80	0.1261	0.2542	0.3867	0.5265	0.6776	0.8461	1.043
100	0.1260	0.2540	0.3864	0.5261	0.6770	0.8452	1.042
200	0.1258	0.2537	0.3859	0.5252	0.6757	0.8434	1.039
400	0.1257	0.2535	0.3856	0.5248	0.6751	0.8425	1.038

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ν	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$	$t_{0.9995}$	
600	0.1257	0.2535	0.3855	0.5247	0.6749	0.8422	1.037
800	0.1257	0.2534	0.3855	0.5246	0.6748	0.8421	1.037
1000	0.1257	0.2534	0.3854	0.5246	0.6747	0.8420	1.037
∞	0.1257	0.2533	0.3853	0.5244	0.6745	0.8416	1.036

ν	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$	$t_{0.9995}$	
1	3.078	6.314	12.71	31.82	63.66	36.62	
2	1.886	2.920	4.303	6.965	9.925	31.60	
3	1.638	2.353	3.182	4.541	5.841	12.92	
4	1.533	2.132	2.776	3.747	4.604	8.610	
5	1.476	2.015	2.571	3.365	4.032	6.869	
6	1.440	1.943	2.447	3.143	3.707	5.959	
7	1.415	1.895	2.365	2.998	3.499	5.408	
8	1.397	1.860	2.306	2.896	3.355	5.041	
9	1.383	1.833	2.262	2.821	3.250	4.781	
10	1.372	1.812	2.228	2.764	3.169	4.587	
11	1.363	1.796	2.201	2.718	3.106	4.437	
12	1.356	1.782	2.179	2.681	3.055	4.318	
13	1.350	1.771	2.160	2.650	3.012	4.221	
14	1.345	1.761	2.145	2.624	2.977	4.140	
15	1.341	1.753	2.131	2.602	2.947	4.073	
16	1.337	1.746	2.120	2.583	2.921	4.015	
17	1.333	1.740	2.110	2.567	2.898	3.965	
18	1.330	1.734	2.101	2.552	2.878	3.922	
19	1.328	1.729	2.093	2.539	2.861	3.883	
20	1.325	1.725	2.086	2.528	2.845	3.850	
21	1.323	1.721	2.080	2.518	2.831	3.819	
22	1.321	1.717	2.074	2.508	2.819	3.792	
23	1.319	1.714	2.069	2.500	2.807	3.768	
24	1.318	1.711	2.064	2.492	2.797	3.745	
25	1.316	1.708	2.060	2.485	2.787	3.725	
26	1.315	1.706	2.056	2.479	2.779	3.707	
27	1.314	1.703	2.052	2.473	2.771	3.690	
28	1.313	1.701	2.048	2.467	2.763	3.674	
29	1.311	1.699	2.045	2.462	2.756	3.659	
30	1.310	1.697	2.042	2.457	2.750	3.646	
40	1.303	1.684	2.021	2.423	2.704	3.551	
60	1.296	1.671	2.000	2.390	2.660	3.460	
80	1.292	1.664	1.990	2.374	2.639	3.416	
100	1.290	1.660	1.984	2.364	2.626	3.390	
200	1.286	1.653	1.972	2.345	2.601	3.340	
400	1.284	1.649	1.966	2.336	2.588	3.315	
600	1.283	1.647	1.964	2.333	2.584	3.307	
800	1.283	1.647	1.963	2.331	2.582	3.303	
1000	1.282	1.646	1.962	2.330	2.581	3.300	
∞	1.282	1.645	1.960	2.326	2.576	3.291	

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C.3 Critical Values of the χ^2-Distribution

Table C.3: Critical values of the χ^2-distribution computed by the MATLAB function `chi2inv(1 − α,ν)` for significance level α and degrees of freedom ν

