# Gravity Recovery by Kinematic State Vector Perturbation from Satellite-to-Satellite Tracking for GRACE-like Orbits over Long Arcs 

by

## Nlingilili Habana



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Geodetic Science
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## Preface

This report was prepared for and submitted to the Graduate School of The Ohio State University as a dissertation in partial fulfillment of the requirements for the Degree of Doctor of Philosophy


#### Abstract

To improve on the understanding of Earth dynamics, a perturbation theory aimed at geopotential recovery, based on purely kinematic state vectors, is implemented. The method was originally proposed in the study by Xu (2008). It is a perturbation method based on Cartesian coordinates that is not subject to singularities that burden most conventional methods of gravity recovery from satellite-to-satellite tracking. The principal focus of the theory is to make the gravity recovery process more efficient, for example, by reducing the number of nuisance parameters associated with arc endpoint conditions in the estimation process. The theory aims to do this by maximizing the benefits of pure kinematic tracking by GNSS over long arcs. However, the practical feasibility of this theory has never been tested numerically.

In this study, the formulation of the perturbation theory is first modified to make it numerically practicable. It is then shown, with realistic simulations, that Xu's original goal of an iterative solution is not achievable under the constraints imposed by numerical integration error. As such, a non-iterative alternative approach is implemented, instead. Finally, the principles of this modified procedure are applied to the Schneider (1968) model, improving the original model by an order of magnitude for high-low satellite-tosatellite tracking (SST). The new model is also adapted to the processing of low-low SST, and a combination thereof, i.e. GRACE-like missions. In validating the linearized model for multiple-day-long arcs, it is revealed (through simulated GRACE-like orbits) to be at least as accurate as (or in some cases better than) the GRACE K-band range-rate nominal precision of $0.1 \mu \mathrm{~m} / \mathrm{s}$. Further application of the model to simulated recovery of spherical harmonic coefficients is shown to achieve accuracies commensurate to other models in practice today.


## Acknowledgements

First and foremost, I convey my sincerest gratitude and appreciation to my advisor, Prof. Christopher Jekeli. His mentorship these past five years came not only with immense intellectual insight on geodesy and mathematics, but also with humility and patience. His door was always open to help with the many hurdles along this journey. For these, and far more I could never possibly exhaust in a single writing, I am grateful. Prof. C.K. Shum, has always been gracious with his time towards me. I thank him for his continuous motivation and overall assistance. I also wish to sincerely thank Dr. Michael Durand and Dr. Steven Lower for their constructive reviews and edits regarding this report. Lectures from Dr. Burkhard Schaffrin were paramount to the statistical analysis found in this research.

I wish to also thank my colleagues in Prof. Shum's laboratory, specifically: Yu Zhang, for assisting in some of the testing of the integrator accuracy, and Chaoyang Zhang, for his assistance in reviewing sections of this report. Furthermore, I wish to thank my officemates for the various fruitful discussion we have had regarding this research, throughout my tenure as a graduate student. Lastly, I am forever indebted to my family and friends all around the globe; your daily messages of nurture were the fuel that got me here.

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## Chapter 1 Introduction

An understanding of our physical Earth along with its shape and size is the foundation of geodesy and geophysics. One path to this understanding has been through the improvement of gravity field recovery methods. It is now a well-established fact that the motion of low Earth orbiting (LEO) satellites is highly dependent on the global mass distribution within Earth's system. Courtesy of this knowledge, modern advancements in satellite-to-satellite tracking (SST) allow for periodically-deduced models of the geopotential to be used to monitor the mass variations of the Earth's surface (see, e.g., (Harig and Simons 2015); (Humphrey, et al. 2016); (Chambers, et al. 2017)). There have even been multiple satellite missions dedicated to Earth's gravity recovery, with another still in orbit at the time of this writing (see Section 1.2.). Improving our knowledge of the gravity field has inevitably been highly beneficial to other fields, including, but not limited to navigation systems, space exploration, hydrology, and climatology. However, to appreciate the current state of gravity recovery, it is important to give a brief history remarking on some of the literature and events that led to the current state of gravity recovery, and finally elaborate on gravity recovery missions past and present.

### 1.1. Historical Background of Gravity Recovery

Perhaps, the most important writing pertaining to celestial mechanics is Kepler's Astronomia nova (1609). It is here, where Kepler notes his laws of planetary motion that he derived from studying the motion of Mars. However, when it comes to studies of the Earth and its gravity field, few studies have had as much impact as Galilei's De motu Antiquiora (1687) and Newton's Philosophice Naturalis Principia Mathematica (1687). Galilei's book thoroughly investigates the motion of falling bodies using the scientific method, as it is understood today. Principia, gives us Newton's laws of motion and gravitation, the latter of which was derived based on Kepler's laws. Subsequently, other contributors have built on Newton's theory of gravitation, notably: (i) Laplace's Mecanique Celeste (1799-1825), applied the three-body problem to gravity, as opposed to Newton's two-body problem; (ii) Lagrange's Mecanique Analytique (1788-1789), developed the principle of spherical harmonics; and (iii) Einstein's Generalized Theory of Relativity (1914), moved from the aforementioned classical approaches and described gravitation as a geometric property of space-time.

As a result of these pioneers, and many other contributors this writing could never exhaust, we were able to launch artificial satellites, and with that began the space age. Shortly afterwards the satellite's orbits where being analyzed to deduce some of the main characteristics of the Earth's gravitational field (see, (Buchar 1958); (O'Keefe, et al.
1959)). One of the most prominent publications to come out during this era of gravity recovery was Kaula's Theory of Satellite Geodesy (1966), in which perturbation theory was elaborated by expressing orbital perturbation in terms of geopotential coefficients. Kaula also developed an analytic expression for a satellite's motion due to non-central terms of the geopotential. In fact, some of the current forms of gravity recovery are still based on refined versions of Kaula's model. It is important to note that during this period the satellite tracking was done using ground stations that were irregularly distributed and not global (Corliss 1967). Arguably, the biggest breakthrough in gravity recovery came with the use of the Global Positioning System (GPS) to track LEO satellites. The tracking was now, not only continuous but also global. This eventually resulted in some of the initial global high degree Earth gravitational models using GPS (Tapley, et al. 1996).

Wolff (1969) improved on a gravitational recovery model first suggested in O'Keefe (1957) by explicitly proposing an in situ observation technique for using the principle of conservation of mechanical energy. Such an approach, known as the energy balance method, was finally realized with the launch of the Challenging Minisatellite Payload (CHAMP) (Reigber, et al. 2003). Another approach to gravity recovery proposed during the space age is the gravity gradient method, Colombo (1986). This method also uses in situ measurements along a satellite's orbit to solve for the gravitational field (Koop 1993). The gradiometry approach was the basis of the Gravity Field and Steady-State Ocean Circulation Explorer (GOCE) satellite mission (Drinkwater, et al. 2003). However, the aspect of determining the gravity field from satellite tracking (rather than gravity gradiometry) is largely made possible by Global Navigation Satellite Systems (GNSS) (specifically GPS) tracking that is virtually continuous.

### 1.2. Dedicated Gravity Missions

The new millennium has seen a culmination of the aforementioned groundwork into four dedicated gravity missions, namely: CHAMP, Gravity Recovery and Climate Experiment (GRACE), GRACE-Follow On (GRACE-FO), and GOCE. These missions have proven invaluable in gravity recovery, with each improving on the last one and furthering our understanding of climate change and Earth dynamics. Each mission not only came with advanced instrumentation, but in the case of GOCE used a different set of observations for mapping the gravity field.

CHAMP was the first mission dedicated towards recovery of the Earth's gravitational field; it was commissioned by the German Aerospace Agency (DLR, née DARA) in 1995 and launched in 2000 under the management of the German research center (GFZ) (Reigber, et al. 2003). The satellite had a near circular, almost polar, and low altitude ( 450 km ) orbit. The mission was given a nominal lifespan of five years but ended up in orbit for a little over a decade. This lifespan exceedance definitely paved the way for future gravity dedicated missions. CHAMP based gravity deductions on the satellite's position derived from onboard equipment, including: high-precision GPS receivers for high-low SST (hl-SST) data, accelerometer for non-gravitational acceleration, retroreflectors for ground based satellite laser ranging (SLR) and a pair of star sensors for orientation of the vector instruments (Liu 2008). Despite its groundbreaking work on
improving our knowledge of the static gravity field, CHAMP was unable to provide timevariable gravity field results as was initially stated in the mission statement. This task was then carried on to the next gravity dedicated mission, GRACE.

The GRACE project is a joint venture between the National Aeronautics and Space Administration (NASA) and DLR. It consists of a pair of satellites on similar trajectories committed to high-precision mapping of the Earth's gravity field, mostly emphasizing its variability over time (Tapley, et al. 2004a). The first pair of satellites on this project were launched in 2002, with a nominal lifespan of five years but ended up surpassing that, threefold. Comparable to CHAMP, the pair had a: near-polar inclination, near-circular eccentricity, and low altitude ( 500 km ), they also had a nominal separation of about 220 km . The satellites were each equipped with a K/Ka-band ranging (KBR) system for lowlow SST (ll-SST). This allowed for continuous tracking of the inter-satellite ranging between the two satellites with precision of micron level. There were also high-precision GPS receivers onboard for hl-SST and synchronizing time tags of the KBR system. To account for some of the dissipative forces, the satellites were also each equipped with a SuperSTAR accelerometer. Each satellite also had a pair of star cameras and magnetic torquers to determine the attitude (McCullough 2017). Due to its high accuracy, ll-SST was effectively used to measure the temporal gravity field during the GRACE mission. In fact, the amalgamation of both low-low and high-low tracking observations has proven to improve the gravity field results than either observation alone (ibid.).

Of all gravity missions, GRACE has, thus far, had the greatest impact on studies of climate change and our understanding of Earth dynamics (see, (Chambers 2008); (Trenberth and Fasullo 2013); (Sakumura, et al. 2014)). During its nearly 16 years in orbit, the mission showed that a significant cause of Earth's temporal gravitation variations is due to the movement of ground water. Over the same period, it also inferred approximately 5000 Gigatons of ice loss from Greenland and Antarctica (Tapley, et al. 2017). Regardless of all these achievements, throughout its lifespan GRACE was contaminated with multiple errors, including systematic errors, that had to be dealt with during post processing to minimize the errors in the final gravity field model (Save, et al. 2012). Addressing these errors was one of the assignments of the recently launched GRACE-FO mission. GRACEFO was launched on May 2018, with initial orbital elements similar to those stated for GRACE (Flechtner, et al. 2014). Each satellite of GRACE-FO was also furnished with more advanced instrumentation from that which was onboard the GRACE pair. Unfortunately initial results did not show much of the anticipated improvement from the GRACE data, mostly due to problems related to the installation of the accelerometers (Landerer, et al. 2018). Though, the accelerometer problems still persist, as of this writing, (temporary) workarounds have been developed to minimize their errors (McCullough, et al. 2019). These include the reconstruction of angular accelerations through a process inherited from GRACE that combines the inertial measurement unit, star camera, and magnetic torquer measurements.

GOCE was a European Space Agency (ESA) mission that launched in March of 2009 (Drinkwater, et al. 2003). The satellite had a near circular, retrograde, and an extremely low altitude ( 250 km ) orbit. The mission was given a life expectancy of 20 months but lasted for nearly 55 months in orbit. Contrary to all aforementioned missions, GOCE used in situ satellite gravity gradiometry (SGG) measurements to deduce the gravity
field. The SGG system consisted of three pairs of highly sensitive 3-axis accelerometers configured around the satellite's center of mass. The satellite was also equipped with a drag-free control system to account for non-gravitational forces, and a high-precision GPS receiver for hl-SST. During its lifespan, GOCE measured highly precise gravity signals at high spatial frequencies (see, e.g., (Baur, et al. 2014); (Wu 2016)).

### 1.3. Motivation, Objectives, and Overview

Geopotential recovery techniques from satellite tracking have traditionally broken the orbit into short arcs to accommodate for the irregular distribution of tracking stations around the globe and constraints associated with linearizing highly non-linear models, among other reasons. This approach though highly acc/urate, is still inefficient compared to the ability to determine the state vector of a satellite continuously by GNSS tracking. It is then that a new approach, of using arcs of arbitrary length, to estimate the geopotential may be considered. Such an approach is suggested mathematically, but in principle only, by Xu (2008), but in this report the formulation is refined in an attempt to make it also numerically practical.

From the start, Xu (2008) performs all derivations in spherical harmonics which inevitably means having to always deal with double summations throughout the development process. In addition, his solution to the equation of motion is given in double integrals. It is emphasized that, neither of these actions is a drawback to the model development process (at least analytically), however it does make it a bit harder to follow, especially when formulating the iterations. Furthermore, this combination of double sums and double integrals is not trivial for applications and just complicates the gravitational estimation problem. Though this report also eventually uses double summations, this extension of the problem towards spherical harmonic parameterization is nevertheless left towards the end of the formulation procedure (Chapter 5). The initial double integral formulation is also avoided altogether, as shown in eq. (2.29). It is noted that, it is not paramount that the estimation of the gravity field be expressed in the form of global spherical harmonics, in fact one could always use any specification applicable to the gravitational potential, for example, Mascons or so-called Slepian functions.

As innovative as the approach proposed by Xu (2008) is, its basic premise is part of a general scheme of recovery techniques known as kinematic integral equation approaches (see Section 2.2.2); these methods generally require time-wise quadrature that are reliant on the solution to orbital arc boundary conditions. Therefore, all things considered, the Xu (2008) model can be reformulated according to other integral equation approaches, specifically the Schneider (1968) method as applied in Mayer-Gürr et al. (2005). It is then noted that the primary focus of this report is the modification of Xu's model towards the aforementioned Mayer-Gürr et al. (2005) approach.

## Chapter 2 Geopotential Theory

The gravitational potential, $V$, of a solid body can be analytically expressed as a harmonic function (Heiskanen and Moritz 1967). For the Earth, this is

$$
\begin{equation*}
V(r, \theta, \lambda)=\frac{G M}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n}\left(\frac{R}{r}\right)^{n+1} C_{n, m} \bar{Y}_{n m}(\theta, \lambda) \tag{2.1}
\end{equation*}
$$

where $(r, \theta, \lambda)$ are the spherical coordinates, namely: radius, co-latitude, and longitude, respectively, GM is the gravitational constant times the Earth's total mass (including the atmosphere), $R$ is an Earth radius (Brillouin sphere radius), $n$ and $m$ are respectively the spherical harmonic degree and order, $C_{n, m}$ is the spherical harmonic coefficient at a specific degree and order, and $\bar{Y}_{n m}(\cdot)$ is the surface spherical harmonic function defined as (Jekeli 2007, p. 22),

$$
\bar{Y}_{n m}(\theta, \lambda)=\bar{P}_{n|m|}(\cos \theta) \begin{cases}\cos (m \lambda), & m \geq 0  \tag{2.2}\\ \sin (|m| \lambda), & m<0\end{cases}
$$

where $\bar{P}_{n m}(\cdot)$ is the fully normalized associated Legendre function. Solving the spherical harmonic coefficients using measurements related to the potential is the usual premise of gravity recovery techniques (Cunningham 1970). In practice one cannot resolve the spherical harmonics to an infinite spatial resolution, as such, solutions truncate the harmonic degrees to a finite value, $n_{\max }$. Furthermore, some low degree harmonics are well determined and can be defined up to $n_{r e f}$. The zeroth-degree harmonic is related to the central source point, and is conventionally set to one, while the first-degree harmonics are related to the geo-center coordinates, and are often set to zero by convention of the coordinate system (but can be deduced from accurate tracking if the satellite and conventional systems are displaced from each other). Another harmonic of note is the second zonal harmonic, $C_{2,0}$, which indicates Earth's equatorial bulge and is also the largest (numerically) by at least three orders of magnitude. Most spherical harmonic estimate applications will just set $n_{r e f}=1$. However, it is noted that this report is not concerned with improving those (low degree) harmonics that are sufficiently determined from ground tracking data (e.g., LAGEOS). Eq. (2.1) is then further truncated between $n_{\text {max }}$ and $n_{r e f}+$ 1. For any given model, the number of unknown coefficients, is then

$$
\begin{equation*}
N_{\text {coef }}:=\left(n_{\max }+1\right)^{2}-\left(n_{\text {ref }}+1\right)^{2} \tag{2.3}
\end{equation*}
$$

It is important to note that generally the gravitational potential, $V$, cannot be measured directly. In lieu of observing the gravitational potential, it (or its spectrum) can be expressed in terms of position, velocity, and/or acceleration (all of which are technically observable). There are generally two schools of thought on how to use these observations to recover the spherical harmonic coefficients (Rummel, et al. 1993). One view, known as the space-wise technique, formulates the problem as solving for a function whose values are spatially and (usually) globally distributed on a set of well-defined points. The other, known as the time-wise technique, solves for the field parameters on the basis of observed quantities related to the solution of the equations of satellite motion.

In this chapter, a relationship between spherical and Cartesian coordinates is developed. This is done to better show how the gradient operator, $\nabla_{\boldsymbol{x}}$, (given in Cartesian coordinates and shown in eq. (2.6)) can be used on the geopotential (given in spherical coordinates and shown in eq. (2.1)). A compendious summary of some of the main procedures of gravity recovery that are in practice and the one suggested by Xu (2008) is also performed here. For more extensive explanations and worked out examples using the methods being practiced, the reader is referred to (Naeimi and Flury 2017).

### 2.1. Local and Global Coordinate Transformations

The relationship between the gravitational potential and position in eq. (2.1) is given in terms of spherical coordinates, whereas (normally) GNSS will provide the position in terms of a geocentric position vector, $\boldsymbol{x}$. The coordinate transformation between these coordinates is given by,

$$
\left[\begin{array}{l}
x  \tag{2.4}\\
y \\
z
\end{array}\right]=r\left[\begin{array}{c}
\sin \theta \cos \alpha \\
\sin \theta \sin \alpha \\
\cos \theta
\end{array}\right] \quad \text { and }
$$

$$
\left[\begin{array}{c}
\alpha \\
\theta \\
r
\end{array}\right]=\left[\begin{array}{c}
\tan ^{-1} \frac{y}{x} \\
\tan ^{-1} \sqrt{\frac{x^{2}+y^{2}}{z^{2}}} \\
\sqrt{x^{2}+y^{2}+z^{2}}
\end{array}\right]
$$

where $(x, y, z)$ are the axes directions of the geocentric position vector, and $\alpha$ is the right ascension, which is related to the longitude, $\lambda$, at each epoch by

$$
\begin{equation*}
\lambda(t)=\alpha(t)-\left(G A S T_{0}+\omega_{E} \cdot\left(t-t_{A}\right)\right) \tag{2.5}
\end{equation*}
$$

where $G A S T_{0}$ is the Greenwich apparent sidereal time at the initial epoch $\left(t_{A}\right), t$ is the time of epoch, and $\omega_{E}$ is the Earth rotation rate (which is assumed to be constant).

The gradient of the potential is the gravitational acceleration, and in a global Cartesian coordinate frame this is,

$$
\boldsymbol{g}(\boldsymbol{x})=\nabla_{\boldsymbol{x}} V(r, \theta, \lambda)=\left[\begin{array}{c}
\frac{\partial}{\partial x}  \tag{2.6}\\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{array}\right] V(r, \theta, \lambda)
$$

where the gradient operator is transformed to derivatives with respect to the spherical coordinates as,

$$
\left[\begin{array}{c}
\frac{\partial}{\partial x}  \tag{2.7}\\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{array}\right]=\left[\begin{array}{lll}
\frac{\partial \alpha}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial r}{\partial x} \\
\frac{\partial \alpha}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial r}{\partial y} \\
\frac{\partial \alpha}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial r}{\partial z}
\end{array}\right]\left[\begin{array}{c}
\frac{\partial}{\partial \alpha} \\
\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial r}
\end{array}\right]=\boldsymbol{R}\left[\begin{array}{c}
\frac{\partial}{r \sin \theta \alpha \alpha} \\
-\frac{\partial}{r \partial \theta} \\
\frac{\partial}{\partial r}
\end{array}\right]=\boldsymbol{R}\left[\begin{array}{c}
\frac{\partial}{\partial u} \\
\frac{\partial}{\partial v} \\
\frac{\partial}{\partial w}
\end{array}\right]
$$

where $\boldsymbol{u}$ is the local coordinate vector, with $(u, v, w)$ as the east-north-up axis directions, and $\boldsymbol{R}$ is the rotation matrix from local to global directions,

$$
\boldsymbol{R}=\boldsymbol{R}_{3}\left(-\alpha-\frac{\pi}{2}\right) \boldsymbol{R}_{1}(-\theta)=\left[\begin{array}{ccc}
-\sin \alpha & \cos \theta \cos \alpha & \sin \theta \cos \alpha  \tag{2.8}\\
\cos \alpha & \cos \theta \sin \alpha & \sin \theta \sin \alpha \\
0 & \sin \theta & \cos \theta
\end{array}\right]
$$

Therefore,

$$
\begin{equation*}
\boldsymbol{g}(\boldsymbol{x})=\boldsymbol{R} \boldsymbol{g}(\boldsymbol{u})=\nabla_{\boldsymbol{u}} V(r, \theta, \lambda) \tag{2.9}
\end{equation*}
$$

where $\boldsymbol{g}(\boldsymbol{u})$ is the gravitation vector related to the local gradients of the gravitational potential.

### 2.2. Gravity Field Modelling Approaches

### 2.2.1. Variational Equations

The variational equations approach (VEA) in its initial conception was applied by concurrently solving for both the dynamic orbit and gravitational parameters, a procedure known as the dynamic method (Tapley, et al. 2004b). Though, this is how it is still mainly implemented, procedures do exist that separate the precise orbit determination (POD) routine from the gravity recovery process. Such procedures are aptly called two-step variational equation approaches, where the first step is the POD and the second step is the use of that orbit to recover the gravity field (Liu 2008). Herein the discussion will only be focused on the second step of the two-step VEA.