ν	α=0.999	0.995	0.99	0.975	0.95	0.90	0.75	0.50	0.25	0.10	0.05	0.025	0.01	0.005	0.001
1	0.000	0.000	0.000	0.001	0.004	0.016	0.102	0.455	1.323	2.706	3.841	5.024	6.635	7.879	10.828
2	0.002	0.010	0.020	0.051	0.103	0.211	0.575	1.386	2.773	4.605	5.991	7.378	9.210	10.597	13.816
3	0.024	0.072	0.115	0.216	0.352	0.584	1.213	2.366	4.108	6.251	7.815	9.348	11.345	12.838	16.266
4	0.091	0.207	0.297	0.484	0.711	1.064	1.923	3.357	5.385	7.779	9.488	11.143	13.277	14.860	18.467
5	0.210	0.412	0.554	0.831	1.145	1.610	2.675	4.351	6.626	9.236	11.070	12.833	15.086	16.750	20.515
6	0.381	0.676	0.872	1.237	1.635	2.204	3.455	5.348	7.841	10.645	12.592	14.449	16.812	18.548	22.458
7	0.598	0.989	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.017	14.067	16.013	18.475	20.278	24.322
8	0.857	1.344	1.646	2.180	2.733	3.490	5.071	7.344	10.219	13.362	15.507	17.535	20.090	21.955	26.124
9	1.152	1.735	2.088	2.700	3.325	4.168	5.899	8.343	11.389	14.684	16.919	19.023	21.666	23.589	27.877
10	1.479	2.156	2.558	3.247	3.940	4.865	6.737	9.342	12.549	15.987	18.307	20.483	23.209	25.188	29.588
11	1.834	2.603	3.053	3.816	4.575	5.578	7.584	10.341	13.701	17.275	19.675	21.920	24.725	26.757	31.264
12	2.214	3.074	3.571	4.404	5.226	6.304	8.438	11.340	14.845	18.549	21.026	23.337	26.217	28.300	32.909
13	2.617	3.565	4.107	5.009	5.892	7.042	9.299	12.340	15.984	19.812	22.362	24.736	27.688	29.819	34.528
14	3.041	4.075	4.660	5.629	6.571	7.790	10.165	13.339	17.117	21.064	23.685	26.119	29.141	31.319	36.123
15	3.483	4.601	5.229	6.262	7.261	8.547	11.037	14.339	18.245	22.307	24.996	27.488	30.578	32.801	37.697
16	3.942	5.142	5.812	6.908	7.962	9.312	11.912	15.338	19.369	23.542	26.296	28.845	32.000	34.267	39.252
17	4.416	5.697	6.408	7.564	8.672	10.085	12.792	16.338	20.489	24.769	27.587	30.191	33.409	35.718	40.790
18	4.905	6.265	7.015	8.231	9.390	10.865	13.675	17.338	21.605	25.989	28.869	31.526	34.805	37.156	42.312
19	5.407	6.844	7.633	8.907	10.117	11.651	14.562	18.338	22.718	27.204	30.144	32.852	36.191	38.582	43.820
20	5.921	7.434	8.260	9.591	10.851	12.443	15.452	19.337	23.828	28.412	31.410	34.170	37.566	39.997	45.315
21	6.447	8.034	8.897	10.283	11.591	13.240	16.344	20.337	24.935	29.615	32.671	35.479	38.932	41.401	46.797
22	6.983	8.643	9.542	10.982	12.338	14.041	17.240	21.337	26.039	30.813	33.924	36.781	40.289	42.796	48.268
23	7.529	9.260	10.196	11.689	13.091	14.848	18.137	22.337	27.141	32.007	35.172	38.076	41.638	44.181	49.728
24	8.085	9.886	10.856	12.401	13.848	15.659	19.037	23.337	28.241	33.196	36.415	39.364	42.980	45.559	51.179
25	8.649	10.520	11.524	13.120	14.611	16.473	19.939	24.337	29.339	34.382	37.652	40.646	44.314	46.928	52.620
26	9.222	11.160	12.198	13.844	15.379	17.292	20.843	25.336	30.435	35.563	38.885	41.923	45.642	48.290	54.052
27	9.803	11.808	12.879	14.573	16.151	18.114	21.749	26.336	31.528	36.741	40.113	43.195	46.963	49.645	55.476
28	10.391	12.461	13.565	15.308	16.928	18.939	22.657	27.336	32.620	37.916	41.337	44.461	48.278	50.993	56.892
29	10.986	13.121	14.256	16.047	17.708	19.768	23.567	28.336	33.711	39.087	42.557	45.722	49.588	52.336	58.301
30	11.588	13.787	14.953	16.791	18.493	20.599	24.478	29.336	34.800	40.256	43.773	46.979	50.892	53.672	59.703
31	12.196	14.458	15.655	17.539	19.281	21.434	25.390	30.336	35.887	41.422	44.985	48.232	52.191	55.003	61.098
32	12.811	15.134	16.362	18.291	20.072	22.271	26.304	31.336	36.973	42.585	46.194	49.480	53.486	56.328	62.487
33	13.431	15.815	17.074	19.047	20.867	23.110	27.219	32.336	38.058	43.745	47.400	50.725	54.776	57.648	63.870
34	14.057	16.501	17.789	19.806	21.664	23.952	28.136	33.336	39.141	44.903	48.602	51.966	56.061	58.964	65.247
35	14.688	17.192	18.509	20.569	22.465	24.797	29.054	34.336	40.223	46.059	49.802	53.203	57.342	60.275	66.619
36	15.324	17.887	19.233	21.336	23.269	25.643	29.973	35.336	41.304	47.212	50.998	54.437	58.619	61.581	67.985
37	15.965	18.586	19.960	22.106	24.075	26.492	30.893	36.336	42.383	48.363	52.192	55.668	59.893	62.883	69.346
38	16.611	19.289	20.691	22.878	24.884	27.343	31.815	37.335	43.462	49.513	53.384	56.896	61.162	64.181	70.703
39	17.262	19.996	21.426	23.654	25.695	28.196	32.737	38.335	44.539	50.660	54.572	58.120	62.428	65.476	72.055
40	17.916	20.707	22.164	24.433	26.509	29.051	33.660	39.335	45.616	51.805	55.758	59.342	63.691	66.766	73.402
41	18.575	21.421	22.906	25.215	27.326	29.907	34.585	40.335	46.692	52.949	56.942	60.561	64.950	68.053	74.745
42	19.239	22.138	23.650	25.999	28.144	30.765	35.510	41.335	47.766	54.090	58.124	61.777	66.206	69.336	76.084
43	19.906	22.859	24.398	26.785	28.965	31.625	36.436	42.335	48.840	55.230	59.304	62.990	67.459	70.616	77.419
44	20.576	23.584	25.148	27.575	29.787	32.487	37.363	43.335	49.913	56.369	60.481	64.201	68.710	71.893	78.750
45	21.251	24.311	25.901	28.366	30.612	33.350	38.291	44.335	50.985	57.505	61.656	65.410	69.957	73.166	80.077
46	21.929	25.041	26.657	29.160	31.439	34.215	39.220	45.335	52.056	58.641	62.830	66.617	71.201	74.437	81.400