Models using the VEA (and all other approaches discussed herein) stem directly from Newton's second law of motion,

$$
\begin{equation*}
\ddot{\boldsymbol{x}}(t):=\boldsymbol{a}(t)+\boldsymbol{g}(t) \tag{2.10}
\end{equation*}
$$

where $t$ is the timestamp, $\ddot{\boldsymbol{x}}$ is the total acceleration of the satellite, $\boldsymbol{a}$ is the specific force (inertial acceleration), and $\boldsymbol{g}$ is the acceleration due to the gravitational field. The equation of motion above may be re-written (as a first order differential equation) using a Lagrangian function formulation,

$$
\begin{equation*}
\frac{d}{d t} \boldsymbol{X}(t):=\boldsymbol{F}(t ; \boldsymbol{x}, \dot{\boldsymbol{x}} ; \boldsymbol{q}, \boldsymbol{p}) \tag{2.11}
\end{equation*}
$$

where the $(3 \times 1)$ vectors $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are, respectively, the satellite's position and velocity observables, therefore $\boldsymbol{X}:=[\boldsymbol{x}, \dot{\boldsymbol{x}}]^{T}$ is a $(6 \times 1)$ vector at each epoch, $\boldsymbol{p}$ is an $N_{\text {coef }}$ size vector that holds the gravitational unknowns, and the $(6 \times 1)$ function $\boldsymbol{F}(\cdot)$ is the dynamical model describing the motion of the satellite, due to both the inertial and gravitational accelerations. Normal convention would have the $\boldsymbol{q}$ vector include the accelerometer bias, scaling coefficients and any other non-gravitational parameters affecting the motion of the satellite. However, for this illustration it is assumed that the orbit is known and along with it, these non-gravitational parameters. By virtue of being an ordinary differentiation equation, the solution to eq. (2.11) requires an estimate of initial values. Therefore, in this instance $\boldsymbol{q}:=\left[\boldsymbol{x}_{A}, \dot{x}_{A}\right]$ is a $(6 \times 1)$ vector holding the initial position and velocity estimates of an orbital arc. The derivative of the dynamic model with respect to the initial vector, $\boldsymbol{q}$, when multiplied by the unit matrix $\partial \boldsymbol{X} / \partial \boldsymbol{X}$, and using the chain rule of differentiation can be shown to be (Liu 2008, p. 21),

$$
\begin{align*}
\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{q}} & =\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{X}(t)} \cdot \frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{q}} \\
& =\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{X}(t)} \cdot \boldsymbol{\Phi}\left(t, t_{A}\right) \tag{2.12}
\end{align*}
$$

where $\boldsymbol{\Phi}$ is the $(6 \times 6)$ state transition matrix, which is expanded as

$$
\boldsymbol{\Phi}\left(t, t_{A}\right)=\frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{q}}=\left[\begin{array}{cc}
\frac{\partial \boldsymbol{x}(t)}{\partial \boldsymbol{x}_{A}} & \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial \boldsymbol{x}_{A}}  \tag{2.13}\\
\frac{\partial \dot{\boldsymbol{x}}(t)}{\partial \dot{\boldsymbol{x}}_{A}} & \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial \dot{\boldsymbol{x}}_{A}}
\end{array}\right]
$$

Using eq. (2.11), then eq. (2.12) can be re-written as,

$$
\begin{equation*}
\frac{\partial}{\partial \boldsymbol{q}} \frac{d}{d t} \boldsymbol{X}(t)=\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{X}(t)} \cdot \boldsymbol{\Phi}\left(t, t_{A}\right) \tag{2.14}
\end{equation*}
$$

and then the state transition matrix can be solved from the first-order differential equation,

$$
\begin{equation*}
\frac{d}{d t} \boldsymbol{\Phi}\left(t, t_{A}\right)=\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{X}(t)} \cdot \boldsymbol{\Phi}\left(t, t_{A}\right) \tag{2.15}
\end{equation*}
$$

with the initial value of the state transition matrix set to an identity matrix, $\boldsymbol{\Phi}\left(t_{A}, t_{A}\right)=$ $\boldsymbol{I}_{6 \times 6}$. The first derivative on the RHS of eq. (2.15) is also the same derivative of the left side of eq. (2.11),

$$
\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{X}(t)}=\left[\begin{array}{ll}
\frac{\partial \dot{\boldsymbol{x}}(t)}{\partial \boldsymbol{x}(t)} & \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial \dot{\boldsymbol{x}}(t)}  \tag{2.16}\\
\frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial \boldsymbol{x}(t)} & \frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial \dot{\boldsymbol{x}}(t)}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{0}_{3 \times 3} & \boldsymbol{I}_{3 \times 3} \\
\frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial \boldsymbol{x}(t)} & \frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial \dot{\boldsymbol{x}}(t)}
\end{array}\right]
$$

The derivative of $\boldsymbol{X}$ with respect to the parameter's vector, $\boldsymbol{p}$, gives the ( $6 \times N_{\text {coef }}$ ) socalled sensitivity matrix,

$$
\boldsymbol{S}(t)=\frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{p}}=\left[\begin{array}{c}
\frac{\partial \boldsymbol{x}(t)}{\partial \boldsymbol{p}}  \tag{2.17}\\
\frac{\partial \dot{\boldsymbol{x}}(t)}{\partial \boldsymbol{p}}
\end{array}\right]
$$

which, analogous to eq. (2.14), is obtained by solving the differential equation,

$$
\begin{equation*}
\frac{\partial}{\partial \boldsymbol{p}} \frac{d}{d t} \boldsymbol{X}(t)=\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{X}(t)} \cdot \frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{p}}+\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{p}} \tag{2.18}
\end{equation*}
$$

that can be re-written as (Montenbruck and Gill 2012, p. 241),

$$
\begin{equation*}
\frac{d}{d t} \boldsymbol{S}(t)=\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{X}(t)} \cdot \boldsymbol{S}(t)+\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{p}} \tag{2.19}
\end{equation*}
$$

where the last derivative may be expanded as,

$$
\begin{align*}
\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{p}} & =\left[\begin{array}{c}
\frac{\partial \dot{\boldsymbol{x}}(t)}{\partial \boldsymbol{p}} \\
\frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial \boldsymbol{p}}
\end{array}\right]  \tag{2.20}\\
& =\left[\begin{array}{c}
\mathbf{0}_{3 \times N_{\text {coef }}} \\
\frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial \boldsymbol{p}}
\end{array}\right]
\end{align*}
$$

It is noted that, from eq. (2.10), in $\ddot{\boldsymbol{x}}$ only $\boldsymbol{g}$ depends on $\boldsymbol{p}$, therefore the above equation can be re-written,

$$
\frac{\partial \boldsymbol{F}(\cdot)}{\partial \boldsymbol{p}}=\left[\begin{array}{c}
\mathbf{0}_{3 \times N_{\text {coef }}}  \tag{2.21}\\
\frac{\partial \boldsymbol{g}(t)}{\partial \boldsymbol{p}}
\end{array}\right]
$$

For the solution to eq. (2.19) the standard procedure has been to set the initial value to zero, i.e., $\boldsymbol{S}\left(t_{0}\right)=\mathbf{0}$, this is primarily based on that $\boldsymbol{X}\left(t_{A}\right)$ does not depend on any force model parameters. Notwithstanding, it is worth pointing out that there is an assertion against the validity of this standard method of operation (see, (Xu 2009); (Xu 2018)). However, since the substance of this argument is beyond the scope of this report, readers interested in more details are referred to the aforementioned references. For the nominal gravitational parameters, $\boldsymbol{p}_{0}$, (which are also required in the solution for $\boldsymbol{S}$ ) it is recommended they are based on some a priori reference field (e.g., EGM2008, (Pavlis, et al. 2012)) which would have to be refined on the basis of new (and/or improved) observations.

The differential equations (2.15) and (2.19) can be combined, and expanded with (2.16) and (2.21) into the first order initial value problem,

Letting the unknown gravitational parameter's vector be ordered, $\boldsymbol{p}=\left(\begin{array}{lll}\cdots & C_{n, m} & \cdots\end{array}\right)^{T}$, the non-zero term on the last matrix above (which is part of sensitivity matrix derivative with respect to time), can be denoted (Liu 2008, p. 27),

$$
\begin{align*}
\frac{\partial \boldsymbol{g}(t)}{\partial \boldsymbol{p}} & =\frac{\partial}{\partial \boldsymbol{p}} \frac{\partial}{\partial \boldsymbol{x}(t)}\left(\frac{G M}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n}\left(\frac{R}{r}\right)^{n+1} C_{n, m} \bar{Y}_{n m}(\theta, \lambda)\right) \\
& =\frac{\partial}{\partial \boldsymbol{x}(t)}\left(\frac{G M}{R}\left(\frac{R}{r(t)}\right)^{n+1} \bar{Y}_{n m}(\theta(t), \lambda(t))\right) \tag{2.23}
\end{align*}
$$

Eq. (2.22) is solved iteratively for the gravitational parameters. This procedure is inevitably computationally costly, especially if given a variety of observations. The computation burden also increases proportional to the increase of maximum degree of the coefficients being computed.

The two-step variational equations approach described above uses high precision orbits to determine the spherical harmonic coefficients. Similar steps are taken if one were to use range rate observations from inter-satellite tracking with a change in the derivative of the observations with respect to position and velocity, eq. (2.18) (ibid.). That being said, it is noted, there exists a one-step procedure that may use raw GNSS observations such as code and carrier phase data, and satellite laser ranging data as the observables (see, (Liu 2008); (McCullough 2017); (Darbeheshti, et al. 2018)). Using the one-step-approach, the positions, $\boldsymbol{x}$, used as the dependent variables of eq. (2.18), would be replaced with models
that are functions of the raw observations. Thus, one would need to solve for an entire arc orbit's trajectory, along with the spherical harmonic coefficients.

### 2.2.2. Kinematic Integral Equations

Similar to the version of the VEA discussed above, the integral equation approach (IEA) uses positions (and can be extended to velocities) as observations, and is based on a solution to the equation of motion, eq. (2.10). The solution to the IEA in its initial application used precise kinematic orbits as opposed to dynamic or reduced dynamic orbits; for more on precise orbit determination, interested readers are referred to Bisnath (2004) and references therein. However, recent adjustments to the IEA do allow for the use of reduced dynamic orbits (Ellmer 2018). Strictly speaking, an orbit computed using a purely kinematic approach is done independently of any gravitational or dissipative force models (e.g., third body perturbations, tidal effects, relativistic effects, and non-gravitational perturbations). But, because the orbit depends on these forces, the kinematic method is ideal for a more straightforward gravity solution than the traditional dynamic approaches that have to deal with the processing and analysis of observations that are used simultaneously to estimate both the orbit and the gravitational field (Švehla and Rothacher 2003). The kinematic approach permits direct integration of the equations of motion, based on independent observations of the orbit. The solutions to equations of motion, e.g., eq. (2.10) can be written formally as integral equations, of which there are two types: the two-boundary-point, and the one-boundary-point formulations. The corresponding solutions are computed by numerical integration.

### 2.2.2.1. Two-Boundary-Point Formulation

The two-boundary-point problem is formulated in the form of a Fredholm-type integral equation of the second kind (see (Schneider 1968); (Mayer-Gürr, et al. 2005)). For boundary values, $\boldsymbol{x}_{A}:=\boldsymbol{x}\left(t_{A}\right)$ and $\boldsymbol{x}_{B}:=\boldsymbol{x}\left(t_{B}\right)$ the solution to eq. (2.10) is (which is easily verified by back-substitution),

$$
\begin{equation*}
\boldsymbol{x}(t)=\frac{t_{B}-t}{T} \boldsymbol{x}_{A}+\frac{t-t_{A}}{T} \boldsymbol{x}_{B}-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{a}\left(t^{\prime}\right)+\boldsymbol{g}\left(t^{\prime}\right)\right) d t^{\prime} \tag{2.24}
\end{equation*}
$$

where the subscripts $A$ and $B$ denote the arc's initial and end points respectively, $\boldsymbol{x}$ is the position vector, $\boldsymbol{a}$ is the specific force, $\boldsymbol{g}$ is the acceleration due to the gravitational field, $T=t_{B}-t_{A}$ is the arc length, and the kernel function,

$$
K\left(t, t^{\prime}\right)=\frac{1}{T^{2}} \begin{cases}\left(t-t_{A}\right)\left(t_{B}-t^{\prime}\right), & t \leq t^{\prime}  \tag{2.25}\\ \left(t^{\prime}-t_{A}\right)\left(t_{B}-t\right), & t^{\prime} \leq t\end{cases}
$$

as defined by Schneider (1968), is a Green function. As aforementioned, the satellite motion is highly non-linear with position. However, in the approach taken by Mayer-Gürr
(2006, ch. 4) no linearization is performed with respect to position (as illustrated in eq. (2.33)), but rather only with respect to the field. The reduced observation equation for eq. (2.24) is then,

$$
\begin{equation*}
\boldsymbol{y}(t)=-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \tag{2.26}
\end{equation*}
$$

where $\boldsymbol{y}$ is the reduced observation, which hosts the assumed known terms: the specific force term, a reference gravitational field, and the boundary values,

$$
\begin{align*}
\boldsymbol{y}(t)=\boldsymbol{x}(t)- & \frac{t_{B}-t}{T} \boldsymbol{x}_{A}-\frac{t-t_{A}}{T} \boldsymbol{x}_{B} \\
& +T \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}\left(t^{\prime}\right)\right)+\boldsymbol{a}\left(t^{\prime}\right)\right) d t^{\prime} \tag{2.27}
\end{align*}
$$

$\Delta \boldsymbol{g}=\boldsymbol{g}-\boldsymbol{g}^{(r e f)}$ is the residual gravitational field, $\boldsymbol{g}^{(r e f)}$ is the reference gravitational field, and $\boldsymbol{p}$ is the vector of gravitational unknowns. Note that the arc boundaries are also observed in $\boldsymbol{x}(t)$, and for eq. (2.26) it is assumed no further improvement is required from their observation. Finally, the integral in eq. (2.26) is discretized on the basis of highly precise numerical quadrature to form a design matrix corresponding to the unknown parameters (Mayer-Gürr 2006, p. 22). A version of this approach using ranges and range rates is also derived in Mayer-Gürr, et al. (2006).

Strictly speaking, the choice of arc length in eq. (2.26) is somewhat arbitrary. However, in practice, when choosing a $t_{B}$, one has to find a balance between increasing arc length and the effects of dissipative forces on the model accuracy (Baur, et al. 2014). This trade-off has to date restricted this approach (in its current implementation) to relatively short arcs, e.g., approximately 30 minutes in the case of Mayer-Gürr, et al. (2005) or up to 3 hours in the case of Ellmer (2018). As a result of this strategy, on top of the gravitational parameters, the model ends up with a great number of additional orbital unknowns to determine. For example, 153000 arc-related unknowns over one year are processed in the case of Mayer-Gürr, et al. (2005). This results in a computationally intensive solution. Since the two-boundary-point integral equation approach discussed here has the observed orbit as part of the integrand that forms the design matrix, one is bound to propagate the error inherent in these observations into the solution when eventually forming the least squares solution. To rectify this, it is recommended one uses field gradients to solve for position errors (Mayer-Gürr 2006, p. 60). This is contrary to the IEA suggested by Xu (2008), discussed below, where the gradients are fundamental to the model design. A closer look into the similarities and differences between the Schneider and Xu models is found in Chapter 4.

### 2.2.2.2. One-Boundary-Point

Alternative to eq. (2.24) and the overall procedure discussed for the two-boundarypoint, Xu (2008) proposes a perturbation model in the form of a Volterra integral equation of the second kind that can be solved iteratively. This approach has never been tested; however (as aforementioned), this report aims at such a test (see Chapter 3). For now, the model is presented as theoretically derived in Xu (2008), with some modifications. The solution to the differential equation (2.10) can be written as (which is easily checked by Leibniz's rule for differentiating an integral),

$$
\begin{equation*}
\boldsymbol{x}(t)=\boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)+\int_{t_{A}}^{t} \int_{t_{A}}^{t^{\prime}}\left(\boldsymbol{a}\left(t^{\prime \prime}\right)+\boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime \prime}\right), \boldsymbol{p}\right)\right) d t^{\prime} d t^{\prime \prime} \tag{2.28}
\end{equation*}
$$

where $\boldsymbol{x}_{A}:=\boldsymbol{x}\left(t_{A}\right)$, and $\dot{\boldsymbol{x}}_{\boldsymbol{A}}:=\boldsymbol{x}\left(t_{A}\right)$ are the initial conditions in position and velocity, and $\boldsymbol{p}$ is still the vector of unknown gravitational parameters. This is equivalent to the formulation in Xu (2008, eq. 8). However, this formulation involves double integrals and considering that numerical integration errors tend to accumulate, this formulation is likely to be burdensome. To alleviate this concern, a substitute (yet equal) solution that only requires one integral is used (which can also be confirmed by Leibniz's rule),

$$
\begin{equation*}
\boldsymbol{x}(t)=\boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right)\left(\boldsymbol{a}\left(t^{\prime}\right)+\boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right)\right) d t^{\prime} \tag{2.29}
\end{equation*}
$$

It is noted that all other derivations and eventual numerical tests are based on the formulation in eq. (2.29) and not (2.28), with no loss in assessment of the method.

Theoretically, for $t=t_{B}$ and assuming no integration error, solutions to models (2.24) and (2.29) are equivalent. Nonetheless, in the linearization of eq. (2.29), an errorfree reference orbit, $\boldsymbol{x}^{(r e f)}$, is introduced. The reference orbit here is essentially arbitrary (but presumably always close to the true orbit). This is contrary to the VEA where the reference orbit is a function of an a priori nominal field. Using a random reference orbit (that is error free), as suggested for the procedure in Xu (2008), mitigates the propagation of observation errors into the field integrals. As previously stated, it also ensures the reference orbit is always close to the true orbit, a feat desirable for any gravitational recovery procedure (Tapley, et al. 2004b). Unfortunately, due to the way the one-boundarypoint IEA is developed in Xu (2008), it does not allow for one to use the observed orbit as a reference orbit, even if it is desired (see, eq. (3.6)). The linearization of eq. (2.29) also includes the introduction of a reference field, $\boldsymbol{g}^{(r e f)}$, and reference specific force, $\boldsymbol{a}^{(r e f)}$,

$$
\begin{align*}
\boldsymbol{x}(t) & =\boldsymbol{x}^{(r e f)}(t)+\Delta \boldsymbol{x}(t)  \tag{2.30}\\
\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{p}) & =\boldsymbol{g}^{(r e f)}(\boldsymbol{x}(t))+\Delta \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{p})  \tag{2.31}\\
\boldsymbol{a}(t) & =\boldsymbol{a}^{(r e f)}(t)+\Delta \boldsymbol{a}(t) \tag{2.32}
\end{align*}
$$

where $\Delta \boldsymbol{x}$ is the orbital position perturbation, $\boldsymbol{g}^{(r e f)}$ is the reference gravitational field (computed using spherical harmonic coefficients up to degree and order $n_{r e f}$ ), $\Delta \boldsymbol{g}$ is the residual gravitational field, $\boldsymbol{a}^{(r e f)}$ is the reference inertial acceleration (computed using the known non-gravitational field parameters), and $\Delta \boldsymbol{a}$ is the residual inertial acceleration. The residual gravitational field term holds the spherical harmonic coefficients to be estimated, while the residual inertial acceleration term holds the unknown parameters (if any) to all the other forces exerted on the satellite, e.g., solar radiation pressure, atmospheric drag, third-body effects, and general relativistic drag (Xu 2018). Such parameters may include, accelerometer bias(es) and scaling factors to the aforementioned forces.

Since there is no a priori constraint that the orbital and field references should be mutually consistent, it is suggested that here they are explicitly made independent of each other, as well as independent of the unknown gravitational or orbital parameters (e.g., $\left.\boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}(t), \boldsymbol{p}\right) \neq \boldsymbol{g}^{(r e f)}\right)$. This is done by, as previously stated, having random reference orbits while the reference fields are from a priori (field) models. Neglecting higher order terms, and linearizing eq. (2.29) with respect to the position, generates,

$$
\begin{align*}
\boldsymbol{x}^{(r e f)}(t)+\Delta \boldsymbol{x} & (t) \\
& =\boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right) \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right)\left(\boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)+\left.\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right)\right|_{\boldsymbol{x}=\boldsymbol{x}^{(r e f)}} \Delta \boldsymbol{x}\left(t^{\prime}\right)\right.  \tag{2.33}\\
& \left.+\boldsymbol{a}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

The next linearization is with respect to both the accelerations, i.e., substitute eq. (2.31) and (2.32) into (2.33),

$$
\begin{align*}
\boldsymbol{x}^{(r e f)}(t)+\Delta \boldsymbol{x} & (t) \\
& =\boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A}\left(t-t_{A}\right) \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right)\left(\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)+\Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)\right.  \tag{2.34}\\
& +\left.\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}\left(t^{\prime}\right)\right)\right|_{\boldsymbol{x}=\boldsymbol{x}^{(r e f)}} \Delta \boldsymbol{x}\left(t^{\prime}\right) \\
& \left.+\left.\frac{\partial}{\partial \boldsymbol{x}} \Delta \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right)\right|_{\boldsymbol{x}=\boldsymbol{x}^{(r e f)}} \Delta \boldsymbol{x}\left(t^{\prime}\right)+\boldsymbol{a}^{(r e f)}\left(t^{\prime}\right)+\Delta \boldsymbol{a}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

The gravitational gradient tensor, $\boldsymbol{\Gamma}$, is given by $\partial \boldsymbol{g} / \partial \boldsymbol{x}$. Inserting this into eq. (2.34) and neglecting the second order terms, produces the linear perturbation approximation,

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\boldsymbol{x}_{A}+ & \dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)-\boldsymbol{x}^{(r e f)}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime}  \tag{2.35}\\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}\left(t^{\prime}\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{a}^{(r e f)}\left(t^{\prime}\right) d t^{\prime}+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{a}\left(t^{\prime}\right) d t^{\prime}
\end{align*}
$$

where $\Gamma^{(r e f)}$ is the reference gravitational gradient tensor (computed up to degree and order $n_{r e f}$ ). Assuming known initial conditions, all exactly known terms in eq. (2.35) could be gathered into a single term,

$$
\begin{align*}
\Delta \boldsymbol{x}^{(0)}(t)=\boldsymbol{x}_{A} & +\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)-\boldsymbol{x}^{(r e f)}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{a}^{(r e f)}\left(t^{\prime}\right) d t^{\prime} \tag{2.36}
\end{align*}
$$

Finally, one obtains the linear one-boundary-point kinematic integral equation perturbation model as a Volterra integral equation of the second kind,

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\Delta \boldsymbol{x}^{(0)}(t) & +\int_{t_{A}}^{t}\left(t-t^{\prime}\right)\left(\Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)+\Delta \boldsymbol{a}\left(t^{\prime}\right)\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}\left(t^{\prime}\right) d t^{\prime} \tag{2.37}
\end{align*}
$$

It is worth pointing out that the observation of the perturbation, $\Delta \boldsymbol{x}$, is on both sides of eq. (2.37). To rectify this, Xu (2008) suggest using a Taylor series first approximation as the perturbation on the right hand side (RHS), i.e.,

$$
\begin{equation*}
\Delta \boldsymbol{x}(t) \approx \Delta \boldsymbol{x}^{(0)}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right)\left(\Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)+\Delta \boldsymbol{a}\left(t^{\prime}\right)\right) d t^{\prime} \tag{2.38}
\end{equation*}
$$

which is similar to the quasi-linear coordinate perturbation denoted in Xu (2008, eq. 11). This gives

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\Delta \boldsymbol{x}^{(0)}(t) & +\int_{t_{A}}^{t}\left(t-t^{\prime}\right)\left(\Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)+\Delta \boldsymbol{a}\left(t^{\prime}\right)\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)\left(\Delta \boldsymbol{x}^{(0)}\left(t^{\prime}\right)\right.  \tag{2.39}\\
& \left.+\int_{t_{A}}^{t^{\prime}}\left(t^{\prime}-t^{\prime \prime}\right)\left(\Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime \prime}\right), \boldsymbol{p}\right)+\Delta \boldsymbol{a}\left(t^{\prime \prime}\right)\right) d t^{\prime \prime}\right) d t^{\prime}
\end{align*}
$$

and can be re-written as,

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\Delta \boldsymbol{x}^{(0)}(t) & +\int_{t_{A}}^{t}\left(t-t^{\prime}\right)\left(\Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)+\Delta \boldsymbol{a}\left(t^{\prime}\right)\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}^{(0)}\left(t^{\prime}\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)\left(\int_{t_{A}}^{t^{\prime}}\left(t^{\prime}-t^{\prime \prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime \prime}\right), \boldsymbol{p}\right) d t^{\prime \prime}\right) d t^{\prime}  \tag{2.40}\\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)\left(\int_{t_{A}}^{t^{\prime}}\left(t^{\prime}-t^{\prime \prime}\right) \Delta \boldsymbol{a}\left(t^{\prime \prime}\right) d t^{\prime \prime}\right) d t^{\prime}
\end{align*}
$$

which is like the linear coordinate perturbation in Xu (2008, eq. 12). Higher order perturbations could be derived from the same iterative process. This approach is justified by the theory of solutions to Volterra integral equations of the second kind, which guarantees convergence to the true solution with such iterated substitutions (Hackbusch 1995, p. 28).

Since this approach has never been verified, it is not clear if convergence could be achieved under conditions of imperfect numerical integration. It has also been suggested (anonymous reviewer of Jekeli and Habana (2019)) that convergence could only happen if the reference orbit is updated with each iteration, but it has already been noted that the reference orbit in this formulation cannot be the true orbit, as the estimability of the gravitational parameters then vanishes. Moreover, this theory still requires nested integrals (despite starting with single integrals in eq. (2.29)), and this might pose a numerical burden if one were to evaluate the perturbations as presented.

### 2.2.3. Energy Balance

The energy balance approach (EBA) in the latest applications uses the basic principle of the conservation of mechanical energy to relate differences in potential of two satellites to their measured inter-satellite range rates, positions, and velocities (see e.g.,
(Wolff 1969); (Jekeli 1999); (Shang, et al. 2015)). The total energy of a pair of satellites using the law of energy conservation in the inertial frame is,

$$
\begin{equation*}
V_{12}^{(K)}+V_{12}-V_{12}^{(r o t)}-V_{12}^{(n g)}=E_{12}^{(0)} \tag{2.41}
\end{equation*}
$$

where the subscript 12 denotes the difference between quantities referring to the leading, 2, and the trailing, 1 , satellite. $V_{12}^{(K)}$ is the (per unit mass) kinetic energy difference, $V_{12}$ is the gravitational potential difference, $V_{12}^{(r o t)}$ is a so-called Earth rotation potential difference, $V_{12}^{(n g)}$ is the difference in potential associated with non-gravitational forces acting on each satellite, and $E_{12}^{(0)}$ is a constant. The model can be expanded and rearranged to solve for the potential energy difference as (Jekeli 2017a),

$$
\begin{align*}
& V_{12}(t)=\frac{1}{2}\left(\dot{\boldsymbol{x}}_{2}(t)+\dot{\boldsymbol{x}}_{1}(t)\right)^{T} \dot{\boldsymbol{x}}_{12}(t) \\
&+\omega_{e}\left(\dot{\boldsymbol{x}}_{12}^{T}(t)\left(\boldsymbol{e}_{3} \times \boldsymbol{x}_{2}(t)\right)-\boldsymbol{x}_{12}^{T}\left(\boldsymbol{e}_{3} \times \dot{\boldsymbol{x}}_{1}(t)\right)\right)  \tag{2.42}\\
& \quad-\int_{t_{0}}^{t}\left(\boldsymbol{a}_{2}(t) \dot{\boldsymbol{x}}_{2}(t)-\boldsymbol{a}_{1}(t) \dot{\boldsymbol{x}}_{1}(t)\right) d t^{\prime}-E_{12}^{(0)}
\end{align*}
$$

where $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are the position and velocity vectors respectively, $\omega_{e}$ is the Earth's rotation rate (assumed constant), and $\boldsymbol{a}$ is the specific force(s). The first (product) term on the left hand side (LHS) of (2.42) is equivalent to $V_{12}^{(K)}$ in (2.41), the term in $\omega_{e}$ is equal to $V_{12}^{(r o t)}$, and the integral term is $V_{12}^{(n g)}$. Acknowledging that range-rate observations are more precise than positions and velocities, Jekeli (1999) derives an expression of model (2.42) that also involves range-rates (as opposed to just positions and velocities).

Model (2.42) shows a linear relationship between the position and velocity observations and gravitational potential differences. This linearity eliminates the need for an iterative solution as required for the VEA and IEA, making the EBA less numerically intensive. Since the EBA is a space-wise technique, it makes for much simpler global and regional mapping of the gravitational field using in situ potential differences (see e.g., (Han 2004); (Shang, et al. 2015)). The main input of model (2.42) is the velocity of the satellites, which is derived from numerical differentiation of the satellites' positions with respect to time, since technically GNSS track positions. This results in noise amplification, and less accurate results (Visser, et al. 2003). Veritably, results from the original EBA and its improved forms are still relatively inaccurate, compared to the other gravitational recovery approaches available (see e.g., (Xu 2008); (Baur, et al. 2014); (Darbeheshti, et al. 2019)).