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47	22.610	25.775	27.416	29.956	32.268	35.081	40.149	46.335	53.127	59.774	64.001	67.821	72.443	75.704	82.720
48	23.295	26.511	28.177	30.755	33.098	35.949	41.079	47.335	54.196	60.907	65.171	69.023	73.683	76.969	84.037
49	23.983	27.249	28.941	31.555	33.930	36.818	42.010	48.335	55.265	62.038	66.339	70.222	74.919	78.231	85.351
50	24.674	27.991	29.707	32.357	34.764	37.689	42.942	49.335	56.334	63.167	67.505	71.420	76.154	79.490	86.661
60	31.738	35.534	37.485	40.482	43.188	46.459	52.294	59.335	66.981	74.397	79.082	83.298	88.379	91.952	99.607
70	39.036	43.275	45.442	48.758	51.739	55.329	61.698	69.334	77.577	85.527	90.531	95.023	100.43	104.22	112.32
80	46.520	51.172	53.540	57.153	60.391	64.278	71.145	79.334	88.130	96.578	101.88	106.63	112.33	116.32	124.84
90	54.155	59.196	61.754	65.647	69.126	73.291	80.625	89.334	98.650	107.57	113.15	118.14	124.12	128.30	137.21
100	61.918	67.328	70.065	74.222	77.929	82.358	90.133	99.334	109.14	118.50	124.34	129.56	135.81	140.17	149.45