### 2.2.4. Acceleration

As with all the aforementioned solutions, the acceleration approach stems from Newton's second law of motion, eq. (2.10). However, in this instance the kinematic
accelerations are considered to be the observations as derived from double derivatives of the observed position with respect to time (Austen, et al. 2001),

$$
\begin{equation*}
\ddot{\boldsymbol{x}}-\boldsymbol{a}:=\nabla_{x} V \tag{2.43}
\end{equation*}
$$

where $\nabla_{\boldsymbol{x}}$ is the gradient, as in eq. (2.6). It is noted that, as shown in model (2.43), the specific forces have to be accounted for and eliminated from the satellite's acceleration, just as in the approaches discussed thus far. However, it is done directly with accelerometer data instead of using integrals of these forces.

Model (2.43) is a second order differential equation (with the differentiation performed along the satellite's trajectory), however, the gradient, $\nabla_{x}$, is a linear operator, thus (2.43) represents a linear model with respect to the spherical harmonic coefficients found in $V$. Therefore, the model is not subject to any linearization errors nor does it require an iterative solution. It is also a space-wise technique, analogous to the energy balance approach. However, the double differentiation results in noise that is amplified by an extra order of magnitude of the frequency compared to the EBA. In the space domain, the noise may (in some cases) surpass the signal to be solved (Liu 2008, p. 34). To rectify this, it is recommended the data weighting and estimation process be performed in the frequency domain (Wu 2016).

## Chapter 3 Model Validation

Current procedures for gravity recovery from satellite tracking, reviewed in the previous chapter, each have advantages and disadvantages. Modern technology is at a stage where LEO satellites can be tracked using GNSS continuously and at higher precision than ever before. In light of these developments, Xu (2008) uses classic perturbation theory to form a version of the integral equations approach that is iterative and can purportedly accommodate for gravity solutions over arbitrarily long-arcs (see Section 2.2.2.2.). The method makes use of a reference orbit that, like the implementation of Mayer-Gürr et al. (2005), is derived directly from the tracking data, although it is not exactly the observed orbit. This unique approach ensures that the reference orbit is not only exactly specified but also completely independent from the unknown field and always close to the true orbit. Despite the approach's novelty, to my knowledge, its practical implementation has yet to be numerically verified. This chapter then looks to follow this verification process. Though Xu's approach as presented and applied here is in rectangular coordinates, it is easily extended to Keplerian elements (Appendix C).

In pursuit of validating the numerical feasibility of the one-boundary-point method (eq. (2.40)), first the linearization error of eq. (2.37) needs to be quantified. This chapter then also looks at generating observations towards that quest, refining some of the formulations for easier computation, and finally determining the linearization errors of various perturbations. As aforementioned, the upper bound of the integral equation approach is meant to be arbitrary. However, due to the (unavoidable) integration error, as well as effects of linearization, for this report, the observations are generated at arc lengths of one day. Appendix D shows the effects of using longer arcs on both the model accuracy and integration error.

### 3.1. Numerical Integration and Observables

To quantify the linearization error of eq. (2.37), preferably, one would use a perfect orbit in a given gravitational field. Unfortunately, such a simulation is attainable only in theory, since the orbit generated as the "true" orbit is bound to have integration error inherited from the solution of eq. (2.10). Therefore, for simulation purposes, the linearization error can only be gauged comparable to the error of the numerical integrator used to create the simulated data, including those of the integrals found in the reduced observations, for example eq. (3.8).

For applications in this report, a multi-step predictor-corrector numerical integrator of the Adams type was used, coded in FORTRAN as a variable order integrator for the numerical solution of ordinary differential equations (DVDQ) (Krogh 1970). The subroutine was developed at the Jet Propulsion Laboratory in the 1960s to compute
numerical integrations at double precision. Its accuracy was tested on the trivial case of a Keplerian orbit, whose solution is known analytically and numerically to the precision of the computer (e.g., double precision). Figure 3.1 shows that the integrator is accurate to better than 0.025 mm in position for a 1-day orbital arc of a LEO satellite in a central gravitational field, with orbital elements as defined in Table 3.1. A similar test was done by Yu Zhang (2019, personal communication) using a fixed-step $10^{\text {th }}$-order Adams-Bashforth-Moulton PECE integrator (fixed at 5-second step length), resulting in the same order of magnitude of accuracy. Similar results were attained when using a Runge-KuttaNyström method developed by Fehlberg (1975) (fixed at 60-second step length) (Yu Zhang 2017, personal communication). The precision of DVDQ has also been verified by comparison to an independent numerical integrator applied to a high degree gravitational field (M. Naeimi, 2016, personal communication with C. Jekeli).


Figure 3.1. Accuracy of numerical integrator for a LEO satellite for a 24 hr long Keplerian orbit using the multi-step predictor-corrector integrator, DVDQ.

Having tested the numerical integrator's precision, the next step is to generate the observable perturbation, $\Delta \boldsymbol{x}$. The "true" orbit positions, $\boldsymbol{x}$, are generated for a low altitude ( 450 km ), near circular, almost polar orbit in a field specified by a maximum harmonic degree and order, $n_{\max }$, using EGM2008 spherical harmonics (see Table 3.1 for the initial Keplerian elements). To generate the reference orbit positions, $\boldsymbol{x}^{(r e f)}$, the true position is perturbed in each axis, at $1 s$ intervals, by a random normally-distributed zero-mean
variable with standard deviation, $\sigma$. These perturbations are then smoothed with a $7^{\text {th }}-$ order B-spline over $100 s$ in each axis. The smoothed output is used as the reference orbit, $\boldsymbol{x}^{(r e f)}$, and the observed perturbation, $\Delta \boldsymbol{x}$, can be computed with eq. (2.30), after some rearrangement. A 1 hr sample of this perturbation, for $n_{\max }=60$ and $\sigma=0.1 \mathrm{~m}$, is shown in Figure 3.2 below. The sample is taken from a day's observation, and is representative of the rest of the day, in terms of magnitude.


Figure 3.2. One hour sample of the simulated perturbation observable, $\Delta x$, for $n_{\max }=60$ and $\sigma=0.1 \mathrm{~m}$.

| Semi-Major axis $a$ | 6808140 m |
| :--- | :--- |
| Eccentricity $e$ | 0.01 |
| Inclination $i$ | $87^{\circ}$ |
| Argument of perigee $\omega$ | $0^{\circ}$ |
| Ascending Node $\Omega$ | $-83^{\circ}$ |
| Time of Perigee $t_{p}$ | 0 s |

Table 3.1. Initial Keplerian elements of the simulated true orbit

### 3.2. One-Boundary-Point Model Reformulation

The specific forces, $\boldsymbol{a}$, in the initial solution to the equation of motion used in Xu's approach (i.e., eq. (2.28)) do not introduce any new mathematical constraints on the model, therefore Xu suggests treating them as reductions to the observed data. Consequently, these forces are neglected in this report, but future work can always emulate them using available models, for example, from Han, et al. (2006). The starting point for the one-boundary-point method is then simply,

$$
\begin{equation*}
\boldsymbol{x}(t)=\boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \tag{3.1}
\end{equation*}
$$

All the other formulations can be re-derived from this step. The first linearization, eq. (2.37), is

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\Delta \boldsymbol{x}^{(0)}(t) & +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}\left(t^{\prime}\right) d t^{\prime} \tag{3.2}
\end{align*}
$$

where the known terms are now,

$$
\begin{equation*}
\Delta \boldsymbol{x}^{(0)}(t)=\boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)-\boldsymbol{x}^{(r e f)}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \tag{3.3}
\end{equation*}
$$

It is reiterated that, the reference orbit and the reference field are not related, i.e., $\boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}(t), \boldsymbol{p}\right) \neq \boldsymbol{g}^{(r e f)}$. In fact, if they were, then according to eq. (3.1), $\Delta \boldsymbol{x}^{(0)}(t)=\mathbf{0}$. The quasi-linear coordinate perturbation is,

$$
\begin{equation*}
\Delta \boldsymbol{x}(t) \approx \Delta \boldsymbol{x}^{(0)}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \tag{3.4}
\end{equation*}
$$

The expression equivalent to Xu's linear coordinate perturbation, eq. (2.40), is

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\Delta \boldsymbol{x}^{(0)}(t) & +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \\
& +\int_{t_{A}}^{t^{\prime}}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}^{(0)}\left(t^{\prime}\right) d t^{\prime}  \tag{3.5}\\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)\left(\int_{t_{A}}^{t \prime}\left(t^{\prime}-t^{\prime \prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime \prime}\right), \boldsymbol{p}\right) d t^{\prime \prime}\right) d t^{\prime}
\end{align*}
$$

If the true orbit is used as the reference orbit as done implicitly by Mayer-Gürr (2006, ch. 4) for the Schneider model, i.e., $\Delta \boldsymbol{x}(t)=\mathbf{0}$, then eq. (3.2) with (3.3) becomes

$$
\begin{aligned}
& \mathbf{0}= \Delta \boldsymbol{x}^{(0)}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \\
&= \boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)-\boldsymbol{x}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}\left(t^{\prime}\right)\right) d t^{\prime} \\
&+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \\
&=-\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime}+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}\left(t^{\prime}\right)\right) d t^{\prime} \\
& \quad+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \\
&= \mathbf{0}
\end{aligned}
$$

where eq. (3.1) is used in the last equality. The implication from eq. (3.6) is that if $\Delta x(t)=$ $\mathbf{0}$ there is nothing to solve for. However, it is worth noting that, using the second equality in eq. (3.6) gives,

$$
\begin{gather*}
\boldsymbol{x}(t)=\boldsymbol{x}_{A}+\dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)-\boldsymbol{x}(t)+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}\left(t^{\prime}\right)\right) d t^{\prime} \\
+\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \tag{3.7}
\end{gather*}
$$

which is just eq. (2.29) (neglecting the inertial acceleration). This is also analogous to the Mayer-Gürr (2006) approach, in that it deals with the observed positions directly. In effect, solutions towards eq. (3.7), and by association eq. (2.28), introduce no such linearization error, as shown, for example in eq. (2.33). Nonetheless, (for an "error free" case) this strategy neglects the gradient term, found in the last integral of eq. (3.2), which would be required if one wishes to use the model at long arcs (even with observations subject to error). This is shown later in this chapter. Therefore, despite the a priori reference orbit, $\boldsymbol{x}^{(r e f)}$, not being required in the integral equations approach, it is highly desirable. The use of an a priori (error-free) reference orbit, may also be mainly preferred because it provides a cleaner procedure for the least-squares procedure to estimate the spherical harmonic coefficients that formulates the problem in terms of a standard Gauss-Markov model.

To further emphasize the benefit of having error-free positions as the input variable on the residual gravitational term, the effect of position error on the gravity anomaly integral versus the gradient integral is shown in Figure 3.3 below. In general, gravitational observations are more sensitive to position than gravitational gradients. It is then expected, as shown in Figure 3.3, that the same level of noise would impact the gravitational term more. Specifically, for random noises in the level of $\sigma=0.01 \mathrm{~m}$, the gravitational term changed by three orders of magnitude more than the gradient term after a day worth of observations.


Figure 3.3. Effect of noise, $\sigma=0.01 \mathrm{~m}$, on the integrals of the gravitational term, $\boldsymbol{g}(\boldsymbol{x}(t)+\sigma(t), \boldsymbol{p}),($ top $)$ and the gravitational gradient, $\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}(t)\right)(\Delta \boldsymbol{x}(t)+$ $\sigma(t))$, (bottom).

### 3.3. Numerical Model Analysis

The plots in this section are based on "true" orbits computed from EGM2008 spherical harmonic coefficients up to degree and order $n_{\max }=60$, with initial state vectors from the Keplerian elements defined in Table 3.1. Therefore, for simulations in this section, it is assumed that the initial conditions are observed exactly, and do not require any further estimation. The reference orbits are perturbed from the "true" orbit as described in the previous section with standard deviation, $\sigma=0.1 \mathrm{~m}$. The reference fields are computed up to degree and order $n_{r e f}=12$. This process is illustrated further in the flow chart in Figure 3.4 below. The integrals on the right side of eq. (3.2) and (3.5) involving the reference gravitational gradient, $\boldsymbol{\Gamma}^{(r e f)}$, also involve the perturbations, $\Delta \boldsymbol{x}$ and $\Delta \boldsymbol{x}^{(0)}$. As noted, the integrator in use is a variable step-size integrator, whereas these perturbations are available only at discrete points. To remedy this, the perturbations are evaluated at the appropriate steps in time by linear interpolation.


Figure 3.4. Flow chart of the process to generate the integration values required to validate the model accuracy

The absolute difference between the LHS and RHS of eq. (3.2), i.e., the error from the linear approximation of the one-boundary-point IEA, is less than 0.1 mm after one day of observations at cm level perturbations (Figure 3.5). This is an order of magnitude worse than the accuracy of the integrator (shown in Figure 3.1), thus indicating eq. (3.2) has a
linearization error. Preferably, this linearization error would be much less than what is depicted, however its assessment is limited by the integration error. Furthermore, $\sigma=$ 0.1 m is the smallest possible perturbation before descending into nominal GNSS error, $\sim 0.01 \mathrm{~m}$ (Švehla and Rothacher 2005). In fact, perturbations this low have no significant difference in linearization error compared to the results depicted in Figure 3.5 (see Appendix D ). Going lower in perturbations will eventually lead to $\sigma \approx 0 \mathrm{~m}$, which (according to eq. (3.6)) is not acceptable.


Figure 3.5. Absolute differences between the LHS and RHS of eq. (3.2), for $n_{\max }=60$, and $n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

The error in the quasi-linear approximation, eq. (3.4), is in the order of $1 m$ (Figure 3.6). This is about four orders of magnitude larger than the linear model approximation error shown above, and shows that if the arc-length is 1-day long, then the integral of the gradient term cannot be neglected, even for millimeter-level accuracy in the model. As shown in Figure 3.6, leaving this term out, would restrict the model to short-arcs of less than 30 mins for the same level of accuracy shown in Figure 3.5 above. The contribution of the gradient term in the linear approximation is illustrated in Figure 3.7.


Figure 3.6. Absolute differences between the LHS and RHS of eq. (3.4), for $n_{\max }=60$, and $n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.


Figure 3.7. Gradient term of eq. (3.2), for $n_{\max }=60$, and $n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

As aforementioned, the coordinate perturbation theory proposed by Xu (and modified here), includes the observation, $\Delta \boldsymbol{x}$, on both the LHS and RHS of the equation(s). Strictly speaking, this prevents the formation of a direct Gauss-Markov model needed for the least-squares solution of the spherical harmonics. The gradient term (which hosts the observations on the RHS), is moved to the LHS to act as a correction to those observation perturbations. Then all observations and their corrections in eq. (3.2) are consolidated into one term,

$$
\begin{equation*}
\boldsymbol{y}^{(x)}(t)=\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \tag{3.8}
\end{equation*}
$$

where the reduced observation is

$$
\begin{equation*}
\boldsymbol{y}^{(x)}(t)=\Delta \boldsymbol{x}(t)-\Delta \boldsymbol{x}^{(0)}(t)-\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}\left(t^{\prime}\right) d t^{\prime} \tag{3.9}
\end{equation*}
$$

and where (again) it is momentarily assumed that the initial conditions require no further adjustments. With the gradient term as part of the reduced observations and given a set of observations, one can use eq. (3.8) to formulate a "pseudo" Gauss-Markov model to estimate the spherical harmonics. In order to compute the perturbation term at a given epoch, $\Delta \boldsymbol{x}(t)$, one first needs to have observed the position and subsequently computed the reference orbit. Therefore, the computation of the gradient term is reliant on an accurate "observation" of $\Delta \boldsymbol{x}(t)$. Concisely, the reduced observations are not fully formed by direct observations, as is the case with the classical Gauss-Markov model (Koch 2002). However, the error models of the pseudo case still remain random. For all intents and purposes one can proceed with this model as they would with a Gauss-Markov model; in fact, in this report, while acknowledging the subtle differences between the two, the terms are used interchangeably.

To further test the accuracy of model (3.8), various perturbations were simulated. Table 3.2 below shows the linearization error from these tests (after one day of orbit observations). As shown in Table 3.2, for the simulated cases, perturbing the true orbit by $\sigma=0.1 \mathrm{~m}$ yielded the best accuracy (sub-millimeter accuracy). It is then of interest to determine the feasibility of recovering the spherical harmonics from such arcs (see Chapter 6). It is reiterated that perturbations lower than $\sigma=0.1 \mathrm{~m}$ are not only impractical as they would be smaller than the anticipated observation error, but such perturbations also provide no significant improvement to the model accuracy (Appendix D).

| $\begin{array}{c}\text { Perturb }, \boldsymbol{\sigma} \\ {[\mathbf{m}]}\end{array}$ | $\boldsymbol{n}_{\text {max }}$ | $\boldsymbol{n}_{\text {ref }}$ | absolute error at $\boldsymbol{t}=\mathbf{8 6 4 0 0 \boldsymbol { s }}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $[\mathbf{m m}]$ |  |  |  |  |  |$]$

Table 3.2. Absolute maximum differences for LHS and RHS of eq. (3.8) for varying fields and perturbations at the end of a 1-day orbit.

According to eq. (3.5), Xu's linear coordinate perturbation, replacing the $\Delta x$ on the RHS of eq. (3.2) with the quasi-linear approximation should (theoretically) improve the model accuracy. However, (in practice) the model error diverges rapidly as $t$ increases, even for centimeter-level perturbations (Figure 3.8). After 1-day the modeling error increased by at least seven orders of magnitude, for the same level of perturbations used in analyzing eq. (3.2). This is from the fact that eq. (3.4) already includes numerical integration error, which is then just compounded into eq. (3.2). It is then obvious that higher iterations will not result in a convergent solution, as suggested by the theory of solutions to Volterra integral equations of the second kind.


Figure 3.8. Absolute differences between the LHS and RHS of eq. (3.5), for $n_{\max }=60$, and $n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

## Chapter 4 Two-Point Boundary Perturbation Models

As aforementioned, barring any integration errors, the kinematic integral equation solutions for the two-boundary-point method, eq. (2.24), and the one-boundary-point, eq. (2.29), are theoretically equivalent. However, how each method is implemented is what makes them unique; each implementation of course having its own set of advantages and disadvantages compared to the other. For instance, model (2.29) requires initial position and velocity boundary values, while model (2.24) requires only position observations (but at two points) for its setup. As previously discussed, positions are more fundamental to GNSS tracking than velocities. Furthermore, Schneider's model does not linearize with respect to position, thus is implemented on the basis that the observed orbit is determined accurately by kinematic tracking (i.e., $\Delta \boldsymbol{x} \approx \mathbf{0}$ ), whereas Xu's model does linearize with respect to position and consequently utilizes (arbitrary) reference orbits.

The version of the modified Xu model with no perturbation, $\Delta \boldsymbol{x}$, i.e., the quasilinear model, can be viewed as analogous to the Schneider model. According to Figure 3.5, in order to maintain the same level of accuracy as that of eq. (3.2) (i.e., $\sim 10 \mu \mathrm{~m}$ ), the length of arcs would have to be kept under 30 min ; a range resembling the typical arc length of the Schneider model discussed in Section 2.2.2.1. (see, e.g., (Mayer-Gürr, et al. 2005); (Baur, et al. 2014)). As established by eq. (3.6), if one wishes to apply the perturbed integral equation approach as modified in this report, i.e., the linear version of the modified Xu model, it is important that the observed orbit cannot be set equal to the reference orbit, i.e. $\Delta \boldsymbol{x} \neq \mathbf{0}$. However, it is worth noting that, this restriction can easily be satisfied (since an approximation to the true orbit from observations can always be determined).

The reduced observation of the integral equation approach with no $\Delta \boldsymbol{x}$, e.g., the Schneider model, involves integrals of functions of the observed orbit (see, e.g., eq. (2.27)). Therefore, the model error accumulates insidiously with the length of integration. On the other hand, since the reference orbit used in the perturbed IEA is error-free, there is no registration error in the residual gravitational vector, eq. (3.8). That the latter case is more desirable, should be obvious by the very fact that one is able to curtail the spread of observation noise into the design matrix when using these error-free reference orbits.

Following well established theory to solutions of Volterra integral equations of the second kind, Xu's intention was to provide a much cleaner least squares procedure for estimating spherical harmonics from SST through a Gauss-Markov model. Unfortunately, the iterative formulation does not hold under numerical analysis, mostly because it does not account for the numerical integration error that would accumulate with each iteration procedure (see, Figure 3.7). Nonetheless, the modified version of the perturbation theory still upholds Xu's original objective albeit through a "pseudo" Gauss-Markov model. The non-perturbed IEA has the advantage that it continues to be practically deployed to successfully estimate the harmonic coefficients (see, e.g., (Mayer-Gürr, et al. 2005); (Baur,
et al. 2014); (Ellmer 2018)). Therefore, it is known to be pragmatic, even though there is still obvious room for improvement (mostly evident in the limited arc length). In light of all this, this chapter then aims to develop a high-low SST method, herein referred to as the GNSS-based perturbation method, which encompasses the main advantages of the nonperturbed IEA, e.g., eq. (2.24), and the perturbed IEA approach, eq. (3.7). This model is validated for various perturbations as done for the modified Xu method. Lastly, a low-low SST approach, herein referred to as the KBR-based kinematic perturbation method, is developed that is still based on the amalgamation of the merits of the Schneider model and the desirability of the modified Xu proposal. This too is validated for various perturbations.

### 4.1. GNSS-based Kinematic Perturbation Model

If the specific forces are treated as corrections to the observations, and are omitted for consideration in the basic model, then the two-boundary-point solution, can be written as,

$$
\begin{equation*}
\boldsymbol{x}(t)=\frac{t_{B}-t}{T} \boldsymbol{x}_{A}+\frac{t-t_{A}}{T} \boldsymbol{x}_{B}-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \boldsymbol{g}\left(\boldsymbol{x}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \tag{4.1}
\end{equation*}
$$

where, as before, subscripts $A$ and $B$ are the arc's initial and end boundary points (i.e., the boundary values are $\boldsymbol{x}_{A}:=\boldsymbol{x}\left(t_{A}\right)$ and $\left.\boldsymbol{x}_{B}:=\boldsymbol{x}\left(t_{B}\right)\right), T=t_{B}-t_{A}$, and the kernel function is still,

$$
K\left(t, t^{\prime}\right)=\frac{1}{T^{2}} \begin{cases}\left(t-t_{A}\right)\left(t_{B}-t^{\prime}\right), & t \leq t^{\prime}  \tag{4.2}\\ \left(t^{\prime}-t_{A}\right)\left(t_{B}-t\right), & t^{\prime} \leq t\end{cases}
$$

Analogous to eq. (3.2), the linearization of eq. (4.1) with respect to position, then with respect to the gravitational field, and neglecting higher order terms gives

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\frac{t_{B}-t}{T} & \boldsymbol{x}_{A}+\frac{t-t_{A}}{T} \boldsymbol{x}_{B}-\boldsymbol{x}^{(r e f)}(t) \\
& -T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)+\Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)\right.  \tag{4.3}\\
& \left.+\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

The observation equation (assuming perfectly observed boundary vectors) is then

$$
\begin{equation*}
\boldsymbol{y}^{(x)}(t)=-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime} \tag{4.4}
\end{equation*}
$$

with reduced observations as,

$$
\begin{align*}
\boldsymbol{y}^{(x)}(t)=\Delta \boldsymbol{x}(t) & +\boldsymbol{x}^{(r e f)}(t)-\frac{t_{B}-t}{T} \boldsymbol{x}_{A}-\frac{t-t_{A}}{T} \boldsymbol{x}_{B} \\
& +T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)\right.  \tag{4.5}\\
+ & \left.\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

### 4.1.1. GNSS-based Kinematic Perturbation Model Validation

Assuming known boundary points (as in the numerical analysis of Xu's approach), for the same orbit used in testing the one-boundary-point approach, the accuracy of eq. (4.4) is tested with the gradient term considered as a correction to the model. The results show accuracies consistent with those from the one-boundary-point approach (see Figure 4.1). Here the model is constrained at both ends of the arc, unlike in eq. (3.2) where only the initial boundary was restricted. As a result, as seen in Figure 4.1, the model tends to zero at both arc boundary ends.