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C.4 Critical Values of the **F**-Distribution

The tables below list critical values of the F-distribution computed by the MATLAB function `finv(1 − α,r₁,r₂)` for level of significance α and degrees of freedom $r_1 = \{1, 2, 3\}$ and r_2, where $\alpha(2)$ pertains to two tails and $\alpha(1)$ pertains to a single tail. The critical values for ∞ were generated by `finv(1 − α,r₁,1.0e6)`.

Table C.4: Critical values of the F-distribution with numerator degrees of freedom $r_1 = 1$

$\alpha(2)$:	0.5000	0.2000	0.1000	0.0500	0.0200	0.0100	0.0050	0.0020	0.0010
$\alpha(1)$:	0.2500	0.1000	0.0500	0.0250	0.0100	0.0050	0.0025	0.0010	0.0005
r_2									
1	5.828	39.86	161.4	647.8	4052.	16210.	64840	405400.	1621000.
2	2.571	8.526	18.51	38.51	98.50	198.5	398.5	998.5	1998.
3	2.024	5.538	10.13	17.44	34.12	55.55	89.58	167.0	266.5
4	1.807	4.545	7.709	12.22	21.20	31.33	45.67	74.14	106.2
5	1.692	4.060	6.608	10.01	16.26	22.78	31.41	47.18	63.61
6	1.621	3.776	5.987	8.813	13.75	18.63	24.81	35.51	46.08
7	1.573	3.589	5.591	8.073	12.25	16.24	21.11	29.25	36.99
8	1.538	3.458	5.318	7.571	11.26	14.69	18.78	25.41	31.56
9	1.512	3.360	5.117	7.209	10.56	13.61	17.19	22.86	27.99
10	1.491	3.285	4.965	6.937	10.04	12.83	16.04	21.04	25.49
11	1.475	3.225	4.844	6.724	9.646	12.23	15.17	19.69	23.65
12	1.461	3.177	4.747	6.554	9.330	11.75	14.49	18.64	22.24
13	1.450	3.136	4.667	6.414	9.074	11.37	13.95	17.82	21.14
14	1.440	3.102	4.600	6.298	8.862	11.06	13.50	17.14	20.24
15	1.432	3.073	4.543	6.200	8.683	10.80	13.13	16.59	19.51
16	1.425	3.048	4.494	6.115	8.531	10.58	12.82	16.12	18.89
17	1.419	3.026	4.451	6.042	8.400	10.38	12.55	15.72	18.37
18	1.413	3.007	4.414	5.978	8.285	10.22	12.32	15.38	17.92
19	1.408	2.990	4.381	5.922	8.185	10.07	12.12	15.08	17.53
20	1.404	2.975	4.351	5.871	8.096	9.944	11.94	14.82	17.19
21	1.400	2.961	4.325	5.827	8.017	9.830	11.78	14.59	16.89
22	1.396	2.949	4.301	5.786	7.945	9.727	11.64	14.38	16.62
23	1.393	2.937	4.279	5.750	7.881	9.635	11.51	14.20	16.38
24	1.390	2.927	4.260	5.717	7.823	9.551	11.40	14.03	16.17
25	1.387	2.918	4.242	5.686	7.770	9.475	11.29	13.88	15.97
26	1.384	2.909	4.225	5.659	7.721	9.406	11.20	13.74	15.79
27	1.382	2.901	4.210	5.633	7.677	9.342	11.11	13.61	15.63
28	1.380	2.894	4.196	5.610	7.636	9.284	11.03	13.50	15.48
29	1.378	2.887	4.183	5.588	7.598	9.230	10.96	13.39	15.35
30	1.376	2.881	4.171	5.568	7.562	9.180	10.89	13.29	15.22
35	1.368	2.855	4.121	5.485	7.419	8.976	10.61	12.90	14.72
40	1.363	2.835	4.085	5.424	7.314	8.828	10.41	12.61	14.35
45	1.358	2.820	4.057	5.377	7.234	8.715	10.26	12.39	14.08
50	1.355	2.809	4.034	5.340	7.171	8.626	10.14	12.22	13.86
60	1.349	2.791	4.001	5.286	7.077	8.495	9.962	11.97	13.55
70	1.346	2.779	3.978	5.247	7.011	8.403	9.838	11.80	13.33
80	1.343	2.769	3.960	5.218	6.963	8.335	9.747	11.67	13.17
90	1.341	2.762	3.947	5.196	6.925	8.282	9.677	11.57	13.05
100	1.339	2.756	3.936	5.179	6.895	8.241	9.621	11.50	12.95
120	1.336	2.748	3.920	5.152	6.851	8.179	9.539	11.38	12.80
140	1.334	2.742	3.909	5.134	6.819	8.135	9.480	11.30	12.70