Figure 4.1. Absolute differences between the LHS and RHS of eq. (4.4), for $n_{\max }=60$, and $n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

As with Table 3.2, model (4.4) is tested for various orbits at multiple perturbations. Again, it is shown that the cm perturbations showed the best model accuracy (see Table 4.1). For these cm perturbations, results were again (generally) commensurate with the accuracy of the one-boundary-point, while at least one order of magnitude improvement is demonstrated for the larger perturbations.

| Perturb, $\boldsymbol{\sigma}$ <br> $[\mathbf{m}]$ | $\boldsymbol{n}_{\text {max }}$ | $\boldsymbol{n}_{\text {ref }}$ | Absolute maximum error <br> $[\mathbf{m m}]$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{z}$ |
| 0.1 | 24 | 4 | 0.022 | 0.036 | 0.027 |
|  | 36 | 12 | 0.020 | 0.005 | 0.015 |
|  | 60 | 12 | 0.006 | 0.018 | 0.032 |
|  | 120 | 12 | 0.012 | 0.012 | 0.014 |
|  |  |  |  |  |  |
| 1 | 36 | 12 | 0.048 | 0.056 | 0.091 |
|  | 60 | 12 | 0.200 | 0.126 | 0.230 |
|  | 120 | 12 | 0.033 | 0.083 | 0.142 |
|  |  |  |  |  |  |
|  | 36 | 12 | 1.917 | 1.292 | 3.372 |
|  | 60 | 12 | 1.787 | 0.931 | 0.730 |
|  | 120 | 12 | 0.545 | 1.449 | 2.238 |

Table 4.1. Absolute maximum differences for LHS and RHS of eq. (4.4) for varying fields and perturbations

### 4.2. KBR-based Kinematic Perturbation Model

To take advantage of the precise inter-satellite ranging observations for low-low SST missions, such as GRACE and GRACE-FO, this section develops a perturbation model for the range-rate observations analogous to the GNSS-based kinematic perturbation method. The inter-satellite range, $\rho$, and range-rate, $\dot{\rho}$, models are derived by projecting the relative position and relative velocities to the line of sight between the two satellites, respectively (Jekeli 1999). The range is also equivalent to the norm of the relative position vector, $\boldsymbol{x}_{12}$.

$$
\begin{align*}
& \rho(t)=\boldsymbol{e}_{12}^{T}(t) \boldsymbol{x}_{12}(t)=\left|\boldsymbol{x}_{12}\right|  \tag{4.6}\\
& \dot{\rho}(t)=\boldsymbol{e}_{12}^{T}(t) \dot{\boldsymbol{x}}_{12}(t)+\dot{\boldsymbol{e}}_{12}^{T}(t) \boldsymbol{x}_{12}(t) \tag{4.7}
\end{align*}
$$

where the subscripts 1 and 2 respectively represent the trailing and leading satellite, and 12 is the difference between results of the two satellites, thus $\boldsymbol{x}_{12}:=\boldsymbol{x}_{2}-\boldsymbol{x}_{1}$ is the relative
true position vector, $\dot{\boldsymbol{x}}_{12}:=\dot{\boldsymbol{x}}_{2}-\dot{\boldsymbol{x}}_{1}$ is the relative true velocity vector, and the unit vector projection in the direction of the line of sight is

$$
\begin{equation*}
e_{12}=\frac{x_{12}}{\left|x_{12}\right|}=\frac{x_{12}}{\rho} \tag{4.8}
\end{equation*}
$$

Using the quotient rule and simplifying with eq. (4.8), the derivative with respect to time of the projection unit vector is,

$$
\begin{equation*}
\dot{\boldsymbol{e}}_{12}=\frac{\dot{\boldsymbol{x}}_{12} \rho-\boldsymbol{x}_{12} \dot{\rho}}{\rho^{2}}=\frac{\dot{\boldsymbol{x}}_{12}}{\rho}-\boldsymbol{e}_{12} \frac{\dot{\rho}}{\rho} \tag{4.9}
\end{equation*}
$$

The second term on the RHS of eq. (4.7) can be expanded as,

$$
\begin{align*}
\dot{\boldsymbol{e}}_{12}^{T} \boldsymbol{x}_{12} & =\boldsymbol{x}_{12}^{T} \frac{\dot{\boldsymbol{x}}_{12}}{\rho}-\boldsymbol{x}_{12}^{T} \boldsymbol{e}_{12} \frac{\dot{\rho}}{\rho}  \tag{4.10}\\
& =\boldsymbol{e}_{12}^{T} \dot{\boldsymbol{x}}_{12}-\dot{\rho}
\end{align*}
$$

where the second equality is derived from applying eq. (4.8) to the first term, and eq. (4.6) to the second term. Substituting eq. (4.10) back into eq. (4.7), yields the range-rate as a projection of the relative velocity into the line of sight between the two satellites,

$$
\begin{equation*}
\dot{\rho}(t)=\boldsymbol{e}_{12}^{T}(t) \dot{\boldsymbol{x}}_{12}(t) \tag{4.11}
\end{equation*}
$$

It is noted that the range is a function of only the satellite positions, $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$. Therefore, analogous to eq. (2.27), a linearization of eq. (4.6) with respect to position, gives

$$
\begin{align*}
& \rho(t)=\left.\rho(t)\right|_{x_{1}=x_{1}^{(r e f)}, x_{2}=x_{2}^{(r e f)}} \\
&+\left(\left.\boldsymbol{e}_{12}^{T}(t)\right|_{x_{1}=x_{1}^{(r e f)}}{ }_{, x_{2}=x_{2}^{(r e f)}} \Delta \boldsymbol{x}_{2}(t)\right.  \tag{4.12}\\
&\left.-\left.\boldsymbol{e}_{12}^{T}(t)\right|_{x_{1}=x_{1}^{(r e f)}}{ }_{, x_{2}=x_{2}^{(r e f)}} \Delta \boldsymbol{x}_{1}(t)\right)
\end{align*}
$$

which can be simplified to,

$$
\begin{equation*}
\rho(t)=\rho^{(r e f)}(t)+\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T} \Delta \boldsymbol{x}_{2}(t)-\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T} \Delta \boldsymbol{x}_{1}(t) \tag{4.13}
\end{equation*}
$$

Let the relative orbital position perturbation be,

$$
\begin{equation*}
\Delta \boldsymbol{x}_{12}:=\Delta \boldsymbol{x}_{2}-\Delta \boldsymbol{x}_{1} \tag{4.14}
\end{equation*}
$$

with $\left.\Delta \boldsymbol{x}_{i}\right|_{i=1,2}$ as the difference between the true and reference orbital position vector of the corresponding satellite. $\Delta \rho:=\rho-\rho^{(r e f)}$ is the range perturbation. Then eq. (4.13) can be re-written as,

$$
\begin{equation*}
\Delta \rho(t)=\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T} \Delta x_{12}(t) \tag{4.15}
\end{equation*}
$$

which is equivalent to Xu (2008, eq. 27). The reference along-track unit vector is,

$$
\begin{equation*}
\boldsymbol{e}_{12}^{(r e f)}(t)=\frac{\boldsymbol{x}_{12}^{(r e f)}(t)}{\left|\boldsymbol{x}_{12}^{(r e f)}(t)\right|}=\frac{\boldsymbol{x}_{12}^{(r e f)}(t)}{\rho^{(r e f)}(t)} \tag{4.16}
\end{equation*}
$$

Similar to eq. (4.6) and (4.11), the reference range, $\rho^{(r e f)}$, and reference range-rate, $\dot{\rho}^{(r e f)}$, are just, respectively, projections of the reference position and velocity difference vectors between the two satellites onto the line joining them,

$$
\begin{align*}
\rho^{(r e f)}(t) & =\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T} \boldsymbol{x}_{12}^{(r e f)}(t)=\left|\boldsymbol{x}_{12}^{(r e f)}(t)\right|  \tag{4.17}\\
\dot{\rho}^{(r e f)}(t) & =\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T} \dot{\boldsymbol{x}}_{12}^{(r e f)}(t) \tag{4.18}
\end{align*}
$$

where $\boldsymbol{x}_{12}^{(r e f)}:=\boldsymbol{x}_{2}^{(r e f)}-\boldsymbol{x}_{1}^{(r e f)}$ is the relative reference position vector, and $\dot{\boldsymbol{x}}_{12}^{(r e f)}$ $:=\dot{\boldsymbol{x}}_{2}^{(r e f)}-\dot{\boldsymbol{x}}_{1}^{(r e f)}$ is the relative reference velocity vector. The linearized range rate residual is derived by differentiating model (4.15) with respect to time,

$$
\begin{equation*}
\Delta \dot{\rho}(t)=\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T} \Delta \boldsymbol{x}_{12}(t)+\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T} \Delta \dot{\boldsymbol{x}}_{12}(t) \tag{4.19}
\end{equation*}
$$

where $\Delta \dot{\rho}:=\dot{\rho}-\dot{\rho}^{(r e f)}$ is the range rate perturbation, and the relative orbital velocity perturbation is,

$$
\begin{equation*}
\Delta \dot{x}_{12}:=\Delta \dot{x}_{2}-\Delta \dot{x}_{1} \tag{4.20}
\end{equation*}
$$

with $\left.\Delta \dot{x}_{i}\right|_{i=1,2}$ as the orbital velocity perturbation of the corresponding satellite, and $\dot{\boldsymbol{e}}_{12}^{(r e f)}$ is the derivative of the reference unit vector with respect to time. From the last equality in eq. (4.16), the derivative with respect to time (according to the quotient rule of derivatives) is,

$$
\begin{equation*}
\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)=\frac{\rho^{(r e f)}(t) \cdot \dot{\boldsymbol{x}}_{12}^{(r e f)}(t)-\boldsymbol{x}_{12}^{(r e f)}(t) \cdot \dot{\rho}^{(r e f)}(t)}{\left(\rho^{(r e f)}(t)\right)^{2}} \tag{4.21}
\end{equation*}
$$

which simplifies to,

$$
\begin{equation*}
\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)=\frac{\dot{x}_{12}^{(r e f)}(t)}{\rho^{(r e f)}(t)}-\frac{\boldsymbol{x}_{12}^{(r e f)}(t) \cdot \dot{\rho}^{(r e f)}(t)}{\left(\rho^{(r e f)}(t)\right)^{2}} \tag{4.22}
\end{equation*}
$$

It is important to note that (as shown in eq. (4.16) and (4.22)) the reference unit vector, $\boldsymbol{e}_{12}^{(\text {ref })}$, and its derivative, $\dot{\boldsymbol{e}}_{12}^{(\text {ref })}$, are both strictly functions of reference variables, namely: reference orbit, reference range, and reference range rates. Inserting (4.16) and (4.22) into (4.19) generates the linearized observation equation,

$$
\begin{gather*}
\Delta \dot{\rho}(t)=\left(\frac{\left(\dot{x}_{12}^{(r e f)}(t)\right)^{T}}{\rho^{(r e f)}(t)}-\frac{\left(x_{12}^{(r e f)}(t)\right)^{T} \dot{\rho}^{(r e f)}(t)}{\left(\rho^{(r e f)}(t)\right)^{2}}\right) \Delta x_{12}(t)  \tag{4.23}\\
+\left(\frac{x_{12}^{(r e f)}(t)}{\rho^{(r e f)}(t)}\right)^{T} \Delta \dot{x}_{12}(t)
\end{gather*}
$$

This is equivalent to Xu (2008, eq 29). Substituting the linearized approximations of the GNSS-based kinematic perturbation method for relative position and velocity perturbations into eq. (4.23) generates the complete KBR-based kinematic perturbation method as follows.

Combining the relative orbital position perturbation of two LEO satellites, eq. (4.14), with the linear perturbation model of each satellite, eq. (4.3), generates,

$$
\begin{align*}
& \Delta \boldsymbol{x}_{12}(t)=\frac{t_{B}-t}{T} \boldsymbol{x}_{2}^{(A)}+\frac{t-t_{A}}{T} \boldsymbol{x}_{2}^{(B)}-\boldsymbol{x}_{2}^{(r e f)} \\
&-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right)+\Delta \boldsymbol{g}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)\right. \\
&\left.+\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}_{2}\left(t^{\prime}\right)\right) d t^{\prime}-\frac{t_{B}-t}{T} \boldsymbol{x}_{1}^{(A)} \\
&-\frac{t-t_{A}}{T_{1}} \boldsymbol{x}_{1}^{(B)}-\boldsymbol{x}_{1}^{(r e f)}  \tag{4.24}\\
&+T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right)+\Delta \boldsymbol{g}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right)\right. \\
&\left.+\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}_{1}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

where the boundaries are $\boldsymbol{x}_{1}^{(A)}:=\boldsymbol{x}_{1}\left(t_{A}\right), \boldsymbol{x}_{1}^{(B)}:=\boldsymbol{x}_{1}\left(t_{B}\right), \boldsymbol{x}_{2}^{(A)}:=\boldsymbol{x}_{2}\left(t_{A}\right)$, and $\boldsymbol{x}_{2}^{(B)}$ $:=\boldsymbol{x}_{2}\left(t_{B}\right)$. Let,

$$
\begin{equation*}
\boldsymbol{x}_{12}^{(A)}=\boldsymbol{x}_{2}\left(t_{A}\right)-\boldsymbol{x}_{1}\left(t_{A}\right), \quad \boldsymbol{x}_{12}^{(B)}=\boldsymbol{x}_{2}\left(t_{B}\right)-\boldsymbol{x}_{1}\left(t_{B}\right) \tag{4.25}
\end{equation*}
$$

$$
\begin{align*}
\boldsymbol{g}_{12}^{(r e f)}(t) & =\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}(t)\right)-\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}(t)\right)  \tag{4.26}\\
\boldsymbol{\Psi}_{12}(t) & =\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}(t)\right) \Delta \boldsymbol{x}_{2}(t)-\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}(t)\right) \Delta \boldsymbol{x}_{1}(t)  \tag{4.27}\\
\Delta \boldsymbol{g}_{12}(t) & =\Delta \boldsymbol{g}\left(\boldsymbol{x}_{2}^{(r e f)}(t), \boldsymbol{p}\right)-\Delta \boldsymbol{g}\left(\boldsymbol{x}_{1}^{(r e f)}(t), \boldsymbol{p}\right) \tag{4.28}
\end{align*}
$$

then, eq. (4.24) can be written as

$$
\begin{align*}
& \Delta \boldsymbol{x}_{12}(t)=\frac{t_{B}-t}{T} \boldsymbol{x}_{12}^{(A)}+\frac{t-t_{A}}{T} \boldsymbol{x}_{12}^{(B)}-\boldsymbol{x}_{12}^{(r e f)} \\
&-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\Delta \boldsymbol{g}_{12}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime} \tag{4.29}
\end{align*}
$$

The model for the relative orbital velocity perturbation, $\Delta \dot{x}_{12}$, is deduced from the derivative of the relative position perturbations, eq. (4.24), with respect to time,

$$
\begin{align*}
& \Delta \dot{\boldsymbol{x}}_{12}(t)=\frac{1}{T}\left(\boldsymbol{x}_{12}^{(B)}-\boldsymbol{x}_{12}^{(A)}\right)-\dot{\boldsymbol{x}}_{12}^{(r e f)} \\
&-T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\Delta \boldsymbol{g}_{12}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime} \tag{4.30}
\end{align*}
$$

where the derivative of the kernel function, eq. (4.2), with respect to time is,

$$
\frac{d}{d t} K\left(t, t^{\prime}\right)=\frac{1}{T^{2}} \begin{cases}t t_{B}-t^{\prime}, & t \leq t^{\prime}  \tag{4.31}\\ t_{A}-t^{\prime}, & t^{\prime} \leq t\end{cases}
$$

Inserting eq. (4.29) and (4.30) into eq. (4.19) gives,

$$
\begin{align*}
& \Delta \dot{\rho}(t)=\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T}\left(\frac{t_{B}-t}{T} \boldsymbol{x}_{12}^{(A)}+\frac{t-t_{A}}{T} \boldsymbol{x}_{12}^{(B)}-\boldsymbol{x}_{12}^{(r e f)}\right. \\
&\left.-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\Delta \boldsymbol{g}_{12}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime}\right)  \tag{4.32}\\
&+\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T}\left(\frac{1}{T}\left(\boldsymbol{x}_{12}^{(B)}-\boldsymbol{x}_{12}^{(A)}\right)-\dot{\boldsymbol{x}}_{12}^{(r e f)}\right. \\
&-\left.T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\Delta \boldsymbol{g}_{12}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime}\right)
\end{align*}
$$

which can be simplified to,

$$
\begin{align*}
\Delta \dot{\rho}(t)=\left(\frac{t_{B}-}{T}\right. & \left.t\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T}-\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T}\right) \boldsymbol{x}_{12}^{(A)} \\
& +\left(\frac{t-t_{A}}{T}\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T}+\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T}\right) \boldsymbol{x}_{12}^{(B)} \\
& -\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T}\left(T \int _ { t _ { A } } ^ { t _ { B } } K ( t , t ^ { \prime } ) \left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right.\right. \\
& \left.\left.+\Delta \boldsymbol{g}_{12}\left(t^{\prime}\right)\right) d t^{\prime}+\boldsymbol{x}_{12}^{(r e f)}(t)\right)  \tag{4.33}\\
& -\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T}\left(T \int _ { t _ { A } } ^ { t _ { B } } \frac { d } { d t } K ( t , t ^ { \prime } ) \left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right.\right. \\
& \left.\left.+\Delta \boldsymbol{g}_{12}\left(t^{\prime}\right)\right) d t^{\prime}+\dot{\boldsymbol{x}}_{12}^{(r e f)}(t)\right)
\end{align*}
$$

From eq. (4.33), the computable terms can be separated out to get the observation equation,

$$
\begin{align*}
y^{(\dot{\rho})}(t)=-T & \left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T} \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \Delta \boldsymbol{g}_{12}\left(t^{\prime}\right) d t^{\prime}  \tag{4.34}\\
& -T\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T} \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right) \Delta \boldsymbol{g}_{12}\left(t^{\prime}\right) d t^{\prime}
\end{align*}
$$

where the boundary point coordinates are (still temporarily) assumed to be perfectly observed, and requiring no further estimation. Therefore, the reduced observation is

$$
\begin{align*}
y^{(\dot{\rho})}(t)=\Delta \dot{\rho}( & t)-\left(\frac{t_{B}-t}{T}\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T}-\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T}\right) \boldsymbol{x}_{12}^{(A)} \\
& -\left(\frac{t-t_{A}}{T}\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T}+\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T}\right) \boldsymbol{x}_{12}^{(B)} \\
& +\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}(t)\right)^{T}\left(T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime}\right. \\
& \left.+\boldsymbol{x}_{12}^{(r e f)}(t)\right)  \tag{4.35}\\
& +\left(\boldsymbol{e}_{12}^{(r e f)}(t)\right)^{T}\left(T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime}\right. \\
& \left.+\dot{\boldsymbol{x}}_{12}^{(r e f)}(t)\right)
\end{align*}
$$

Similar to the high-low SST models, eq. (3.2) and (4.3), if one where to set the reference range-rate, $\dot{\rho}^{(r e f)}$, reference relative orbital position, $\boldsymbol{x}_{12}^{(r e f)}$, and reference relative orbital velocity, $\dot{\boldsymbol{x}}_{12}^{(r e f)}$, equal to their true counterparts (i.e., $\Delta \dot{\rho}=0, \Delta \boldsymbol{x}_{12}=\mathbf{0}$ and $\Delta \dot{\boldsymbol{x}}_{12}=\mathbf{0}$ ) there would be nothing to estimate in eq. (4.34). Hence, the model also relies on the use of reference state vectors that are very close to their true states, but not too close (analogous to eq. (3.6)).

### 4.2.1. KBR-based Kinematic Perturbation Model Validation

For the validation of the KBR-based kinematic perturbation approach, the trailing satellite initial state vectors are derived from the Keplerian elements stated in Table 3.1. For the leading satellite, all the initial Keplerian elements defining the orbit's shape (eccentricity, $e$ ), size (semi-major axis, $a$ ), and orientation (inclination, $i$, argument of perigee, $\omega$, and ascending node, $\Omega$ ) are kept similar to those in Table 3.1. However, the parameter defining the satellite's position (time of perigee, $t_{p}$ ) is changed to $t_{P}=30 s$ for the leading satellite. This ensures that, at the initial epoch, both satellites are experiencing the same field, just at different locations. All this corresponds to a nominal separation of $\sim 230 \mathrm{~km}$.

Table 4.2 demonstrates that, for a day's worth of observations, model (4.23) has accuracy comparable to the nominal range-rate precision of the KBR system for the GRACE satellites (which is in the order of microns per second), even for perturbations in the order of decameters, i.e., $\sigma=10 \mathrm{~m}$ (Loomis, et al. 2012). It is worth emphasizing that
the model, eq. (4.23), is only the preliminary linearization in the development of the KBRbased kinematic perturbation model (as it is not yet formulated in terms of unknown parameters). Consequently, results shown in Table 4.2 do not include any of the linearization errors from formulating the relative state vector perturbations in terms of the unknowns (i.e., errors from using the RHS of eq. (4.24) and (4.25)).

| Perturbation of GNSS <br> Orbits, $\boldsymbol{\sigma}[\mathbf{m}]$ | RMS $(\boldsymbol{\Delta} \dot{\boldsymbol{\rho}})$ <br> $[\mathbf{m} / \mathbf{s}]$ | Abs. max. $\boldsymbol{\Delta} \dot{\boldsymbol{\rho}}$ linearization error <br> in $\mathbf{m o d e l}[\mathbf{m} / \mathbf{s}]$ |
| :---: | :---: | :---: |
| 0.1 | $2.9 \times 10^{-4}$ | $2.9 \times 10^{-11}$ |
| 1.0 | $2.7 \times 10^{-3}$ | $2.6 \times 10^{-9}$ |
| 10.0 | $2.7 \times 10^{-2}$ | $2.5 \times 10^{-7}$ |

Table 4.2. Range-rate model (4.19) error (maximum absolute) disregarding the linearization of the hl-SST perturbations, for $n_{\max }=60$, \& $n_{\text {ref }}=12$ for 24 hr long orbits

The simulation accuracy above is limited by the numerical integration error in the generation of the "true" orbit state vectors. It is reiterated that, in the case of real GNSS data the fundamental observable is the position, and no such integration error will be present to determine the position. However, for the present application the velocities are generated from the integration of eq. (2.10) with respect to time (neglecting $\boldsymbol{a}$, of course), and the position is obtained from the integral of the velocity (with respect to time). Consequently, the biggest impact of the integration error falls on the position. For eq. (4.23), the main contributor to the linearization error is $\Delta \boldsymbol{x}_{12}=O\left(10^{-1} \mathrm{~m}\right)$, which is multiplied by $\dot{\boldsymbol{e}}_{12}^{(r e f)}=O\left(10^{-3} / s\right)$. For the second term in eq. (4.23), the contributions are $\Delta \dot{\boldsymbol{x}}_{12}=O\left(10^{-2} \mathrm{~m} / \mathrm{s}\right)$, which is multiplied by $\boldsymbol{e}_{12}^{(\text {ref })}=O\left(10^{0}\right)$. The contribution of each component is shown in Figure 4.2. From Figure 4.2, it can be deduced that for eq. (4.23), the first term on the RHS has a magnitude of $O\left(10^{-4} \mathrm{~m} / \mathrm{s}\right)$ while the second term has a magnitude of $O\left(10^{-2} \mathrm{~m} / \mathrm{s}\right)$. It can then be expected that for the error propagation analysis, the greatest contribution would come from the relative velocity perturbation. Fortunately, expanding for the relative position and velocity perturbations eliminates any error sources from the satellite's velocities (see, eq. (4.32)).


Figure 4.2. Norms of the RHS components of eq. (4.23), for $n_{\max }=60$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

Obviously, the substitution of the linearized hl-SST perturbations into eq. (4.14) deteriorates the model accuracies shown in Table 4.2. Despite this, Figure 4.3 below shows the model can still yield accuracies commensurate to the nominal GRACE KBR precisions of $\sigma=0.1 \mu \mathrm{~m} / \mathrm{s}$, if the high-low SST perturbations are set at $\sigma=0.1 \mathrm{~m}$ for GNSS positions. Unfortunately, the same distinction does not hold for the higher level perturbations, as shown in Table 4.3. Figure 4.4 shows the magnitude of the components of eq. (4.33), with the largest term being the integral of the gravity field, $T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \boldsymbol{g}_{12}\left(t^{\prime}\right) d t^{\prime}=O\left(10^{6} m\right)$ which is multiplied by $\dot{\boldsymbol{e}}_{12}^{(r e f)}=O\left(10^{-3}\right)$.


Figure 4.3. Absolute differences between the LHS and RHS of eq. (4.33), for $n_{\max }=60$, and $n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

| Perturbation of GNSS <br> Orbits, $\boldsymbol{\sigma}[\mathbf{m}]$ | RMS $(\boldsymbol{\Delta} \dot{\boldsymbol{\rho}})$ <br> $[\mathbf{m} / \mathbf{s}]$ | Abs. max. $\boldsymbol{\Delta} \dot{\boldsymbol{\rho}}$ linearization error in <br> $\mathbf{m o d e l}[\mathbf{m} / \mathbf{s}]$ |
| :---: | :---: | :---: |
| 0.1 | $2.9 \times 10^{-4}$ | $6.8 \times 10^{-8}$ |
| 1.0 | $2.7 \times 10^{-3}$ | $4.5 \times 10^{-7}$ |
| 10.0 | $2.7 \times 10^{-2}$ | $5.0 \times 10^{-6}$ |

Table 4.3. Range-rate model (4.32) errors (maximum absolute) considering the linearization of the hl-SST perturbations, for $n_{\max }=60$, \& $n_{\text {ref }}=12$ for 24 hr long orbits


Figure 4.4. Norm of the components of eq. (4.33), for $n_{\max }=60$, \& $n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

### 4.2.1.1. Differential Analysis

The only sources of errors in observation model (4.35) are the range rate, $\dot{\rho}$, and orbits of the trailing, $\boldsymbol{x}_{1}$, and leading, $\boldsymbol{x}_{2}$, satellite, as entering in the term, $\boldsymbol{\Psi}_{12}$. The error model contribution can be represented as partial derivatives of eq. (4.35) and the error propagation as dispersions from these contributions (see, e.g,, (Jekeli 2017); (Ghilani 2018)). The error model is then,

$$
\begin{equation*}
\partial y^{(\dot{\rho})}=a_{1} \partial \dot{\rho}+\boldsymbol{a}_{2} \partial \boldsymbol{x}_{1}+\boldsymbol{a}_{3} \partial \boldsymbol{x}_{2} \tag{4.36}
\end{equation*}
$$

where $a_{1}$ is the derivative of eq. (4.35) with respect to the range rate, $\dot{\rho}$, and the coefficient vectors $\boldsymbol{a}_{2}$, and $\boldsymbol{a}_{3}$ are the partial derivatives with respect to the trailing and leading satellite, respectively. These are computed as,

$$
\begin{equation*}
a_{1}=1 \tag{4.37}
\end{equation*}
$$

$$
\begin{align*}
& \boldsymbol{a}_{2}= \frac{\partial}{\partial \boldsymbol{x}_{1}}\left(-\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)^{T}\left(T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}_{1}\left(t^{\prime}\right) d t^{\prime}\right)\right. \\
&\left.\left.-\left(\boldsymbol{e}_{12}^{(r e f)}\right)^{T}\left(T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}_{1}\left(t^{\prime}\right) d t^{\prime}\right)\right)\right)  \tag{4.38}\\
&=-\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)^{T} T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \\
& \quad\left(\boldsymbol{e}_{12}^{(r e f)}\right)^{T} T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \\
& \boldsymbol{a}_{3}= \frac{\partial}{\partial \boldsymbol{x}_{2}}\left(\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)^{T}\left(T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}_{2}\left(t^{\prime}\right) d t^{\prime}\right)\right. \\
&\left.\left.+\left(\boldsymbol{e}_{12}^{(r e f)}\right)^{T}\left(T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}_{2}\left(t^{\prime}\right) d t^{\prime}\right)\right)\right)  \tag{4.39}\\
&=\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)^{T} T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \\
& \quad+\left(\boldsymbol{e}_{12}^{(r e f)}\right)^{T} T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right) \boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

Assuming equal variances for both satellites' positions, the observation variance can be shown to be,

$$
\begin{equation*}
\sigma_{y^{(\dot{\rho})}}^{2}=\sigma_{\dot{\rho}}^{2}+\left(\boldsymbol{a}_{2} \boldsymbol{a}_{2}^{T}+\boldsymbol{a}_{3} \boldsymbol{a}_{3}^{T}\right) \sigma_{x}^{2} \tag{4.40}
\end{equation*}
$$

with the coefficient for $\sigma_{x}^{2}$ depicted in Figure 4.5. Considering the nominal standard deviations of KBR and GNSS measurements are $\sigma_{\dot{\rho}}=0.1 \mu \mathrm{~m} / \mathrm{s}$ and $\sigma_{\boldsymbol{x}}=0.01 \mathrm{~m}$ respectively, and $\boldsymbol{a}_{2} \boldsymbol{a}_{2}^{T}+\boldsymbol{a}_{3} \boldsymbol{a}_{3}^{T}=O\left(10^{-1} / \mathrm{s}\right)$, it can be concluded that position errors will dominate the KBR-perturbation model error analysis, and therefore cannot be ignored in the dispersion of eq. (4.35).


Figure 4.5. Coefficient of the orbital variance in eq. (4.40), for $n_{\max }=60, \& n_{r e f}=12$ from EGM2008 for 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

### 4.2.1.2 Comparison to the Schneider Model

Analogous to eq. (4.1), the relative position of the satellites can be denoted as

$$
\begin{align*}
\boldsymbol{x}_{12}(t)=\frac{t_{B}-t}{T} & \boldsymbol{x}_{12}^{(A)}+\frac{t-t_{A}}{T} \boldsymbol{x}_{12}^{(B)} \\
& -T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}\left(\boldsymbol{x}_{2}\left(t^{\prime}\right), \boldsymbol{p}\right)-\boldsymbol{g}\left(\boldsymbol{x}_{1}\left(t^{\prime}\right), \boldsymbol{p}\right)\right) d t^{\prime} \tag{4.41}
\end{align*}
$$

and the derivative of eq. (4.41) gives the relative velocity,

$$
\begin{equation*}
\dot{\boldsymbol{x}}_{12}(t)=\frac{1}{T}\left(\boldsymbol{x}_{12}^{(B)}-\boldsymbol{x}_{12}^{(A)}\right)-T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}\left(\boldsymbol{x}_{2}\left(t^{\prime}\right), \boldsymbol{p}\right)-\boldsymbol{g}\left(\boldsymbol{x}_{1}\left(t^{\prime}\right), \boldsymbol{p}\right)\right) d t^{\prime} \tag{4.42}
\end{equation*}
$$

Inserting this into eq. (4.11), generates the equivalent to the Schneider model, for range rates

$$
\begin{align*}
\dot{\rho}(t)=\boldsymbol{e}_{12}^{T}(t) & \left(\frac{1}{T}\left(\boldsymbol{x}_{12}^{(B)}-\boldsymbol{x}_{12}^{(A)}\right)\right.  \tag{4.43}\\
& \left.-T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}\left(\boldsymbol{x}_{2}\left(t^{\prime}\right), \boldsymbol{p}\right)-\boldsymbol{g}\left(\boldsymbol{x}_{1}\left(t^{\prime}\right), \boldsymbol{p}\right)\right) d t^{\prime}\right)
\end{align*}
$$

which simplifies to,

$$
\begin{align*}
& \dot{\rho}(t)=\frac{1}{T}\left(\rho\left(t_{B}\right)-\rho\left(t_{A}\right)\right) \\
&-T \boldsymbol{e}_{12}^{T}(t) \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t, t^{\prime}\right)\left(\boldsymbol{g}\left(\boldsymbol{x}_{2}\left(t^{\prime}\right), \boldsymbol{p}\right)-\boldsymbol{g}\left(\boldsymbol{x}_{1}\left(t^{\prime}\right), \boldsymbol{p}\right)\right) d t^{\prime} \tag{4.44}
\end{align*}
$$

Comparable to eq. (4.32) the range rate model above is absolved of errors from the velocity observations, and the source of errors will be from the position observations. The error model of the Schneider model involves linearization with respect to position (see, e.g. (Mayer-Gürr 2006, p. 61)). This procedure is analogous to the perturbation model, and effectively the error analysis is the same.

## Chapter 5 Gravitational Field Estimation Procedure

In this chapter a procedure for using the observation equations from Chapter 4 to estimate the spherical harmonics is developed, specifically using the GNSS-based kinematic perturbation method, the KBR-based kinematic perturbation method, and a combination thereof. The main focus is to develop the respective design matrices for each approach, the system of normal equations, and a technique for adjusting multiple day-long observations.

Assuming that the number of orbital arcs is $S$, and considering a finite number of observables, $N$, per arc at discrete times, $t_{i}, i=0, \ldots, N-1$, then Figure 5.1 below shows the schematic diagram for reading multiple arcs, where $S=5$ and $N=5$.


Figure 5.1. Schematic diagram for the numbering of multiple orbital arcs, and components of an arc, shown with 5 arcs, and 5 observations per arc.

### 5.1. GNSS-based Kinematic Perturbation Model Theory

### 5.1.1. Design Matrix

As stated for eq. (2.25), the residual gravitational field, $\Delta \boldsymbol{g}$, actually holds the unknown gravitational parameters to be estimated. Since the gravitational potential is linearly related to the spherical harmonics, the residual gravitational field can be parameterized as

$$
\begin{equation*}
\Delta \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{p})=\sum_{n=n_{r e f}+1}^{n_{\max }} \sum_{m=-n}^{n} C_{n, m} \Delta \boldsymbol{g}_{n, m}(\boldsymbol{x}(t)) \tag{5.1}
\end{equation*}
$$

where the unknown gravitational parameters' vector is arranged as,

$$
\boldsymbol{p}=\left(\begin{array}{lllllll}
C_{n_{r e f}+1,-n_{r e f}-1} & \ldots & C_{n_{r e f}+1, n_{r e f}+1} & \ldots & C_{n_{\max }, n_{\max }} & \ldots & C_{n_{\max }, n_{\max }} \tag{5.2}
\end{array}\right)^{T}
$$

The $C_{n, m}$, as used in eq. (2.1), are the individual spherical harmonic coefficients at a given degree, $n$, and order, $m$, and $\Delta \boldsymbol{g}_{n, m}$ is the gradient of potential harmonics, which is expressed in terms of the solid harmonics, $\left(\frac{R}{r}\right)^{n+1} \bar{Y}_{n, m}(\cdot)$,

$$
\begin{equation*}
\Delta \boldsymbol{g}_{n, m}(\boldsymbol{x}(t))=\frac{G M}{R} \nabla_{\boldsymbol{x}}\left(\left(\frac{R}{r(t)}\right)^{n+1} \bar{Y}_{n, m}(\theta(t), \lambda(t))\right) \tag{5.3}
\end{equation*}
$$

Now assuming the boundary points are observed with errors (hence, they need to be adjusted), then model (4.4) (with errors included) can be re-written (for observations at discrete times, $t_{i}, i=0, \ldots, N-1$, that is, for one arc) as

$$
\begin{equation*}
\boldsymbol{y}_{i}^{(x)}=\frac{t_{B}-t_{i}}{T} \boldsymbol{x}_{A}+\frac{t_{i}-t_{A}}{T} \boldsymbol{x}_{B}-T \sum_{n=n_{r e f}+1}^{n_{m a x}} \sum_{m=-n}^{n} C_{n, m} \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right) \Delta \boldsymbol{g}_{n, m}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime}+\boldsymbol{\varepsilon}_{i} \tag{5.4}
\end{equation*}
$$

where $\boldsymbol{y}_{i}^{(\boldsymbol{x})}$ is the reduced observation term

$$
\begin{align*}
\boldsymbol{y}_{i}^{(\boldsymbol{x})}=\Delta \widetilde{\boldsymbol{x}}\left(t_{i}\right) & +\boldsymbol{x}^{(r e f)}\left(t_{i}\right) \\
& +T \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right)\right.  \tag{5.5}\\
& \left.+\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \widetilde{\boldsymbol{x}}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

where

$$
\begin{equation*}
\Delta \widetilde{\boldsymbol{x}}\left(t_{i}\right)=\Delta \boldsymbol{x}\left(t_{i}\right)+\boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{i}\right) \tag{5.6}
\end{equation*}
$$

is the perturbation of the observed orbit, $\widetilde{\boldsymbol{x}}$, and $\boldsymbol{\varepsilon}_{\boldsymbol{x}}$ is the error in these observations. Collecting a set of observations for eq. (5.5) gives the vector-matrix form,

$$
\begin{equation*}
\boldsymbol{Y}^{(x)}=\boldsymbol{A X}+\boldsymbol{G} \boldsymbol{p}+\boldsymbol{E} \tag{5.7}
\end{equation*}
$$

Recalling that the number of orbital arcs is $S$, and letting each arc be numbered as $s=$ $0, \ldots, S-1$, then it is noted for each arc, $t_{A}:=t_{s N}$, and $t_{B}:=t_{(s+1) N}$. For eq. (5.7), $\boldsymbol{Y}^{(\boldsymbol{x})}$ is a $3(S N+1) \times 1$ reduced observation vector, $\boldsymbol{A}$ is a $3(S N+1) \times 3(S+1)$ orbital coefficients matrix, $\boldsymbol{X}$ is a $3(S+1) \times 1$ vector of boundary points estimates, $\boldsymbol{G}$ is a
$3(S N+1) \times N_{\text {coef }}$ gravitational mapping matrix, the dimension of $\boldsymbol{p}$ is $N_{\text {coef }} \times 1$. The unknown boundary values are grouped as,

$$
\boldsymbol{X}^{T}=\left(\begin{array}{llll}
\hat{\boldsymbol{x}}_{0} & \hat{\boldsymbol{x}}_{N} & \ldots & \hat{\boldsymbol{x}}_{S N} \tag{5.8}
\end{array}\right)^{T}
$$

Partitioning each arc, the complete reduced observations, $\boldsymbol{Y}^{(x)}$, are

$$
\boldsymbol{Y}^{(x)}=\left[\begin{array}{c}
\boldsymbol{y}_{0}^{(\boldsymbol{x})}  \tag{5.9}\\
\vdots \\
\boldsymbol{y}_{N-1}^{(x)} \\
\vdots \\
\boldsymbol{y}_{(S-1) N}^{(x)} \\
\vdots \\
\boldsymbol{y}_{S N-1}^{(\boldsymbol{x})} \\
\hdashline \boldsymbol{y}_{S N}^{(\boldsymbol{x})}
\end{array}\right]
$$

The coefficient matrix of the arc-related (local) unknowns are

$$
\boldsymbol{A}=\frac{1}{T}\left[\begin{array}{ccccc}
\left(t_{N}-t_{0}\right) \boldsymbol{I}_{3} & \left(t_{0}-t_{0}\right) \boldsymbol{I}_{3} & \mathbf{0}_{3} & \cdots & \mathbf{0}_{3}  \tag{5.10}\\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\left(t_{N}-t_{N-1}\right) \boldsymbol{I}_{3} & \left(t_{N-1}-t_{0}\right) \boldsymbol{I}_{\mathbf{3}} & \mathbf{0}_{3} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0}_{3} & \cdots & \cdots & \left(t_{S N}-t_{(S-1) N}\right) \boldsymbol{I}_{3} & \left(t_{(S-1) N}-t_{(S-1) N}\right) \boldsymbol{I}_{3} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0}_{3} & \cdots & \cdots & \left(t_{S N}-t_{S N-1}\right) \boldsymbol{I}_{3} & \left(t_{S N-1}-t_{(S-1) N}\right) \boldsymbol{I}_{3} \\
\hdashline \mathbf{0}_{3} & \cdots & \cdots & \mathbf{0}_{3} & \left(t_{S N}-t_{(S-1) N}\right) \boldsymbol{I}_{3}
\end{array}\right]
$$

where $\mathbf{0}_{3}$ is a $3 \times 3$ null matrix, and $\boldsymbol{I}_{3}$ is a $3 \times 3$ identity matrix. The design matrix for gravitational (global) unknowns is

### 5.1.2. Dispersion

It is further emphasized that the reference orbits are completely error free, thus for the observation equation of the GNSS-based kinematic perturbation approach, e.g., eq. (5.4), the error model shall be sourced only from the observed orbits, $\widetilde{x}$ (which are part of the perturbation term, $\Delta \widetilde{\boldsymbol{x}}$ ). Then, from eq. (5.5), the complete error at each epoch can be formulated as

$$
\begin{equation*}
\boldsymbol{\varepsilon}_{i}=\boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{i}\right)+T \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t^{\prime}\right) d t^{\prime} \tag{5.12}
\end{equation*}
$$

where the dispersion matrix, $\boldsymbol{\Sigma}_{i}^{\left(\boldsymbol{y}^{(x)}\right)}$, of the reduced observation model at each epoch, $t_{i}$, will be a $3 \times 3$ fully populated matrix, i.e.,

$$
\boldsymbol{\Sigma}_{i}^{\left(\boldsymbol{y}^{(x)}\right)}=\left[\begin{array}{lll}
\boldsymbol{\Sigma}_{x_{i} x_{i}} & \boldsymbol{\Sigma}_{x_{i} y_{i}} & \boldsymbol{\Sigma}_{x_{i} z_{i}}  \tag{5.13}\\
\boldsymbol{\Sigma}_{y_{i} x_{i}} & \boldsymbol{\Sigma}_{y_{i} y_{i}} & \boldsymbol{\Sigma}_{y_{i} z_{i}} \\
\boldsymbol{\Sigma}_{z_{i} x_{i}} & \boldsymbol{\Sigma}_{z_{i} y_{i}} & \boldsymbol{\Sigma}_{z_{i} z_{i}}
\end{array}\right]
$$

where $\left(x_{i}, y_{i}, z_{i}\right)$ are the axes directions of the reduced observation vector, $\boldsymbol{y}^{(x)}\left(t_{i}\right)$. The diagonal elements of $\boldsymbol{\Sigma}_{i}^{\left({ }^{(x)}\right)}$ represent the variance components along the axis shown in the subscript, while the off-diagonal elements represent the covariance between the subscripted axes. In order to obtain the statistics of these errors at the discrete points in eq. (5.12), one must propagate the statistics of the position errors through the integral of the whole arc.

This implies a correlation amongst points within an arc, or at the very least, between adjacent points. Therefore, for a given $\operatorname{arc}$ of $\boldsymbol{Y}^{(x)}$, the variance-covariance matrix is,

$$
\begin{align*}
& \mathbf{\Sigma}^{\left(\boldsymbol{y}_{s}^{(x)}\right)} \\
& =\left[\begin{array}{cccc}
D\left\{\boldsymbol{y}_{(s-1) N+0}^{(x)}\right\} & C\left\{\boldsymbol{y}_{(s-1) N+0}^{(x)}, \boldsymbol{y}_{(s-1) N+1}^{(x)}\right\} & \ldots & C\left\{\boldsymbol{y}_{(s-1) N+0}^{(x)}, \boldsymbol{y}_{s N-1}^{(x)}\right\} \\
C\left\{\boldsymbol{y}_{(s-1) N+1}^{(x)}, \boldsymbol{y}_{(s-1) N+0}^{(x)}\right\} & D\left\{\boldsymbol{y}_{(s-1) N+1}^{(x)}\right\} & \ldots & C\left\{\boldsymbol{y}_{(s-1) N+1}^{(x)}, \boldsymbol{y}_{s N-1}^{(x)}\right\} \\
\vdots & \vdots & \ddots & \vdots \\
C\left\{\boldsymbol{y}_{s N-1}^{(x)}, \boldsymbol{y}_{(s-1) N+0}^{(x)}\right\} & \ldots & \ldots & D\left\{\boldsymbol{y}_{s N-1}^{(x)}\right\}
\end{array}\right] \tag{5.14}
\end{align*}
$$

where $D\{\cdot\}:=\boldsymbol{\Sigma}_{i}^{\left(\boldsymbol{y}^{(x)}\right)}$ is the dispersion operator, and $C\{\cdot\}$ is the covariance operator. Fundamentally, the variance-covariance matrix of the full dataset, $\boldsymbol{Y}^{(x)}$, should also be fully populated. However, in practice, it is not always easy to get such a fully populated matrix. For example, consider the GRACE Level 1B data, GNV1B and KBR1B (which provide the location coordinates of the GRACE satellites' orbit in an Earth fixed frame as tracked by GNSS, as well as the range-rate between the satellites), the former of which are given only with standard deviations per observation (Case, et al. 2010). In other words, at each epoch, the GNV1B state vectors are only given with the diagonal elements of eq. (5.13). To then get the fully populated dispersion matrix one must estimate it, for example, by iteratively using the post-fit residuals (Wu 2016). Such an approach is prone to only further complicate the estimation procedure and tends to be impractical for very long datasets (Mayer-Gürr, et al. 2005).

While the GNV1B state vector is supplied with standard deviations (but no correlation), the full correlation matrix of the errors modeled by eq. (5.12) could be obtained by error propagation (see Appendix E). However, for the present application, it turns out that the diagonal matrix of eq. (5.14) is sufficient to use as the dispersion matrix for the full dataset. Appendix E shows a short case study on this assertion; the spherical harmonic coefficients are estimated using a fully populated dispersion matrix compared to when using a diagonal matrix. No clear distinction is found between these estimates. Given a dispersion matrix that is positive definite, and assuming one variance factor for all axes, the weight matrix, $\boldsymbol{P}_{\boldsymbol{x}}$, is simply the inverse of the product of eq. (5.14) and the variance component, $\sigma_{x}^{2}$, i.e., $\boldsymbol{P}_{\boldsymbol{x}}=\frac{1}{\sigma_{x}^{2}}\left(\mathbf{\Sigma}^{\left(Y^{(x)}\right)}\right)^{-1}$.

### 5.1.3. Normal Equations

The system of normal equations is derived from minimizing the sum of weighted errors, which is a quadratic equation. In the form presented here, this minimization results in a least-squares solution. The Gauss-Markov model in eq. (5.7) can be re-written as

$$
\begin{equation*}
\boldsymbol{Y}^{(x)}=\boldsymbol{H}_{x} \xi+\boldsymbol{E} \tag{5.15}
\end{equation*}
$$

where $\boldsymbol{H}_{\boldsymbol{x}}=\left[\begin{array}{ll}\boldsymbol{A} & \boldsymbol{G}\end{array}\right]$ is a collection of the (orbital and gravitational) design matrices for the GNSS-based kinematic perturbation method, and $\boldsymbol{\xi}=\left[\begin{array}{ll}\boldsymbol{X}^{T} & \boldsymbol{p}^{T}\end{array}\right]$ is a collection of the (local and global) unknowns. The Lagrange target function to minimize is then,

$$
\begin{equation*}
\boldsymbol{\Phi}(\xi)=\left(\boldsymbol{Y}^{(x)}-\boldsymbol{H}_{x} \xi\right)^{T} \boldsymbol{P}_{x}\left(\boldsymbol{Y}^{(x)}-\boldsymbol{H}_{x} \xi\right) \tag{5.16}
\end{equation*}
$$

The target function is minimized if its derivative with respect to $\xi$ is zero. Therefore, the Euler-Lagrange necessary condition is set to

$$
\begin{equation*}
\frac{1}{2} \frac{\partial \boldsymbol{\Phi}}{\partial \xi}=\left(\boldsymbol{H}_{x}^{T} \boldsymbol{P}_{x} \boldsymbol{H}_{x}\right) \hat{\xi}-\boldsymbol{H}_{x}^{T} \boldsymbol{P}_{\boldsymbol{x}} \boldsymbol{Y}^{(\boldsymbol{x})} \doteq \mathbf{0} \tag{5.17}
\end{equation*}
$$

and the sufficient condition to find the multivariate minimum, is found from the second derivative of the target function, eq. (5.16), with respect to the error vector,

$$
\begin{equation*}
\frac{1}{2} \frac{\partial^{2} \boldsymbol{\Phi}}{\partial \boldsymbol{E} \partial \boldsymbol{E}^{T}}=\boldsymbol{P}_{\boldsymbol{x}} \tag{5.18}
\end{equation*}
$$

This condition is satisfied if the computed derivative is positive-definite, and from the assumptions made when developing the dispersion matrix, it follows that $\boldsymbol{P}_{\boldsymbol{x}}$ is positivedefinite by definition. From eq. (5.17) the solution to the parameter vector can be shown as a system of normal equations

$$
\begin{align*}
\widehat{\xi} & =\left(\boldsymbol{H}_{x}^{T} \boldsymbol{P}_{x} \boldsymbol{H}\right)^{-1} \boldsymbol{H}_{x}^{T} \boldsymbol{P}_{\boldsymbol{x}} \boldsymbol{Y}^{(\boldsymbol{x})} \\
& =\boldsymbol{N}^{-1} \boldsymbol{c} \tag{5.19}
\end{align*}
$$

where $\boldsymbol{N}$ is a normal matrix and $\boldsymbol{c}$ is a normal vector. In matrix form this system can be expanded back to

$$
\begin{gather*}
{\left[\begin{array}{cc}
\boldsymbol{A}^{T} \boldsymbol{P}_{\boldsymbol{x}} \boldsymbol{A} & \boldsymbol{A}^{T} \boldsymbol{P}_{\boldsymbol{x}} \boldsymbol{G} \\
\boldsymbol{G}^{T} \boldsymbol{P}_{\boldsymbol{x}} \boldsymbol{A} & \boldsymbol{G}^{T} \boldsymbol{P}_{\boldsymbol{x}} \boldsymbol{G}
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{X} \\
\boldsymbol{p}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{A}^{T} \\
\boldsymbol{G}^{T}
\end{array}\right] \boldsymbol{P}_{\boldsymbol{x}} \boldsymbol{Y}^{(x)}} \\
{\left[\begin{array}{cc}
\boldsymbol{N}_{\boldsymbol{x}} & \boldsymbol{N}_{x \boldsymbol{p}} \\
\boldsymbol{N}_{x p}^{T} & \boldsymbol{N}_{\boldsymbol{p}}^{(x)}
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{X} \\
\boldsymbol{p}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{c}_{\boldsymbol{x}} \\
\boldsymbol{c}_{\boldsymbol{p}}^{(x)}
\end{array}\right]} \tag{5.20}
\end{gather*}
$$

where the subscript $\boldsymbol{x}$ represents the orbit-related normal equations, while the subscript $\boldsymbol{p}$ denotes normal equations related to the gravitational unknowns. Detailed expansions of these are shown in Appendix B.