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$\alpha(2)$:	0.5000	0.2000	0.1000	0.0500	0.0200	0.0100	0.0050	0.0020	0.0010
$\alpha(1)$:	0.2500	0.1000	0.0500	0.0250	0.0100	0.0050	0.0025	0.0010	0.0005
r_2									
160	1.333	2.737	3.900	5.120	6.796	8.102	9.437	11.24	12.63
180	1.332	2.734	3.894	5.109	6.778	8.077	9.403	11.19	12.57
200	1.331	2.731	3.888	5.100	6.763	8.057	9.377	11.15	12.52
300	1.328	2.722	3.873	5.075	6.720	7.997	9.297	11.04	12.38
500	1.326	2.716	3.860	5.054	6.686	7.950	9.234	10.96	12.28
∞	1.323	2.706	3.841	5.024	6.635	7.879	9.141	10.83	12.12

Continued from previous page, $r_1 = 1$

Table C.5: Critical values of the F-distribution with numerator degrees of freedom $r_1 = 2$

$\alpha(2)$:	0.5000	0.2000	0.1000	0.0500	0.0200	0.0100	0.0050	0.0020	0.0010
$\alpha(1)$:	0.2500	0.1000	0.0500	0.0250	0.0100	0.0050	0.0025	0.0010	0.0005
r_2									
1	7.500	49.50	199.5	799.5	5000.	20000.	80000.	500000.	2000000.
2	3.000	9.000	19.00	39.00	99.00	199.0	399.0	999.0	1999.0
3	2.280	5.462	9.552	16.04	30.82	49.80	79.93	148.5	236.6
4	2.000	4.325	6.944	10.65	18.00	26.28	38.00	61.25	87.44
5	1.853	3.780	5.786	8.434	13.27	18.31	24.96	37.12	49.78
6	1.762	3.463	5.143	7.260	10.92	14.54	19.10	27.00	34.80
7	1.701	3.257	4.737	6.542	9.547	12.40	15.89	21.69	27.21
8	1.657	3.113	4.459	6.059	8.649	11.04	13.89	18.49	22.75
9	1.624	3.006	4.256	5.715	8.022	10.11	12.54	16.39	19.87
10	1.598	2.924	4.103	5.456	7.559	9.427	11.57	14.91	17.87
11	1.577	2.860	3.982	5.256	7.206	8.912	10.85	13.81	16.41
12	1.560	2.807	3.885	5.096	6.927	8.510	10.29	12.97	15.30
13	1.545	2.763	3.806	4.965	6.701	8.186	9.839	12.31	14.43
14	1.533	2.726	3.739	4.857	6.515	7.922	9.475	11.78	13.73
15	1.523	2.695	3.682	4.765	6.359	7.701	9.173	11.34	13.16
16	1.514	2.668	3.634	4.687	6.226	7.514	8.918	10.97	12.69
17	1.506	2.645	3.592	4.619	6.112	7.354	8.701	10.66	12.29
18	1.499	2.624	3.555	4.560	6.013	7.215	8.513	10.39	11.94
19	1.493	2.606	3.522	4.508	5.926	7.093	8.349	10.16	11.64
20	1.487	2.589	3.493	4.461	5.849	6.986	8.206	9.953	11.38
21	1.482	2.575	3.467	4.420	5.780	6.891	8.078	9.772	11.16
22	1.477	2.561	3.443	4.383	5.719	6.806	7.965	9.612	10.95
23	1.473	2.549	3.422	4.349	5.664	6.730	7.863	9.469	10.77
24	1.470	2.538	3.403	4.319	5.614	6.661	7.771	9.339	10.61
25	1.466	2.528	3.385	4.291	5.568	6.598	7.687	9.223	10.46
26	1.463	2.519	3.369	4.265	5.526	6.541	7.611	9.116	10.33
27	1.460	2.511	3.354	4.242	5.488	6.489	7.542	9.019	10.21
28	1.457	2.503	3.340	4.221	5.453	6.440	7.478	8.931	10.09
29	1.455	2.495	3.328	4.201	5.420	6.396	7.419	8.849	9.992
30	1.452	2.489	3.316	4.182	5.390	6.355	7.365	8.773	9.898
35	1.443	2.461	3.267	4.106	5.268	6.188	7.145	8.470	9.519
40	1.435	2.440	3.232	4.051	5.179	6.066	6.986	8.251	9.247
45	1.430	2.425	3.204	4.009	5.110	5.974	6.865	8.086	9.042
50	1.425	2.412	3.183	3.975	5.057	5.902	6.770	7.956	8.883
60	1.419	2.393	3.150	3.925	4.977	5.795	6.632	7.768	8.651