### 5.2. KBR-based Kinematic Perturbation Model Theory

### 5.2.1. Design Matrix

Analogous to eq. (5.4), the observation equation of the range rate perturbations, eq. (4.34), can be expressed for discrete epochs in time as

$$
\begin{align*}
y_{i}^{(\dot{\rho})}=\left(\frac{t_{B}-t_{i}}{T}\right. & \left.\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T}-\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T}\right) \boldsymbol{x}_{12}^{(A)} \\
& +\left(\frac{t_{i}-t_{A}}{T}\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T}+\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T}\right) \boldsymbol{x}_{12}^{(B)} \\
& -T\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T} \int_{t_{A_{A}}}^{t_{B}} K\left(t_{i}, t^{\prime}\right) \Delta \boldsymbol{g}_{12}\left(t^{\prime}\right) d t^{\prime}  \tag{5.21}\\
& -T\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T} \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t_{i}, t^{\prime}\right) \Delta \boldsymbol{g}_{12}\left(t^{\prime}\right) d t^{\prime}+\epsilon_{i}
\end{align*}
$$

where $y_{i}^{(\rho)}$ is the reduced observation term at discrete times, $t_{i}, i=0, \ldots, N-1$,

$$
\begin{align*}
y_{i}^{(\dot{\rho})}=\Delta \dot{\rho}\left(t_{i}\right) & +\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T}\left(T \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime}\right. \\
& \left.+\boldsymbol{x}_{12}^{(r e f)}\left(t_{i}\right)\right)  \tag{5.22}\\
& +\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T}\left(T \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{g}_{12}^{(r e f)}\left(t^{\prime}\right)+\boldsymbol{\Psi}_{12}\left(t^{\prime}\right)\right) d t^{\prime}\right. \\
& \left.+\dot{\boldsymbol{x}}_{12}^{(r e f)}\left(t_{i}\right)\right)
\end{align*}
$$

Collecting a set of observations gives the vector form

$$
\begin{equation*}
\boldsymbol{Y}^{(\dot{\rho})}=\boldsymbol{B} \boldsymbol{X}_{12}+\boldsymbol{W} \boldsymbol{p}+\boldsymbol{\epsilon} \tag{5.23}
\end{equation*}
$$

where $\boldsymbol{\epsilon}$ is the $S N+1$ sized vector of errors of the reduced observations, $\boldsymbol{p}$ is still the vector of the gravitational unknowns, $\boldsymbol{X}_{12}$ is the $3(S+1)$ sized vector of arc boundary residual estimates,

$$
\boldsymbol{X}_{12}^{T}=\left(\begin{array}{llll}
\widehat{\boldsymbol{x}}_{12}^{(0)} & \widehat{\boldsymbol{x}}_{12}^{(N)} & \ldots & \widehat{\boldsymbol{x}}_{12}^{(S N)} \tag{5.24}
\end{array}\right)^{T}
$$

$\boldsymbol{Y}^{(\dot{\rho})}$ is an $S N+1$ sized vector of reduced range rate perturbation observations,

$$
\boldsymbol{Y}^{(\dot{\rho})}=\left[\begin{array}{c}
y_{0}^{(\dot{\rho})}  \tag{5.25}\\
\vdots \\
y_{N-1}^{(\dot{\rho})} \\
\vdots \\
\hdashline y_{(S-1) N}^{(\dot{\rho})} \\
\vdots \\
y_{S N-1}^{(\rho)} \\
\hdashline y_{S N}^{(\dot{\rho})}
\end{array}\right]
$$

$\boldsymbol{W}$ is an $(S N+1) \times N_{\text {coef }}$ design matrix for the gravitational unknowns,

$$
\boldsymbol{W}=\left[\begin{array}{c}
\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{0}^{T} \boldsymbol{G}_{12}\left(t_{0}\right)+\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{0}^{T} \dot{\boldsymbol{G}}_{12}\left(t_{0}\right)  \tag{5.26}\\
\vdots \\
\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{N-1}^{T} \boldsymbol{G}_{12}\left(t_{N-1}\right)+\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{N-1}^{T} \dot{\boldsymbol{G}}_{12}\left(t_{N-1}\right) \\
\hdashline\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{(S-1) N}^{T} \boldsymbol{G}_{12}\left(t_{(S-1) N}\right)+\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{(S-1) N}^{T} \dot{\boldsymbol{G}}_{12}\left(t_{(S-1) N}\right) \\
\vdots \\
\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{S N-1}^{T} \boldsymbol{G}_{12}\left(t_{S N-1}\right)+\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{S N-1}^{T} \dot{\boldsymbol{G}}_{12}\left(t_{S N-1}\right) \\
\hdashline 0
\end{array}\right]
$$

with

$$
\begin{align*}
\boldsymbol{G}_{12}\left(t_{(s-1) N+i}\right)= & -T \int_{t_{(s-1) N}}^{t_{(s-1) N+i}} K\left(t_{(s-1) N+i}, t^{\prime}\right)\left(\Delta \boldsymbol{g}_{n, m}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right)\right.  \tag{5.27}\\
& \left.-\Delta \boldsymbol{g}_{n, m}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right)\right) d t^{\prime}
\end{align*}
$$

and

$$
\begin{align*}
\dot{\boldsymbol{G}}_{12}\left(t_{(s-1) N+i}\right)= & -T \int_{t_{(s-1) N}}^{t_{(s-1) N+i}} \frac{d}{d t} K\left(t_{(s-1) N+i}, t^{\prime}\right)\left(\Delta \boldsymbol{g}_{n, m}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right)\right.  \tag{5.28}\\
& \left.-\Delta \boldsymbol{g}_{n, m}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right)\right) d t^{\prime}
\end{align*}
$$

From eq. (5.21), let the coefficients of the initial and end arc-boundary residual unknowns be respectively denoted by,

$$
\begin{equation*}
\boldsymbol{b}_{i}^{(A)}=\frac{t_{B}-t_{i}}{T}\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T}-\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T} \quad \text { and } \quad \boldsymbol{b}_{i}^{(B)}=\frac{t_{i}-t_{A}}{T}\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T}+\frac{1}{T}\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T} \tag{5.29}
\end{equation*}
$$

Then, the coefficient matrix of these boundary unknowns, $\boldsymbol{B}$, which is of size $(S N+1) \times 3(S+1)$ is computed as,

$$
\boldsymbol{B}=\left[\begin{array}{ccccc}
\boldsymbol{b}_{0}^{(A)} & \boldsymbol{b}_{0}^{(B)} & \mathbf{0}_{3} & \ldots & \mathbf{0}_{3}  \tag{5.30}\\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\boldsymbol{b}_{N-1}^{(A)} & \boldsymbol{b}_{N-1}^{(B)} & \mathbf{0}_{3} & \cdots & \mathbf{0}_{3} \\
\hdashline \vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0}_{3} & \ldots & \ldots & \boldsymbol{b}_{(S-1) N}^{(A)} & \boldsymbol{b}_{(S-1) N}^{(B)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0}_{3} & \ldots & \ldots & \boldsymbol{b}_{S N-1}^{(A)} & \boldsymbol{b}_{S N-1}^{(B)} \\
\hdashline \mathbf{0}_{3} & \ldots & \ldots & \boldsymbol{b}_{S N}^{(A)} & \boldsymbol{b}_{S N}^{(B)}
\end{array}\right]
$$

### 5.2.2. Dispersion

Similar to (5.4), the only sources of error in eq. (5.21) are the observations. These include the range rate, $\dot{\rho}$, which is found in the range rate perturbation term, i.e., $\Delta \dot{\rho}:=\dot{\rho}-$ $\dot{\rho}^{(r e f)}$, and the observed orbits, $\left.\widetilde{x}_{i}\right|_{i=1,2}$, which are found in the gradient terms, $\left.\boldsymbol{\Gamma}_{r e f}\left(\boldsymbol{x}_{i}^{(r e f)}\right) \Delta \widetilde{\boldsymbol{x}}_{i}\right|_{i=1,2}$. These observations are assumed to be uncorrelated. Let $\epsilon_{\dot{\rho}}$ be the error from the range rates. Then, from eq. (4.35), the complete error in the reduced observation at each epoch is determined by

$$
\begin{align*}
\epsilon_{i}=\epsilon_{\dot{\rho}}\left(t_{i}\right)+ & T\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T} \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{2}}\left(t^{\prime}\right)\right. \\
& \left.-\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{1}}\left(t^{\prime}\right)\right) d t^{\prime} \\
& -T\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T} \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{2}}\left(t^{\prime}\right)\right.  \tag{5.31}\\
& \left.-\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{1}}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

Analogous to the generation of the dispersion matrix, eq. (5.14), for the KBR procedure the dispersion matrix for each arc is

$$
\boldsymbol{\Sigma}^{\left(Y_{s}^{(\rho)}\right)}=\left[\begin{array}{cccc}
\sigma_{(s-1) N+0}^{2} & \sigma_{(s-1) N+0,(s-1) N+1} & \ldots & \sigma_{(s-1) N+0, S N-1}  \tag{5.32}\\
\sigma_{(s-1) N+1,(s-1) N+0} & \sigma_{(s-1) N+1}^{2} & \ldots & \sigma_{(s-1) N+1, S N-1} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{S N-1,(s-1) N+0} & \ldots & \cdots & \sigma_{S N-1}^{2}
\end{array}\right]
$$

where $\sigma_{i}^{2}$ is the variance of the reduced observation at $t_{i}$, and $\sigma_{i, j}$ is the covariance between the reduced observation at $t_{i}$ and $t_{j}$. Analogous to the example from Appendix E, the complete dispersion for the full dataset is assumed to be diagonal for the present application. The weight matrix is then just $\boldsymbol{P}_{\dot{\rho}}=\frac{1}{\sigma_{\dot{\rho}}^{2}}\left(\boldsymbol{\Sigma}^{\left(\boldsymbol{Y}^{(\dot{\rho})}\right)}\right)^{-1}$, with the variance factor assumed to be equal for all range rate errors per arc. The observation variances, $\sigma_{i}^{2}$, are not to be confused with the variance factors: $\sigma_{\boldsymbol{x}}^{2}$ and $\sigma_{\dot{\rho}}^{2}$.

### 5.2.3. Normal Equations

As with the hl-SST model, the system of normal equations is computed by defining a performance index to minimize the sum of squares of the weighted residuals. Eq. (5.20) can be re-written as

$$
\begin{equation*}
\boldsymbol{Y}^{(\dot{\rho})}=\boldsymbol{H}_{\dot{\rho}} \boldsymbol{\xi}+\boldsymbol{\epsilon} \tag{5.33}
\end{equation*}
$$

where $\boldsymbol{H}_{\dot{\rho}}=\left[\begin{array}{ll}\boldsymbol{B} & \boldsymbol{W}\end{array}\right]$ is a collection of the gravitational and orbital design matrices, and the values to be estimated are still stored in $\boldsymbol{\xi}=\left[\begin{array}{ll}\boldsymbol{X}_{12}^{T} & \boldsymbol{p}^{T}\end{array}\right]$. Following a similar process to the one described for eq. (5.16) through (5.19), the unknown values solution can be shown to be

$$
\begin{align*}
\boldsymbol{\xi} & =\left(\boldsymbol{H}_{\dot{\rho}}^{T} \boldsymbol{P}_{\dot{\rho}} \boldsymbol{H}_{\dot{\rho}}\right)^{-1} \boldsymbol{H}_{\dot{\rho}}^{T} \boldsymbol{P}_{\dot{\rho}} \boldsymbol{Y}^{(\dot{\rho})}  \tag{5.34}\\
& =\boldsymbol{N}^{-1} \boldsymbol{c}
\end{align*}
$$

Extending these normal equations back to matrix form

$$
\begin{gather*}
{\left[\begin{array}{cc}
\boldsymbol{B}^{T} \boldsymbol{P}_{\dot{\rho}} \boldsymbol{B} & \boldsymbol{B}^{T} \boldsymbol{P}_{\dot{\rho}} \boldsymbol{W} \\
\boldsymbol{W}^{T} \boldsymbol{P}_{\dot{\rho}} \boldsymbol{B} & \boldsymbol{W}^{T} \boldsymbol{P}_{\dot{\rho}} \boldsymbol{W}
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{X}_{12} \\
\boldsymbol{p}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{B}^{T} \\
\boldsymbol{W}^{T}
\end{array}\right] \boldsymbol{P}_{\dot{\rho}} \boldsymbol{Y}^{(\dot{\rho})}} \\
{\left[\begin{array}{cc}
\boldsymbol{N}_{\dot{\rho}} & \boldsymbol{N}_{\dot{\rho} \boldsymbol{p}} \\
\boldsymbol{N}_{\dot{\rho} \boldsymbol{p}}^{T} & \boldsymbol{N}_{\boldsymbol{p}}^{(\dot{\rho})}
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{X}_{12} \\
\boldsymbol{p}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{c}_{\dot{\rho}} \\
\boldsymbol{c}_{\boldsymbol{p}}^{(\dot{\rho})}
\end{array}\right]} \tag{5.35}
\end{gather*}
$$

where the subscript $\dot{\rho}$ represents the orbit-related normal equations, and $\boldsymbol{p}$ still denotes the gravity-related normal equations. Detailed expansions of these are also shown in Appendix B.

### 5.3. Combination of GNSS and KBR Perturbation Solutions

Gravity solutions for GRACE-type missions, usually incorporate both high-low and low-low SST observations, so these will need to be combined into the final least squares solution. To ensure both systems are solving for identical parameters, the hl-SST model, eq. (5.7), is re-written in terms of relative positioning

$$
\begin{equation*}
\boldsymbol{Y}_{12}^{(\boldsymbol{x})}=\boldsymbol{A} \boldsymbol{X}_{12}+\boldsymbol{G}_{12} \boldsymbol{p}+\boldsymbol{E}_{12} \tag{5.36}
\end{equation*}
$$

where the parameters to be computed are the estimates of the boundary points residuals and the unknown spherical harmonic coefficients, found in $\boldsymbol{X}_{12}$ and $\boldsymbol{p}$, respectively. The observations include the positions of the leading and trailing satellite (used to compute the relative reduced observation, $\boldsymbol{Y}_{12}$ ). All derivations for dispersions and normal equations follow suit to Sections 5.1.2. and 5.1.3., respectively. The combined observation model is then denoted

$$
\left[\begin{array}{l}
\boldsymbol{Y}_{12}^{(x)}  \tag{5.38}\\
\boldsymbol{Y}^{(\dot{\rho})}
\end{array}\right]=\left[\begin{array}{cc}
\boldsymbol{A} & \boldsymbol{G}_{12} \\
\boldsymbol{B} & \boldsymbol{W}
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{X}_{12} \\
\boldsymbol{p}
\end{array}\right]
$$

or, similar to eq. (5.23), as

$$
\begin{equation*}
L=C X_{12}+D p \tag{5.39}
\end{equation*}
$$

where

$$
\boldsymbol{L}=\left[\begin{array}{l}
\boldsymbol{Y}_{12}^{(x)}  \tag{5.40}\\
\boldsymbol{Y}^{(\dot{\rho})}
\end{array}\right], \quad \boldsymbol{C}=\left[\begin{array}{c}
\boldsymbol{A} \\
\boldsymbol{B}
\end{array}\right], \quad \boldsymbol{D}=\left[\begin{array}{c}
\boldsymbol{G}_{12} \\
\boldsymbol{W}
\end{array}\right]
$$

Since the observations have different accuracies, with the range rates being much more accurate, they will need to be weighted differently. The dispersion matrix is then

$$
\boldsymbol{\Sigma}=\boldsymbol{P}^{-1}=\left[\begin{array}{cc}
\sigma_{\boldsymbol{x}_{12}}^{2} \boldsymbol{P}_{\boldsymbol{x}_{12}}^{-1} & \mathbf{0}  \tag{5.41}\\
\mathbf{0} & \sigma_{\dot{\rho}}^{2} \boldsymbol{P}_{\dot{\rho}}^{-1}
\end{array}\right]
$$

Normal equation system for eq. (5.32) can be derived to be

$$
\left[\begin{array}{cc}
\boldsymbol{C}^{T} \boldsymbol{P C} & \boldsymbol{C}^{T} \boldsymbol{P D}  \tag{5.42}\\
\boldsymbol{D}^{T} \boldsymbol{P C} & \boldsymbol{D}^{T} \boldsymbol{P D}
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{X}_{12} \\
\boldsymbol{p}
\end{array}\right]=\left[\begin{array}{l}
\boldsymbol{C}^{T} \\
\boldsymbol{D}^{T}
\end{array}\right] \boldsymbol{P L}
$$

which gives an estimate for the orbital arc boundaries,

$$
\begin{equation*}
\widehat{\boldsymbol{X}}_{12}=\left(\boldsymbol{C}^{T} \boldsymbol{P} \boldsymbol{C}\right)^{-1}\left(\boldsymbol{C}^{T} \boldsymbol{P L}-\boldsymbol{C}^{T} \boldsymbol{P} \boldsymbol{D} \widehat{\boldsymbol{p}}\right) \tag{5.43}
\end{equation*}
$$

Substituting this into the bottom equation of eq. (5.42),

$$
\begin{equation*}
\boldsymbol{D}^{T} \boldsymbol{P L}=\boldsymbol{D}^{T} \boldsymbol{P C}\left(\boldsymbol{C}^{T} \boldsymbol{P C}\right)^{-1}\left(\boldsymbol{C}^{T} \boldsymbol{P L}-\boldsymbol{C}^{T} \boldsymbol{P} \boldsymbol{D} \widehat{\boldsymbol{p}}\right)+\boldsymbol{D}^{T} \boldsymbol{P} \boldsymbol{D} \widehat{\boldsymbol{p}} \tag{5.44}
\end{equation*}
$$

and solving for $\widehat{\boldsymbol{p}}$,

$$
\begin{equation*}
\widehat{\boldsymbol{p}}=\left(\boldsymbol{D}^{T} \boldsymbol{P} \boldsymbol{D}-\boldsymbol{D}^{T} \boldsymbol{P C}\left(\boldsymbol{C}^{T} \boldsymbol{P C}\right)^{-1} \boldsymbol{C}^{T} \boldsymbol{P} \boldsymbol{D}\right)^{-1}\left(\boldsymbol{D}^{T} \boldsymbol{P}-\boldsymbol{D}^{T} \boldsymbol{P} \boldsymbol{C}\left(\boldsymbol{C}^{T} \boldsymbol{P} \boldsymbol{C}\right)^{-1} \boldsymbol{C}^{T} \boldsymbol{P}\right) \boldsymbol{L} \tag{5.45}
\end{equation*}
$$

gives an estimation for $\boldsymbol{p}$, without explicitly needing to find the best estimates for the boundary observations. In fact, estimate (5.45) is just a Schur complement of the parameter related normal equations, which can be expressed in terms of the normal equations as,

$$
\begin{equation*}
\hat{\boldsymbol{p}}=\left(\boldsymbol{N}^{\left(Y^{(x)}\right)}+\boldsymbol{N}^{\left(Y^{(\hat{\rho})}\right)}\right)^{-1}\left(\boldsymbol{c}^{\left(Y^{(x)}\right)}+\boldsymbol{c}^{\left(Y^{(\rho)}\right)}\right) \tag{5.46}
\end{equation*}
$$

where,

$$
\begin{gather*}
N^{\left(Y^{(x)}\right)}=N_{p}^{(x)}-N_{x p}^{T}\left(N_{x}\right)^{-1} N_{x p}  \tag{5.47}\\
N^{\left(Y^{(\dot{\rho})}\right)}=N_{p}^{(\dot{\rho})}-N_{\dot{\rho} p}^{T}\left(N_{\dot{\rho}}\right)^{-1} N_{\dot{\rho} p}  \tag{5.48}\\
c^{\left(Y^{(x)}\right)}=c_{p}^{(x)}-N_{x p}^{T}\left(N_{x}\right)^{-1} c_{x}  \tag{5.49}\\
c^{\left(Y^{(\dot{\rho})}\right)}=c_{p}^{(\dot{\rho})}-N_{\dot{\rho} p}^{T}\left(N_{\dot{\rho}}\right)^{-1} c_{\dot{\rho}} \tag{5.50}
\end{gather*}
$$

### 5.4. Problems with III-Conditioned Matrices

Estimation of the gravitational field is part of a set of problems known as inverse problems. Unfortunately, due to several factors, normal matrices to such problems tend to be ill-conditioned (Koch and Kusche 2002). For the present application, these include, but are not limited to, poor resolution and the attenuation of the field at shorter halfwavelengths. This is analogous to estimation problems with downward continuation (Ilk, et al. 1995, p. 36). Ill-conditioned problems convolute the physical meaning of a solution. To ameliorate the effects of such problems multiple solutions have been suggested and implemented, however in this section only Tikhonov regularization is considered. For a more detailed approach on solutions to ill conditioned problems, the reader is directed to Cicci (1987) and references therein.

### 5.4.1. Tikhonov Regularization for Gravity Recovery

Contrary to eq. (5.46), where the solution is just a minimization of the error in eq. (5.38), here the minimization is based on knowledge of prior information about the unknown parameters. Then the Lagrange target function of eq. (5.39), using Tikhonov regularization is (Ilk, et al. 1995, p. 39),

$$
\begin{equation*}
\Phi(\xi, \lambda)=\left\|\boldsymbol{L}-\boldsymbol{C} \boldsymbol{X}_{12}-\boldsymbol{D} \boldsymbol{p}\right\|_{P}^{2}+\lambda\left\|\xi-\xi^{(\mathbf{0})}\right\|_{S}^{2} \tag{5.51}
\end{equation*}
$$

where $\lambda$ is the Lagrange multiplier (regularization parameter), $\boldsymbol{\xi}^{(\boldsymbol{0})}$ is the prior information vector for the unknown parameters, and $\boldsymbol{S}$ is the symmetric and positive definite weight matrix of the a priori estimates. In the case where there is no prior information, then $\boldsymbol{\xi}^{(\mathbf{0})}$ is set to zero.

Following the approach derived for the solution to eq. (5.46): the orbit related normal equations are eliminated, and the so-called Kaula stabilization matrix, $\boldsymbol{K}$, is used as the weight matrix for the unknown gravitational parameters. The system of normal equations for gravity related parameters is (Koch and Kusche 2002),

$$
\begin{equation*}
\left(\boldsymbol{N}^{\left(Y^{(x)}\right)}+\omega \boldsymbol{N}^{\left(Y^{(\rho)}\right)}+\lambda \boldsymbol{K}\right) \widehat{\boldsymbol{p}}=\boldsymbol{c}^{\left(Y^{(x)}\right)}+\omega \boldsymbol{c}^{\left(Y^{(\rho)}\right)}+\lambda \boldsymbol{K} s \tag{5.52}
\end{equation*}
$$

where $\boldsymbol{s}$ is the prior information vector of the unknown gravitational parameters. The normal equations can be computed from eq. (5.47) to (5.50), and the coefficients $\omega$, and $\lambda$ are ratios of the variance components, with

$$
\begin{equation*}
\lambda=\frac{\sigma_{x}^{2}}{\sigma_{s}^{2}} \quad \text { and } \quad \omega=\frac{\sigma_{x}^{2}}{\sigma_{\dot{\rho}}^{2}} \tag{5.53}
\end{equation*}
$$

where $\sigma_{s}^{2}$ is the variance component of the spherical harmonic prior estimates. Alternatively, the regularization parameter can be computed from general cross-validation as described in Whaba (1990), and the variance components can be estimated iteratively, as shown in Koch and Kusche (2002). The Kaula matrix is a diagonal matrix derived using Kaula's rule of thumb for degree variances (Kaula 1966, p. 98),

$$
\boldsymbol{K}=\left[\begin{array}{ccc}
\frac{1}{\sigma_{n_{r e f}+1,-n_{r e f}-1}^{2}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{1}{\sigma_{n_{\max }, n_{\max }}^{2}}
\end{array}\right]
$$

and

$$
\begin{equation*}
\sigma_{n m}^{2}=\frac{10^{-10}}{n^{4}} \tag{5.55}
\end{equation*}
$$

## Chapter 6 Gravity Recovery

During a one day period, as the Earth rotates under the orbit, the ground track for the polar LEO satellites described in sections 3.1. and 4.1.2., whose orbital period is around 90 minutes, traces a path that divides the Earth into 16 sections spaced approximately $22.5^{\circ}$ apart at the equator. This corresponds to a half-wavelength of 2500 m and a gravitational resolution of spherical harmonic degree and order of $n_{\max }=8$. However, in practice spherical harmonics of much higher resolution than that need to be recovered, i.e., resolutions that obviously require many more densely spaced ground tracks. In light of that, this chapter is divided into two major sections for recovering the gravity field; one section uses one day's worth of data to validate the basic procedure, while the other uses multiple days corresponding to a higher gravitational resolution. For both sections it is emphasized that the boundary conditions are assumed to be observed precisely and as such are taken as true values, therefore only the spherical harmonic coefficients are estimated. Furthermore, the results of both sections are based on observations derived from constructed perturbations at the level of $\sigma=0.1 \mathrm{~m}$ in orbital position.

The estimates are evaluated in terms of a square-root of degree variances (SDV) of the error per degree, indicating a statistic of the absolute error in the estimation, and the root mean square (RMS) per degree of the relative errors, which approximately indicates the number of accurate digits of the estimated harmonics per degree, i.e.,

$$
\begin{equation*}
S D V_{n}=\sqrt{\sum_{m=-n}^{n}\left(\hat{C}_{n, m}-C_{n, m}\right)^{2}} \text { and } \quad R M S_{n}=\sqrt{\frac{1}{2 n+1} \sum_{m=-n}^{n}\left(\frac{\left|\hat{C}_{n, m}-C_{n, m}\right|}{C_{n . m}}\right)^{2}} \tag{6.1}
\end{equation*}
$$

where $C_{n, m}$ are the given EGM2008 spherical harmonic coefficients and $\hat{C}_{n, m}$ are the spherical harmonics estimated using the GNSS- and/or KBR-perturbation approaches. The (reduced) observation residuals for both perturbation approaches will also be computed. From eq. (5.36) and (5.23) these residuals are respectively

$$
\begin{gather*}
\boldsymbol{E}_{12}=\boldsymbol{Y}_{12}^{(\boldsymbol{x})}-\left(\boldsymbol{A} \widehat{\boldsymbol{X}}_{12}+\boldsymbol{G}_{12} \widehat{\boldsymbol{p}}\right)  \tag{6.2}\\
\boldsymbol{\epsilon}=\boldsymbol{Y}^{(\dot{\rho})}-\left(\boldsymbol{B} \widehat{\boldsymbol{X}}_{12}+\boldsymbol{W} \widehat{\boldsymbol{p}}\right) \tag{6.3}
\end{gather*}
$$

These residuals hold the consolidated error sources associated with the GNSS- and KBRperturbation models, respectively. They (residuals) then indicate the fit of the model to the observations.

The process of generating the values of the integrals needed to estimate the gravitational harmonics is illustrated in the flowchart in Figure 6.1. These values are generated in a similar procedure to the ones used for the previous validations (e.g. see Section 4.1.1. and 4.2.1.). Using initial orbital state vectors described in Table 3.1 (for the leading satellite), and in section 4.2.1. (for the trailing satellite), and EGM2008 spherical harmonics up to the relevant degree as truth, true orbit observations were simulated in the inertial frame. These satellite ephemerides are sampled at 1 s for the one-day studies, which corresponds to an along track spatial resolution of around 7 km ; and, at 45 s for the multiple day studies, which is around 300 km in spatial resolution. It is emphasized that, at this stage, the only effects on the system are assumed to be the gravitational forces (from EGM2008) and Earth rotation (from eq. (2.5)), while the non-gravitational forces and all observational errors are omitted. For each satellite, the true position is perturbed (by $\sigma$ $=0.1 \mathrm{~m})$ and fit with a spline to generate the corresponding reference orbit, which is differentiated with respect to time to generate the reference velocity. The reference orbits are then run through the numerical integrator to generate the relevant field parameter integration values, including the gradient of potential harmonics, $\Delta \boldsymbol{g}_{n, m}$. The true intersatellite ranges and range rates are computed from the relative true positions and velocities (eq. (4.6) and (4.11), respectively) and the reference KBR observations are computed using the corresponding relative reference quantities (eq. (4.17) and (4.18), respectively). In the present application, for each estimation the orbit is generated in a field with maximum harmonic degree equal to the maximum degree of the estimated harmonics. This is done to avoid aliasing (see Appendix F for more details). However, in practice, one does not have the luxury of picking the harmonic degree of the observed orbit; it is an orbit in the true total gravitational field. Figure 6.2 below shows the difference between orbits in various fields, specifically: $n_{\max }=24,36,60$, and 120 , and an orbit in a field of $n_{\max }=180$. The figure shows that, as the orbit gets closer to $n_{\max }=180$, the difference between the orbits decreases. This means that, in practice when one estimates high degrees, the problem of the difference between the observed orbit and the true orbit of the required gravitational signal, is less significant.