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$\alpha(2)$:	0.5000	0.2000	0.1000	0.0500	0.0200	0.0100	0.0050	0.0020	0.0010
$\alpha(1)$:	0.2500	0.1000	0.0500	0.0250	0.0100	0.0050	0.0025	0.0010	0.0005
r_2									
70	1.414	2.380	3.128	3.890	4.922	5.720	6.535	7.637	8.489
80	1.411	2.370	3.111	3.864	4.881	5.665	6.463	7.540	8.371
90	1.408	2.363	3.098	3.844	4.849	5.623	6.409	7.466	8.281
100	1.406	2.356	3.087	3.828	4.824	5.589	6.365	7.408	8.209
120	1.402	2.347	3.072	3.805	4.787	5.539	6.301	7.321	8.103
140	1.400	2.341	3.061	3.788	4.760	5.504	6.255	7.260	8.029
160	1.398	2.336	3.053	3.775	4.740	5.478	6.222	7.215	7.974
180	1.397	2.332	3.046	3.766	4.725	5.457	6.195	7.180	7.931
200	1.396	2.329	3.041	3.758	4.713	5.441	6.175	7.152	7.897
300	1.393	2.320	3.026	3.735	4.677	5.393	6.113	7.069	7.797
500	1.390	2.313	3.014	3.716	4.648	5.355	6.064	7.004	7.718
∞	1.386	2.303	2.996	3.689	4.605	5.298	5.992	6.908	7.601

Continued from previous page, $r_1 = 2$

Table C.6: Critical values of the F-distribution with numerator degrees of freedom $r_1 = 3$