Figure 6.1. Flow chart of the process to generate the values of the integrals required to estimate the gravitational field using simulated "true" orbital state vectors, "true" ranging data, reference orbits, and reference ranging data.


Figure 6.2. Difference between a "true" orbit in a gravitational field of $n_{\max }=180$ and various orbits in gravitational fields: (a) $n_{\max }=24$, (b) $n_{\max }=36$, (c) $n_{\max }=60$ and (d) $n_{\max }=120$.

### 6.1. One Day

As aforementioned, one day of data is meant to be sufficient to estimate a gravitational field with resolution of $n_{\max }=8$ in degree and order. In light of this, three sets of GNSS and KBR observations with maximum harmonic degrees: $n_{\max }=8,24$, and 36 , are set up, all with the reference field defined by maximum degree and order, $n_{r e f}=$ 4 , in order to validate the process developed in Chapter 5, but also to determine to what resolution the gravitational field can be recovered using the perturbation approach developed here, with just one day's worth of data. It is restated that in this section all estimations will be from data sampled at an interval of 1 second. The accuracies of the hlSST and ll-SST perturbation models have been well established in Chapter 4. For the same orbital perturbation, the former model is accurate to $10^{-5} \mathrm{~m}$ in position (Figure 4.1) and the latter is accurate to $10^{-8} \mathrm{~m} / \mathrm{s}$ in range-rate (Figure 4.3). In both cases the major limitation to the model accuracy is the linearization (with respect to the reference orbit) and the integration error. Where both sets of observations are used, despite not yet having added noise to the observations, they are respectively assigned the following variances (based on the aforementioned model accuracies): $\sigma_{x_{12}}^{2}=10^{-10} \mathrm{~m}^{2}$ and $\sigma_{\rho}^{2}=10^{-16} \mathrm{~m}^{2} / \mathrm{s}^{2}$.

This ensures the GNSS observables do not completely overwhelm the KBR observables, in which case it would appear as if only hl-SST was involved in the estimation procedure.

Figure 6.3 compares the results from an estimation where the true gravitational field has maximum harmonic degree of $n_{\max }=8$. In this case, the hl-SST and 1l-SST observations (as well as their combinations) generally performed at the same level of accuracy. With both observations, the approach can recover at least 4-digits of accuracy in harmonic coefficient estimations. Figure 6.4 shows the residual plots of the hl-SST procedure and its ll-SST counterpart. The residuals are shown to be consistent with the model accuracies established in Chapter 4.


Figure 6.3. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking two satellites along a one day orbit in the true gravitational field defined by $n_{\max }=8$, and $n_{r e f}=4$.


Figure 6.4. Observation residuals of the GNSS (top) and KBR (bottom) perturbation approaches for $n_{\max }=8$, and $n_{\text {ref }}=4$ from EGM2008 for one 24 hr long orbit at random perturbations, $\sigma=0.1 \mathrm{~m}$, sampled at 1 s .

Having confirmed the model can recover gravitational coefficients required for one day worth of data, i.e., $n_{\max }=8$, the next step is to estimate higher gravitational resolutions, starting with coefficients up to $n_{\max }=24$ (still from a single day-long observation set, but always with a true field generated from true coefficients up to degree and order $n_{\max }=24$ ). For the range $n \leq n_{\max }=8$, Figure 6.5 shows that when using the KBR observables the rms of the relative error of the estimations are similar to the previous case, Figure 6.3. However, for the GNSS observables, the same comparison shows the estimations improved by an order of magnitude. Overall, the estimation errors from hl-SST were less than those from ll-SST by up to two orders of magnitude. Though this does seem counter intuitive (since the ll-SST observations are generally more accurate than hl-SST), it is important to note that inter-satellite observations are better suited for short-wavelength gravity recovery (McCullough 2017, p. 54), which is reflected in the condition number of the normal matrix, discussed in more detail below. The RMS values of the relative errors, on the bottom plot of Figure 6.5, show the spherical harmonics can be estimated up to 6digits of accuracy for the GNSS and combined observations, especially in the lower degrees. The observation residual plots in Figure 6.6, further emphasize the model consistency and accuracy, similar to that shown in Figures 4.1 and 4.3.


Figure 6.5. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking two satellites along a one day orbit in the true gravitational field defined by $n_{\max }=24$, and $n_{\text {ref }}=4$.


Figure 6.6. Observation residuals of the GNSS (top) and KBR (bottom) perturbation approaches for $n_{\max }=24$, and $n_{r e f}=4$ from EGM2008 for one 24 hr long orbit at random perturbations, $\sigma=0.1 \mathrm{~m}$, sampled at 1 s .

For further emphasis, the one day of observations is used to estimate spherical harmonics up to $n_{\max }=36$, while maintaining the same reference field as in the previous examples. From the square-root of degree variances and root mean squares of the relative error, shown in Figure 6.7, it is clear the results of such an estimation are intolerable for KBR-only observations. The results from the ll-SST observations are not even able to recover the coefficients to any decimal of accuracy. The observation residual plots of the $n_{\max }=36$ estimation (Figure 6.8), also show that, even though the data are not adequate to recover the coefficients from KBR-only observations, the model is still as accurate as previously shown (section 4.1.1. and 4.2.1.); see the discussion on the condition number of the normal matrix, below.


Figure 6.7. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking two satellites along a one day orbit in the true gravitational field defined by $n_{\max }=36$, and $n_{r e f}=4$.


Figure 6.8. Observation residuals of the GNSS (top) and KBR (bottom) perturbation approaches for $n_{\max }=36$, and $n_{\text {ref }}=4$ from EGM2008 for one 24 hr long orbit at random perturbations, $\sigma=0.1 \mathrm{~m}$, sampled at 1 s

Table 6.1 shows that the KBR-only solution in Figure 6.7 is highly ill-conditioned, i.e. it has a large condition number. This means that, despite the accuracy of the KBRperturbation model, $\boldsymbol{Y}^{(\dot{\rho})}$, even the slightest change in the normal vector, $\boldsymbol{c}^{\left(\boldsymbol{Y}^{(\dot{\rho})}\right)}$, causes a large error in the spherical harmonic solution. The reverse is true for the GNSS-only and combined solutions, due to the relatively low condition numbers of their normal matrices. It is worth noting, that in all instances, the combined solution is better than either of the models individually. This is of course due to the fact that, for the combined solutions the observations are weighted according their model accuracies. The notably higher condition numbers for the $n_{\max }=24$ and $n_{\max }=36$ cases can simply be ascribed to the insufficient data.

The computation of the residuals (in this chapter) is analogous to the model accuracy determination in Chapter 4, with the only exception being the spherical harmonic coefficients used on the residual gravitational field, $\Delta \boldsymbol{g}$. For Chapter 6, the estimated harmonic coefficients, $\widehat{\boldsymbol{p}}$, were used, whereas Chapter 4 used EGM2008 harmonic coefficients, $\boldsymbol{p}$. The magnitude of the parameters being estimated is $\boldsymbol{p}=O\left(10^{-7}\right)$. Therefore, when also considering the magnitudes of the design matrices ( $\boldsymbol{G}_{12}=O\left(10^{8} \mathrm{~m}\right)$ and $\boldsymbol{W}=O\left(10^{-2} \mathrm{~m} / \mathrm{s}\right)$ for the GNSS- and KBR- perturbation models, respectively), it then suffices that to maintain the same order of accuracy demonstrated in Chapter 4, $R M S_{n}(\widehat{\boldsymbol{p}}) \leq O\left(10^{7}\right)$. Since this is always the case, even for the KBR-only estimations with $n_{\max }=36$, it is then reasonable that, despite the high condition numbers, the residual plots demonstrated accuracies that are consistent with the original model accuracies.

| $\boldsymbol{n}_{\max }$ | $\boldsymbol{n}_{\text {ref }}$ | Normal Matrix Condition Number |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | GNSS - only | KBR - only | Combined Solution |
| 8 | 4 | $6.90 \times 10^{2}$ | $2.26 \times 10^{2}$ | $1.45 \times 10^{2}$ |
| 24 | 4 | $6.49 \times 10^{8}$ | $1.28 \times 10^{13}$ | $3.48 \times 10^{8}$ |
| 36 | 4 | $1.23 \times 10^{11}$ | $4.24 \times 10^{22}$ | $9.22 \times 10^{10}$ |

Table 6.1. Condition numbers of the normal matrices for the spherical harmonic estimations from one day's worth of data

### 6.2. Multiple Days

For an orbital dataset over a given period, it can be estimated that the maximum degree and order of the field that is recoverable for that orbit over that period is

$$
\begin{equation*}
n_{\max }=\frac{180^{\circ}}{\left(\frac{360^{\circ}}{\text { orbits per day } \times \text { no.of days }}\right)} \tag{6.4}
\end{equation*}
$$

In this instance, it is known the satellite makes about 16 orbits per day, therefore the minimum number of days required for a given $n_{\max }$ is given by

$$
\begin{equation*}
\text { no.of days }=\frac{n_{\max }}{8} \tag{6.5}
\end{equation*}
$$

Thus for $n_{\max }=24$ and 36 , one needs only a minimum of 3 and 5 days' worth of observations, respectively. Given $n_{\max }=24$, the highest spatial resolution of the data grid at the equator is $180^{\circ} / 24=7.5^{\circ}$, which corresponds to about 833 km . A low Earth orbit satellite travels with a velocity of about $7 \mathrm{~km} / \mathrm{s}$. It is then enough to use data sampled at around $100 s$ for the recovery of spherical harmonics up to $n_{\max }=24$. For $n_{\max }=36$, a sampling rate of $75 s$ is acceptable. However, for the sake of consistency (and because it does not significantly affect the estimation) the sampling rate is set to 45 s for both harmonic recoveries. The observables for this section were generated by numerically integrating the orbit in segments of 24 hours, where the initial state vector for each segment after the first one was taken equal to the computed state vector of the previous integration. The least-squares estimation, however, was performed using all observations from all days in one batch process.

Figure 6.9 (in comparison to Figure 6.3) shows an improvement in the results when using multiple arcs to estimate the harmonic coefficients up to degree and order, $n_{\max }=$ 24. The most improvement is seen in the ll-SST results, where at least an extra order of magnitude in accuracy was gained. The observation residuals are still consistent with the model accuracy, which as aforementioned is still also limited by the accuracy of the numerical integrator. Since, a resolution appropriate for the recovery of $n_{\max }=24$ is being used, Table 6.2 shows an improvement in the condition numbers of the normal matrices, with the biggest improvement (7 orders of magnitude) in the KBR-only solution.


Figure 6.9. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking two satellites for a duration of 3 days at 24 hr intervals (with known arc boundaries), in the true gravitational field defined by $n_{\max }=24$, and $n_{\text {ref }}=4$.


Figure 6.10. Observation residuals of the GNSS (top) and KBR (bottom) perturbation approaches for $n_{\max }=24$, and $n_{\text {ref }}=4$ from EGM2008 for three 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$, sampled at 45 s

Similarly, there was a significant improvement to the estimation results when using five days' worth of observations for the $n_{\max }=36$ estimation case. The KBR observations gained about six digits of accuracy compared to the single day case (Figure 6.11). This is further indicative of the consistency of the model. For the range $n \leq n_{\max }=24$, Figure 6.11 (compared to Figure 6.9) shows improved results in terms of the rms values of the relative error, especially for the GNSS observations. Since Figure 6.11 is generated from 5 days worth of observations (and estimates higher degree harmonics), as opposed to the 3 days used for Figure 6.9, this gives additional confidence that the aliasing error diminishes with more data and estimation of spherical harmonics of higher degrees. The observation residuals shown in Figure 6.12, still resemble the same order of magnitude shown in all past plots. In the next chapter, the GNSS/KBR perturbation models are used to study spherical harmonic estimations determined from data with uncorrelated noise. This is done to check if the consistency and accuracy emphasized in this chapter (in conjunction with Chapter 4 ) will still hold under different conditions.


Figure 6.11. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking two satellites along 24 hr long orbits (with known arc boundaries) for a duration of 5 days, in a true gravitational field defined by $n_{\max }=36$, and $n_{r e f}=4$.


Figure 6.12. Observation residuals of the GNSS (top) and KBR (bottom) perturbation approaches for $n_{\max }=36$, and $n_{r e f}=4$ from EGM2008 for five 24 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$, sampled at 45 s

| $\boldsymbol{n}_{\max }$ | $\boldsymbol{n}_{\text {ref }}$ | No. of Arcs | Normal Matrix Condition Number |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | GNSS - only | KBR-only | Combined Solution |
| 24 | 4 | 3 | $1.34 \times 10^{8}$ | $4.57 \times 10^{6}$ | $4.99 \times 10^{6}$ |
| 36 | 4 | 5 | $1.39 \times 10^{9}$ | $1.67 \times 10^{7}$ | $3.01 \times 10^{7}$ |

Table 6.2. Condition numbers of the normal matrices for the spherical harmonic estimations from multiple arcs worth of data.

## Chapter 7 Using Data with Uncorrelated Noise for Gravity Recovery

The application presented thus far has involved the use of observations free of any (random) (white) noise. This was useful in the analysis of the observation minus model residuals for various estimation cases and acts as a good supplement to the error propagation examination that will form a part of this chapter. The simulations showed consistency in the residuals independent of the parameters to be estimated for orbits up to 5 days in length. In this chapter higher degree and order spherical harmonic coefficients (specifically, $n_{\max }=60$ ) are also estimated using a month's worth of orbits with random white noise. Figure 7.1 shows the flow chart for generating the necessary integration values to perform this task. Analogous to Chapter 6, the data are sampled at intervals of $45 s$ from 24 hr long integral periods (with $1 s$ integration step size). The use of 24 hr integration periods is meant to reduce the integration error which, as shown in Appendix D, can accumulate rapidly with longer observation arcs. Similar to the multiple days' estimations in section 6.2. the least squares procedure in this chapter is done in one batch process, as shown in Chapter 5.

For Chapter 6, regularization was not applied because the normal matrices became ill-conditioned only when an attempt was made to estimate more harmonic coefficients than allowed by the resolution of the data (according to eq. (6.5)). However, if the data are corrupted by noise (as in this chapter), even if they have proper resolution, it is desirable to also include some form of regularization on the estimation process. It is noted that, for all estimations in this chapter the Tikhonov regularization was applied to the combined solution (as discussed in section 5.4.1.).


Figure 7.1. Flow chart of the process to add noise to the orbital data.

### 7.1. Error Propagation

The noise models generated here are in agreement with GRACE instrument simulations in Darbeheshti, et al. (2017). For positions, the amplitude spectral density (ASD) of the white noise is in the level of $\mathrm{cm} / \sqrt{\mathrm{Hz}}$, while for velocities it is just the numerical differential of that with respect to time (which corresponds to the level of $10 \mu \mathrm{~m} / \mathrm{s} / \sqrt{\mathrm{Hz}}$ ). The range ASD is in the order of a few $\mu \mathrm{m} / \sqrt{\mathrm{Hz}}$, and the range rate white noise is the numerical differential of the range noise with respect to time (which is just $2 \pi f$ multiplied by the ASD of the range, where $f$ is the frequency) (Thomas 1999, p. $\mathrm{B}-2$ ). Each error model is then added to its respective simulated observation.

A sample of the white noise for the first day of simulations is shown in Figure 7.2, for position (along each axis) and for range rates. Given that the data used is sampled at 45 s , the noise is averaged over moving windows of the same interval. Since all work thus
far has been in the spatial domain, the plots are in the same domain as opposed to the spectral domain. The time series can be derived from the spectral domain by taking the inverse Fourier transform of the power spectral density (PSD), where the PSD is the square of the ASD. This relationship is laid out elegantly in Jekeli (2017b, ch. 5) thus, interested readers are directed there for more details.





Figure 7.2. Time series of the simulated white noise input averaged over a 45 s moving window for the GNSS observations (top) and the inter-satellite range rates (bottom)

Recall that the complete errors at each epoch for the reduced observations of the GNSS- and KBR-perturbation models are respectively given by,

$$
\begin{align*}
\boldsymbol{\varepsilon}_{i}=\boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{i}\right)+ & T \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t^{\prime}\right) d t^{\prime}  \tag{7.1}\\
\epsilon_{i}=\epsilon_{\dot{\rho}}\left(t_{i}\right)+ & T\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T} \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{2}}\left(t^{\prime}\right)\right. \\
& \left.-\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{1}}\left(t^{\prime}\right)\right) d t^{\prime}  \tag{7.2}\\
& -T\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T} \int_{t_{A}}^{t_{B}} \frac{d}{d t} K\left(t_{i}, t^{\prime}\right)\left(\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{2}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{2}}\left(t^{\prime}\right)\right. \\
& \left.-\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}_{1}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}_{1}}\left(t^{\prime}\right)\right) d t^{\prime}
\end{align*}
$$

where the variables are as defined for eq. (5.12) and (5.31). Inserting the simulated errors of Figure 7.2 into the above equations gives the error models for the GNSS- and KBRperturbation reduced observations, respectively, the output of which is shown in Figure 7.3. It is noted that since eq. (7.1) and (7.2) involve integrals in time, the results shown in Figure 7.3 are also corrupted by the integration error (which cannot be avoided). The error for the GNSS model (Figure 7.3) increased by two orders of magnitude from the error in the GNSS observations (Figure 7.2). In the case of the KBR error model, compared to the white noise above, the error is larger by four orders of magnitude. These increases are a result of the aforementioned integration error accumulated from including the gradient term. This (along with Figure 4.5) further emphasizes that the gradient term cannot be neglected in the error propagation process. During the model validation process, the gradient term was able to act as a correction to the position perturbations, $\Delta \boldsymbol{x}$. This ensured the perturbation model could be used over day long arcs. However, since $\Delta \boldsymbol{x}$ contains position errors, the KBR reduced observations' error is completely dominated by the gradient term.


Figure 7.3. Time series of the propagated errors for the GNSS reduced observations in eq. (7.1) (top) and KBR reduced observation in eq. (7.2) (bottom).

### 7.2 Parameter Estimation

Before moving to higher degree estimations, it is important to give a perspective on the impact of observation errors to the estimation procedure. This can easily be shown by repeating estimation procedures from Chapter 6, but this time with noise corrupted observations. For this application, the procedures repeated are those analogous to Figure 6.9 and 6.11: using three and five days' worth of observations to estimate spherical harmonics defined by $n_{\max }=24$, and $n_{\max }=36$, respectively. For both cases $n_{r e f}=4$, as in the aforementioned figures. See Table 3.1 and section 4.2 .1 for the satellites' orbital elements description. While the simulated noise in the reduced observations is evidently highly correlated, the weight matrix in the least squares adjustment for the sake of expediency is kept diagonal. Modeling the correlation is an area of future research; however, see also Appendix E. The variance component estimation, which is part of the regularization process, is iterated up to 20 times or terminated when

$$
\begin{equation*}
\widehat{\boldsymbol{p}}^{(k+1)}-\widehat{\boldsymbol{p}}^{(k)}<10^{-13} \tag{7.3}
\end{equation*}
$$

where $\widehat{\boldsymbol{p}}$ is still the estimated spherical harmonic coefficients and $k$ is the iteration count. The convergence of $10^{-13}$ is chosen on the basis that it was the best accuracy attained from the error-free observations. As aforementioned, the regularization was only conducted on the combined solution.

Figure 7.4 shows that, by using data with uncorrelated noise to estimate spherical harmonic coefficients up to degree and order $n_{\max }=24$, the parameter estimates using only KBR observations lost up to two digits of accuracy compared to when the data were "error-free". Whereas with GNSS data, the rms of the relative errors remained consistent between the "error-free" observations and the uncorrelated data. The same is true for the square root of the degree variances, i.e. the parameter estimations from KBR data deteriorated while the GNSS results remained relatively in the same order of magnitude. This indicates that, the KBR observations are more sensitive to noise inclusion, mostly (as aforementioned) due to the stronger effect of the position errors which enter via the integral of the gradient term, on the resultant errors in the reduced KBR observations. In terms of magnitude, the regularized solution is generally similar to the GNSS-only solution, and in some instances the former solution is better, especially in the higher degrees.


Figure 7.4. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking two satellites for a duration of three days' worth of observations (with observed arc boundaries), in the true gravitational field defined by $n_{\max }=24$, and $n_{r e f}=4$ with 0.01 m noise in position and $0.1 \mu \mathrm{~m} / \mathrm{s}$ noise in range rates.

Figure 7.5 shows the statistics from the estimation of spherical harmonics up to degree and order $n_{\max }=36$ from five days' worth of continuously tracking two LEO satellites. Similar to Figure 7.4, the satellites' GNSS and KBR observations are corrupted by noise. The SDV and RMS of the relative errors in Figure 7.5 can be compared to those in Figure 6.11, where the observations where "error-free". The accuracy of estimating parameters using GNSS observations was largely unaffected by the inclusion of the uncorrelated noise; the lower degrees remain in the order of $O\left(10^{-13}\right)$ and the higher degrees are at $O\left(10^{-11}\right)$ (as was the case for Figure 6.11). The RMS of the relative error generally lost one digit of accuracy, with the previous case, Figure 6.11 , being in the range of $O\left(10^{-7}\right) \leq R M S_{n}(\widehat{\boldsymbol{p}}) \leq O\left(10^{-3}\right)$ and for the current case, Figure 7.5, $O\left(10^{-6}\right) \leq$ $R M S_{n}(\widehat{\boldsymbol{p}}) \leq O\left(10^{-2}\right)$. This gives further confidence in the conclusions of Figure 6.11, where it was asserted that the aliasing error will shrink as higher harmonic degrees are estimated, and more data are used. Despite the gains of the GNSS-perturbation model, the position errors found in the KBR-perturbation error model continue to be insurmountable. The estimation accuracy dropped by at least two orders of magnitude, and for some of the higher degrees, one could not recover the harmonics to any digit of accuracy. However, when combining the KBR observations with those from GNSS, and applying Tikhonov regularization, one is able improve the KBR solutions to a level akin to the GNSS-only solution.


Figure 7.5. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated from five days' worth of observations of two LEO satellites, in the true gravitational field defined by $n_{\max }=36$, and $n_{r e f}=4$ with 0.01 m noise in position and $0.1 \mu \mathrm{~m} / \mathrm{s}$ noise in range rates. The arc boundary points are also observed with error.

For $n_{\max }=60$ the half-wavelength is $\sim 300 \mathrm{~km}$, and according to eq. (6.5) one needs about 8 days' worth of observations to reasonably recover the spherical harmonics. In keeping with the sampling rate of the previous multiple day estimations, the step-size here is $45 s$ (the noise is also averaged with a moving window of the same size). Figure 7.6 shows the statistics of estimating spherical harmonics $n_{\max }=60$, and $n_{\text {ref }}=12$ using 30 days' worth of observations; which is almost four times the minimum requirement. The best results were again obtained from using GNSS observations corrupted by uncorrelated noise. In fact the square root of degree variances, from GNSS observations, are generally in the same order of magnitude as those shown for lower degree spherical harmonic estimates in Figure 7.4. and 7.5. For parameter estimates in the range $n \geq n_{\max }=50$, using only KBR observations did not recover spherical harmonics to any digit of accuracy (as shown by the rms of the relative errors).


Figure 7.6. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking two satellites for a duration of 15 days, in the true gravitational field defined by $n_{\max }=60$, and $n_{r e f}=12$ with 0.01 m noise in position and $0.1 \mu \mathrm{~m} / \mathrm{s}$ noise in range rates.

### 7.3 Comparison to Other Simulations

Kusche and Springer (2017, p. 28) simulate a month's worth of orbits of a single GRACE satellite with gravitational potential (eq. (2.1)) as the observable. This is done in an effort, to demonstrate the importance of regularization in parameter estimation. The estimation procedure is carried out up to degree and order $n_{\max }=30$ for three different cases: (i) error free simulated gravitational potential, (ii) simulated gravitational potential disturbed white noise ( $100 \mathrm{~m}^{2} / \mathrm{s}^{2}$ ), and (iii) disturbed potential and using the Kaula regularization matrix. The corresponding SDV are shown in Figure 7.7.

For the interval $4<n \leq 30$, the case (i) results of Figure 7.7 show at least four orders of magnitude less in SDV compared to any of the estimations in Chapter 6 where the data used had an adequate resolution. However, when there was insufficient data the perturbation theory did perform worse than the Kusche and Springer error free simulations, but only for the KBR- perturbation theory though (which, as already shown, is very susceptible to data insufficiency and error inclusion). For the same interval, the regularized solution from the perturbation theory, Figure 7.5, is three orders of magnitude better than the Kusche and Springer simulations, Figure 7.7.