$\alpha(2)$:	0.5000	0.2000	0.1000	0.0500	0.0200	0.0100	0.0050	0.0020	0.0010
$\alpha(1)$:	0.2500	0.1000	0.0500	0.0250	0.0100	0.0050	0.0025	0.0010	0.0005
r_2									
1	8.200	53.59	215.7	864.2	5403.0	21610.	86460.	540400.2	2162000.
2	3.153	9.162	19.16	39.17	99.17	199.2	399.2	999.2	1999.
3	2.356	5.391	9.277	15.44	29.46	47.47	76.06	141.1	224.7
4	2.047	4.191	6.591	9.979	16.69	24.26	34.96	56.18	80.09
5	1.884	3.619	5.409	7.764	12.06	16.53	22.43	33.20	44.42
6	1.784	3.289	4.757	6.599	9.780	12.92	16.87	23.70	30.45
7	1.717	3.074	4.347	5.890	8.451	10.88	13.84	18.77	23.46
8	1.668	2.924	4.066	5.416	7.591	9.596	11.98	15.83	19.39
9	1.632	2.813	3.863	5.078	6.992	8.717	10.73	13.90	16.77
10	1.603	2.728	3.708	4.826	6.552	8.081	9.833	12.55	14.97
11	1.580	2.660	3.587	4.630	6.217	7.600	9.167	11.56	13.65
12	1.561	2.606	3.490	4.474	5.953	7.226	8.652	10.80	12.66
13	1.545	2.560	3.411	4.347	5.739	6.926	8.242	10.21	11.89
14	1.532	2.522	3.344	4.242	5.564	6.680	7.910	9.729	11.27
15	1.520	2.490	3.287	4.153	5.417	6.476	7.634	9.335	10.76
16	1.510	2.462	3.239	4.077	5.292	6.303	7.403	9.006	10.34
17	1.502	2.437	3.197	4.011	5.185	6.156	7.205	8.727	9.989
18	1.494	2.416	3.160	3.954	5.092	6.028	7.035	8.487	9.686
19	1.487	2.397	3.127	3.903	5.010	5.916	6.887	8.280	9.424
20	1.481	2.380	3.098	3.859	4.938	5.818	6.757	8.098	9.196
21	1.475	2.365	3.072	3.819	4.874	5.730	6.642	7.938	8.994
22	1.470	2.351	3.049	3.783	4.817	5.652	6.539	7.796	8.816
23	1.466	2.339	3.028	3.750	4.765	5.582	6.447	7.669	8.657
24	1.462	2.327	3.009	3.721	4.718	5.519	6.364	7.554	8.515
25	1.458	2.317	2.991	3.694	4.675	5.462	6.289	7.451	8.386
26	1.454	2.307	2.975	3.670	4.637	5.409	6.220	7.357	8.269
27	1.451	2.299	2.960	3.647	4.601	5.361	6.158	7.272	8.163
28	1.448	2.291	2.947	3.626	4.568	5.317	6.100	7.193	8.066
29	1.445	2.283	2.934	3.607	4.538	5.276	6.048	7.121	7.977

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$\alpha(2)$:	0.5000	0.2000	0.1000	0.0500	0.0200	0.0100	0.0050	0.0020	0.0010
$\alpha(1)$:	0.2500	0.1000	0.0500	0.0250	0.0100	0.0050	0.0025	0.0010	0.0005
r_2									
30	1.443	2.276	2.922	3.589	4.510	5.239	5.999	7.054	7.894
35	1.432	2.247	2.874	3.517	4.396	5.086	5.802	6.787	7.565
40	1.424	2.226	2.839	3.463	4.313	4.976	5.659	6.595	7.329
45	1.418	2.210	2.812	3.422	4.249	4.892	5.551	6.450	7.151
50	1.413	2.197	2.790	3.390	4.199	4.826	5.466	6.336	7.013
60	1.405	2.177	2.758	3.343	4.126	4.729	5.343	6.171	6.812
70	1.400	2.164	2.736	3.309	4.074	4.661	5.256	6.057	6.673
80	1.396	2.154	2.719	3.284	4.036	4.611	5.193	5.972	6.571
90	1.393	2.146	2.706	3.265	4.007	4.573	5.144	5.908	6.493
100	1.391	2.139	2.696	3.250	3.984	4.542	5.105	5.857	6.432
120	1.387	2.130	2.680	3.227	3.949	4.497	5.048	5.781	6.341
140	1.385	2.123	2.669	3.211	3.925	4.465	5.008	5.728	6.277
160	1.383	2.118	2.661	3.199	3.906	4.441	4.977	5.689	6.230
180	1.381	2.114	2.655	3.189	3.892	4.423	4.954	5.658	6.193
200	1.380	2.111	2.650	3.182	3.881	4.408	4.936	5.634	6.164
300	1.377	2.102	2.635	3.160	3.848	4.365	4.881	5.562	6.078
500	1.374	2.095	2.623	3.142	3.821	4.330	4.838	5.506	6.010
∞	1.369	2.084	2.605	3.116	3.782	4.279	4.773	5.422	5.910

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