Figure 7.7. SDV of harmonic coefficients defined by $n_{\max }=30$ and $n_{r e f}=1$ estimated by tracking a single satellite for a duration of 30 days and simulating gravitational potential that is error free and disturbed with white noise (Kusche and Springer 2017, p. 28).

## Chapter 8 Conclusion

As a contribution towards furthering the understanding of the geophysical nature of Earth, a technique to estimate the gravitational field using satellite tracking technology that purportedly works over arcs of arbitrary length was investigated in this report. The approach was originally proposed in a study by Xu (2008) and it was motivated by the ability of modern technology to continuously (and kinematically) track LEO satellites with GNSS. Below is a compendious summary of the work carried out as well as possible future works to further this research.

### 8.1. Summary

Xu's (2008) perturbation theory for gravity recovery was modified to make it computationally practical and analyzed in terms of its numerical feasibility. The modifications were of an analytic nature, so as to make the theory more realizable in practice while fully maintaining the conceptual aspects of the original method. The originally conceived perturbation model is in the form of a Volterra-integral equation of the second kind, and is highly reliant on the innovative idea that the reference orbit and reference gravitational field should be independent of each other. Using the theory of solutions to such equations to validate the (analytically) modified perturbation theory proved that, in its original conception, the perturbation theory is impractical. This is due to the accumulation of the integration error at each iteration stage, an error which was originally not considered. Instead of pursuing the formal, but numerically impractical solution to the Volterra integral equations, which might then be treated as a strict GaussMarkov model for estimating gravitational parameters, a rearrangement of terms leads to a model that is of nearly Gauss-Markov type and numerically practical and accurate. Using the Schneider (1968) model to maintain the same level of accuracy one would have to cap arc lengths at 30 mins, which implies over 40 additional boundary related nuisance parameters. Xu's modified perturbation model shows no such limit in arc length, but it was further modified in order to take advantage of the feature in Schneider's model that applies observational constraints at both ends of the arc. The final model was validated to be accurate in position to $O\left(10^{-5} \mathrm{~m}\right)$ for day-long orbits. A perturbation procedure for a lowlow SST model was developed and showed levels of accuracy commensurate to nominal GRACE - KBR accuracy, i.e. $O\left(10^{-8} \mathrm{~m} / \mathrm{s}\right)$.

To further validate the perturbation model, it was used to estimate spherical harmonics to varying degrees, specifically: $n_{\max }=8,24$, and 36 . Initially the simulated data were assumed to be "error-free" and only by the model linearization and integration errors entered the estimation process. When given error-free data of adequate resolution,
the combined GNSS and KBR observations improved the estimation of the spherical harmonic coefficients over the cases of either observation type alone. The combined solution was able to recover the spherical harmonics up to six digits of accuracy.

Following the use of the "error-free" data, uncorrelated noise was added to the observations, with amplitude spectral densities in the order of $\mathrm{cm} / \sqrt{\mathrm{Hz}}$ for GNSS positions and $\mu m / \sqrt{H z}$ for the KBR range observations. The noisy observations were used to estimate spherical harmonics up to degree and order $n_{\max }=24,36$ and 60 . The $n_{\max }=$ 24 estimation was computed with three days' worth of observations, $n_{\max }=36$ used five days, while $n_{\max }=60$ used 30 days. The noise was averaged over a moving window equivalent to the data sampling size, i.e. 45 s . Compared to the use of error-free observations, all showed a decline of accuracy in the ability to recover the spherical harmonic coefficients. This was most evident in the low-low SST observations were, even with adequate data, at least three orders of magnitude where lost in the rms of the relative errors of the spherical harmonic coefficients. This meant that, compared to error-free cases, the KBR-only coefficient estimations using uncorrelated noise, could only (at best) recover three digits less in the EGM2008 coefficients. This is in part due to the error contribution of the gradient term, which is three orders of magnitude larger than the nominal KBR error. The loss in accuracy was less significant the GNSS-only estimations and combined (regularized) solutions. In fact, for the cases that were conducted for both error-free and uncorrelated noise data, the accuracy remained consistent, especially when given adequate data.

For all its advantages in improving the model accuracy, when it comes to error propagation, the gradient term in fact corrupted the reduced observations to the point that it (gradient term) cannot be neglected, especially over day-long arcs. For example, the nominal error for GNSS positioning is $\sigma=0.01 \mathrm{~m}$ but the gradient term contributes the order of $\sigma=0.1 \mathrm{~m}$. Nonetheless these errors can be estimated and reasonably rectified to the point that one is still able to recover the gravity field with high accuracy, at least for the GNSS observations. However, it can be concluded that the GNSS estimations (with or without uncorrelated noise) were fairly reasonable especially if one looks at results from other gravity recovery methods (see, e.g. (Kusche and Springer 2017, p. 28)).

### 8.2. Future Recommendations

The work done so far has gone a long way in proving the consistency of the perturbation theory towards gravity recovery and developing avenues of improving the original theory. However, the theory tested here only involved orbits affected by gravitational forces and Earth rotation. It would then be of interest to include other perturbing forces into the orbit determination process, these include but are not limited to: third-body effects, atmospheric drag, general relativistic drag, and solar radiation pressure. Models for these are available through the notes from the International Earth Rotation and Reference Systems Service (IERS) (Petit and Luzum 2010). One would have to model these and treat them as reductions to the total accelerations, in order to solve for the gravitational parameters. Multiple studies have already carried out similar procedures, therefore emulating them should be a relatively straightforward task.

As noted in Chapter 5, in reality the dispersion matrix is fully populated. For the perturbation theory modified in this report, the gradient integrals involved in the reduced observation error models are highly correlated. Therefore, one may wish to look into modelling the correlation and studying the effects of using fully populated matrices especially over longer arcs. This would most necessarily be desired when one investigates the use of low-low SST, which showed the largest effect of adding noise to the observations. This would obviously require much higher computing power than the one accessible for investigating this report. Lastly, a more thorough investigation into the aliasing problem discussed briefly in Chapter 6 and shown in Appendix F, is also highly warranted.

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## Appendix A The Integral of the Kernel Function

The observation equations of the GNSS- and KBR-based perturbation models involve integrals with the Green function as a kernel, in the form of,

$$
\begin{equation*}
F(t)=-T \int_{t_{A}}^{t_{B}} K\left(t, t^{\prime}\right) f\left(t^{\prime}\right) d t^{\prime} \tag{A.1}
\end{equation*}
$$

It is however noted that integrals of this kernel have a discontinuity at the point of computation, $t$, thus one needs to break integrals like eq. (A.1) into two intervals $\left[t_{A}, t\right.$ ] and $\left[t, t_{B}\right]$ (Mayer-Gürr et al. 2005). This expansion results in,

$$
\begin{align*}
F(t) & =\int_{t_{A}}^{t}\left(t-t^{\prime}\right) f\left(t^{\prime}\right) d t^{\prime}-\frac{t-t_{A}}{T} \int_{t}^{t_{B}}\left(t_{B}-t^{\prime}\right) f\left(t^{\prime}\right) d t^{\prime}  \tag{A.2}\\
& =G(t)-\frac{t-t_{A}}{T} G\left(t_{B}\right)
\end{align*}
$$

Differentials of these integrals with respect to time are expanded as,

$$
\begin{align*}
\dot{F}(t) & =\frac{d}{d t} F(t) \\
& =\int_{t_{A}}^{t} f\left(t^{\prime}\right) d t^{\prime}-\frac{1}{T} \int_{t}^{t_{B}}\left(t_{B}-t^{\prime}\right) f\left(t^{\prime}\right) d t^{\prime}  \tag{A.3}\\
& =\dot{G}(t)-\frac{1}{T} G\left(t_{B}\right)
\end{align*}
$$

## Appendix B Normal Equations Matrices

The submatrices of the normal equations in eq. (5.20) and (5.35) are divided into coefficients of the boundary coordinates, $\boldsymbol{N}_{\boldsymbol{x}}$ and $\boldsymbol{N}_{\dot{\rho}}$, and coefficients of the unknown field parameters, $\boldsymbol{N}_{\boldsymbol{p}}^{(\boldsymbol{x})}$ and $\boldsymbol{N}_{\boldsymbol{p}}^{(\dot{\rho})}$, along the diagonal. The off diagonal submatrices combine coefficients of both unknowns in $\boldsymbol{N}_{\boldsymbol{x p}}$ and $\boldsymbol{N}_{\dot{\rho} \boldsymbol{p}}$. For GNSS-based kinematic perturbation, the boundary coordinates submatrix, $\boldsymbol{N}_{\boldsymbol{x}}$, is shown in the next page.
where as before, $S$ is the number of arcs, $N$ is the number of epochs, and $\boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})}$ is the dispersion matrix of the reduced observations. Let the gravitational unknowns design matrix, eq. (5.11), be

$$
\mathbf{G}=\left[\begin{array}{ccc}
\boldsymbol{h}_{0,1} & \ldots & \boldsymbol{h}_{0, N_{\text {coef }}}  \tag{B.2}\\
\vdots & \vdots & \vdots \\
\boldsymbol{h}_{N-1,1} & \cdots & \boldsymbol{h}_{N-1, N_{c o e f}} \\
\hdashline \boldsymbol{h}_{(S-1) N, 1} & \cdots & \boldsymbol{h}_{(S-1) N, N_{c o e f}} \\
\vdots & \vdots & \vdots \\
\boldsymbol{h}_{S N-1,1} & \cdots & \boldsymbol{h}_{S N-1, N_{\text {coef }}} \\
\hdashline \mathbf{0}_{3} & \cdots & \mathbf{0}_{3}
\end{array}\right]
$$

where

$$
\begin{equation*}
\boldsymbol{h}_{i, j}=-T \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right) \Delta \boldsymbol{g}_{n, m}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \tag{B.3}
\end{equation*}
$$

Then the off-diagonal submatrix, $\boldsymbol{N}_{\boldsymbol{x p}}$, is

$$
\boldsymbol{N}_{x \boldsymbol{p}}=\frac{1}{T}\left[\begin{array}{ccc}
\sum_{i=0}^{N-1}\left(t_{N}-t_{i}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{h}_{i, 1} & \ldots & \sum_{i=0}^{N-1}\left(t_{N}-t_{i}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{h}_{i, N_{c o e f}}  \tag{B.4}\\
\sum_{i=0}\left(t_{i}-t_{0}\right) \boldsymbol{\Sigma}_{i}^{(y))^{-1}} \boldsymbol{h}_{i, 1} & \ldots & \sum_{i=0}^{N-1}\left(t_{i}-t_{0}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{h}_{i, N_{c o e f}} \\
\vdots & \ldots & \vdots \\
\sum_{i=(S-1) N}^{S N}\left(t_{S N}-t_{i}\right) \boldsymbol{\Sigma}_{i}^{(y)-1} \boldsymbol{h}_{i, 1} & \ldots & \sum_{i=(S-1) N}^{S N}\left(t_{S N}-t_{i}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{h}_{i, N_{c o e f}} \\
\sum_{i=(S-1) N}^{S N}\left(t_{i}-t_{(S-1) N}\right) \boldsymbol{\Sigma}_{i}^{(y)^{-1}} \boldsymbol{h}_{i, 1} & \ldots & \sum_{i=(S-1) N}^{S N}\left(t_{i}-t_{(S-1) N}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{h}_{i, N_{c o e f}}
\end{array}\right]
$$

and finally the gravitational related unknowns submatrix is,

$$
\boldsymbol{N}_{p}^{(x)}=\left[\begin{array}{cccc}
\sum_{s=1}^{S} \sum_{i=0}^{N-1} \boldsymbol{h}_{(s-1) N+i, 1}^{T}\left(\boldsymbol{\Sigma}_{(s-1) N+i}^{(y)}\right)^{-1} \boldsymbol{h}_{(s-1) N+i, 1} & \cdots & \sum_{s=1}^{S} \sum_{i=0}^{N-1} \boldsymbol{h}_{(s-1) N+i, 1}^{T}\left(\boldsymbol{\Sigma}_{(s-1) N+i}^{(y)}\right)^{-1} \boldsymbol{h}_{(s-1) N+i, N \text { coef }}  \tag{B.5}\\
\vdots & \vdots & \vdots \\
\sum_{s=1}^{S} \sum_{i=0}^{N-1} \boldsymbol{h}_{(s-1) N+i, N_{\text {coef }}}^{T}\left(\boldsymbol{\Sigma}_{(s-1) N+i}^{(y)}\right)^{-1} \boldsymbol{h}_{(s-1) N+i, 1} & \cdots & \sum_{s=1}^{S} \sum_{i=0}^{N-1} \boldsymbol{h}_{(s-1) N+i, N_{c o e f}}^{T}\left(\boldsymbol{\Sigma}_{(s-1) N+i}^{(y)}\right)^{-1} \boldsymbol{h}_{(s-1) N+i, N_{\text {coef }}}
\end{array}\right]
$$

Similarly, the normal vectors are,

$$
\boldsymbol{c}_{\boldsymbol{x}}=\left[\begin{array}{c}
\sum_{i=0}^{N-1}\left(t_{N}-t_{i}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{y}_{i}^{(\boldsymbol{x})}  \tag{B.6}\\
\sum_{i=0}^{N-1}\left(t_{i}-t_{0}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{y}_{i}^{(\boldsymbol{x})} \\
\vdots \\
\sum_{i=(S-1) N}^{S N}\left(t_{S N}-t_{i}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{y}_{i}^{(\boldsymbol{x})} \\
\sum_{i=(S-1) N}^{S N}\left(t_{i}-t_{(S-1) N}\right) \boldsymbol{\Sigma}_{i}^{(\boldsymbol{y})^{-1}} \boldsymbol{y}_{i}^{(\boldsymbol{x})}
\end{array}\right]
$$

and

$$
\boldsymbol{c}_{\boldsymbol{p}}^{(\boldsymbol{x})}=\left[\begin{array}{c}
\sum_{s=1}^{S} \sum_{i=0}^{N-1} \boldsymbol{h}_{(s-1) N+i, 1}^{T}\left(\boldsymbol{\Sigma}_{(s-1) N+i}^{(\boldsymbol{y})}\right)^{-1} \boldsymbol{y}_{(s-1) N+i}^{(\boldsymbol{x})}  \tag{B.7}\\
\vdots \\
\sum_{s=1}^{S} \sum_{i=0}^{N-1} \boldsymbol{h}_{(s-1) N+i, N_{\text {coef }}}^{T}\left(\boldsymbol{\Sigma}_{(s-1) N+i}^{(\boldsymbol{y})}\right)^{-1} \boldsymbol{y}_{(s-1) N+i}^{(\boldsymbol{x})}
\end{array}\right]
$$

In the case of the low-low SST the submatrices of the normal system of equations can be expanded as shown on the next page.

$$
\boldsymbol{N}_{\bar{\rho}}=\left[\begin{array}{ccccc}
\sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(A)} & \sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(B)} & \mathbf{0}_{3} & \ldots & \mathbf{0}_{3} \\
\sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(A)} & \sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(B)} & \mathbf{0}_{3} & \ldots & \mathbf{0}_{3} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ldots & \sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(A)}+\left(\boldsymbol{b}_{S N}^{(A)}\right)^{T} \sigma_{S N}^{-2} \boldsymbol{b}_{S N}^{(A)} & \sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(B)}+\left(\boldsymbol{b}_{S N}^{(A)}\right)^{T} \sigma_{S N}^{-2} \boldsymbol{b}_{S N}^{(B)} \\
\mathbf{0}_{3} & \ldots & \ldots & \sum_{i=(S-1) N}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(A)}+\left(\boldsymbol{b}_{S N}^{(B)}\right)^{T} \sigma_{S N}^{-2} \boldsymbol{b}_{S N}^{(A)} & \sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} \boldsymbol{b}_{i}^{(B)}+\left(\boldsymbol{b}_{S N}^{(B)}\right)^{T} \sigma_{S N}^{-2} \boldsymbol{b}_{S N}^{(B)}
\end{array}\right]
$$

Analogous to eq. (B.2), let the design matrix for the KBR-perturbation approach be,

$$
\boldsymbol{W}=\left[\begin{array}{ccc}
w_{0,1} & \cdots & w_{0, N_{\text {coef }}}  \tag{B.9}\\
\vdots & \vdots & \vdots \\
w_{N-1,1} & \cdots & w_{N-1, N_{\text {coef }}} \\
\vdots & \vdots & \vdots \\
\hdashline w_{(S-1) N, 1} & \cdots & w_{(S-1)}, N_{\text {coef }} \\
\vdots & \vdots & \vdots \\
w_{S N-1,1} & \cdots & w_{S N-1, N_{\text {coef }}} \\
0_{3} & \cdots & 0_{3}
\end{array}\right]
$$

where

$$
\begin{equation*}
w_{i, j}=\left(\dot{\boldsymbol{e}}_{12}^{(r e f)}\right)_{i}^{T} \boldsymbol{h}_{i, j}^{(12)}+\left(\boldsymbol{e}_{12}^{(r e f)}\right)_{i}^{T} \dot{\boldsymbol{h}}_{i, j}^{(12)} \tag{B.10}
\end{equation*}
$$

Then the off-diagonal matrix is,

$$
\boldsymbol{N}_{\dot{\rho} \boldsymbol{p}}=\left[\begin{array}{ccc}
\sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} w_{i, 1} & \ldots & \sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} w_{i, N_{c o o f}}  \tag{B.11}\\
\sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} w_{i, 1} & \ldots & \sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} w_{i, N_{c o e f}} \\
\vdots & \ldots & \vdots \\
\sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} w_{i, 1}+\left(\boldsymbol{b}_{S N}^{(A)}\right)^{T} \sigma_{S N}^{-2} w_{S N, 1} & \ldots & \sum_{\substack{i=(S-1) N}}^{S N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} w_{i, N_{c o o f}}+\left(\boldsymbol{b}_{S N}^{(A)}\right)^{T} \sigma_{S N}^{-2} w_{S N, N_{c o e f}} \\
\sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} w_{i, 1}+\left(\boldsymbol{b}_{S N}^{(A)}\right)^{T} \sigma_{S N}^{-2} w_{S N, 1} & \ldots & \sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} w_{i, N_{c o e f}}+\left(\boldsymbol{b}_{S N}^{(B)}\right)^{T} \sigma_{S N}^{-2} w_{S N, N c o e f}
\end{array}\right]
$$

and finally the gravitational related unknowns submatrix is,

$$
\begin{align*}
& \boldsymbol{N}_{\boldsymbol{p}}^{(\dot{\rho})} \\
& =\left[\begin{array}{ccc}
\sum_{s=1}^{S} \sum_{i=0}^{N-1} w_{(s-1) N+i, 1} \sigma_{(s-1) N+i}^{-2} w_{(s-1) N+i, 1} & \cdots & \sum_{s=1}^{S} \sum_{i=0}^{N-1} w_{(s-1) N+i, 1} \sigma_{(s-1) N+i}^{-2} w_{(s-1) N+i, N_{\text {coef }}} \\
\vdots & \vdots & \vdots \\
\sum_{s=1}^{S} \sum_{i=0}^{N-1} w_{(s-1) N+i, N_{\text {coef }}} \sigma_{(s-1) N+i}^{-2} w_{(s-1) N+i, 1} & \cdots & \sum_{s=1}^{S} \sum_{i=0}^{N-1} w_{(s-1) N+i, N_{\text {coef }}} \sigma_{(s-1) N+i}^{-2} w_{(s-1) N+i, N_{c o e f}}
\end{array}\right] \tag{B.12}
\end{align*}
$$

The normal vectors are then,

$$
\boldsymbol{c}_{\dot{\rho}}=\left[\begin{array}{c}
\sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} y_{i}^{(\dot{\rho})}  \tag{B.13}\\
\sum_{i=0}^{N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} y_{i}^{(\dot{\rho})} \\
\vdots \\
\sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(A)}\right)^{T} \sigma_{i}^{-2} y_{i}^{(\dot{\rho})}+\left(\boldsymbol{b}_{S N}^{(A)}\right)^{T} \sigma_{S N}^{-2} y_{S N}^{(\dot{\rho})} \\
\sum_{i=(S-1) N}^{S N-1}\left(\boldsymbol{b}_{i}^{(B)}\right)^{T} \sigma_{i}^{-2} y_{i}^{(\dot{\rho})}+\left(\boldsymbol{b}_{S N}^{(B)}\right)^{T} \sigma_{S N}^{-2} y_{S N}^{(\hat{\rho})}
\end{array}\right]
$$

and

$$
\boldsymbol{c}_{\boldsymbol{p}}^{(\dot{\rho})}=\left[\begin{array}{c}
\sum_{s=1}^{S} \sum_{i=0}^{N-1} w_{(s-1) N+i, 1} \sigma_{(s-1) N+i}^{-2} y_{(s-1) N+i}^{(\dot{\rho})}  \tag{B.14}\\
\vdots \\
\sum_{s=1}^{S} \sum_{i=0}^{N-1} w_{(s-1) N+i, N_{c o e f}} \sigma_{(s-1) N+i}^{-2} y_{(s-1) N+i}^{(\dot{\rho})}
\end{array}\right]
$$

## Appendix C Keplerian Element Perturbations

The perturbation theory developed and validated in this report, has been solely based on inertial frame Cartesian coordinates and intersatellite observations that can be derived from such coordinates. This theory can just as equivalently be defined for Keplerian elements: semi-major axis, $a$, eccentricity, $e$, inclination, $i$, argument of perigee, $\omega$, ascending node, $\Omega$, and the time of perigee, $t_{p}$ (Xu 2008). However, for present applications, the mean anomaly, $M$, is more commonly used in place of the time of perigee. Therefore, let $\boldsymbol{k}=\left[\begin{array}{llllll}a & e & i & \Omega & \omega & M\end{array}\right]^{T}$ be the vector of Keplerian elements.

The mean anomaly can be deduced from the time of perigee through the relationship,

$$
\begin{equation*}
M=\tilde{n} \cdot\left(t-t_{p}\right) \tag{C.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{n}=\sqrt{\frac{G M}{a^{3}}} \tag{C.2}
\end{equation*}
$$

is the so-called mean motion of the satellite and $t$ as always is the computation time. It is noted that the tilde accent as used through Appendix C is not meant to imply an estimation, as is usually the case, but rather to distinguish affected variables from similar variables that have already been used abundantly through the report. In that same regard, let $\widetilde{\boldsymbol{q}}$ and $\dot{\widetilde{\boldsymbol{q}}}$ be the position and velocity coordinates in some $q$-frame; which is a coordinate frame with $z$ axis perpendicular to the orbital plane, and $x$-axis pointing to the perigee,

$$
\widetilde{\boldsymbol{q}}=\left[\begin{array}{c}
a(\cos E-e)  \tag{C.3}\\
a \sqrt{1-e^{2}} \sin E \\
0
\end{array}\right] \quad \text { and } \quad \dot{\widetilde{\boldsymbol{q}}}=\frac{\tilde{n} a}{1-e \cos E}\left[\begin{array}{c}
-\sin E \\
\sqrt{1-e^{2}} \cos E \\
0
\end{array}\right]
$$

where $E$ is the eccentric anomaly, and is computed iteratively from the mean anomaly using,

$$
\begin{equation*}
M=E-e \sin E \tag{C.4}
\end{equation*}
$$

The transformation between the inertial frame coordinates and the $q$-frame (orbital elements) is given by,

$$
\left[\begin{array}{l}
\boldsymbol{x}  \tag{C.5}\\
\dot{x}
\end{array}\right]=\boldsymbol{R}_{3}(-\Omega) \boldsymbol{R}_{1}(-i) \boldsymbol{R}_{3}(-\omega)\left[\begin{array}{c}
\widetilde{\boldsymbol{q}} \\
\dot{\tilde{\boldsymbol{q}}}
\end{array}\right]
$$

where $\boldsymbol{R}_{1}(\cdot)$ and $\boldsymbol{R}_{3}(\cdot)$ are the clockwise three-dimensional rotation matrices in the $x$ - and z-axis direction, respectively.

From eq. (C.5) one can always obtain the Keplerian elements from the Cartesian coordinates or vice-versa. The perturbation relationship can as promptly be derived from,

$$
\Delta \boldsymbol{k}=\boldsymbol{T}_{x k}^{-1} \cdot\left[\begin{array}{l}
\Delta \boldsymbol{x}  \tag{C.6}\\
\Delta \dot{\boldsymbol{x}}
\end{array}\right]
$$

where

$$
T_{x k}=\frac{\partial\left[\begin{array}{l}
x  \tag{C.7}\\
\dot{x}
\end{array}\right]}{\partial k}
$$

which can be computed according to the procedure described in Kaula (1966, p. 67).

## Appendix D Perturbations Approaching Zero

This section looks at two of the tests that have been foregone in the main text towards the verification of the Xu-model. Firstly, the effect of using longer arcs on the modified one-boundary-point perturbation method (compared to Figure 3.4). Secondly, the result of using lower perturbations (compared to Table 3.2 and Figure 3.4).

As in section 3.1., the initial step towards testing the perturbation model accuracy with multiple-day long arcs is to validate the numerical integrator, DVDQ, over such long arcs. Using the same LEO satellite described for Table 3.1, Figure D. 1 shows that the integrator is accurate to better than 0.4 mm in position, after a week-long orbit. This is an order of magnitude worse off than the 1-day results (see Figure 3.1). Significant depletion in accuracy can be seen after 1.5 days, however, according to Figure D. 1 in order to maintain the same level of integration accuracy as before, one needs only to keep the arc length, under 3 days.


Figure D.1. Accuracy of numerical integrator for a LEO satellite for a 7 Day long Keplerian orbit using the multi-step predictor-corrector integrator, DVDQ.

In light of this, tests using 3-day long arcs are done on the one-boundary-point perturbation method, restated here,

$$
\begin{align*}
\Delta \boldsymbol{x}(t)=\boldsymbol{x}_{A}+ & \dot{\boldsymbol{x}}_{A} \cdot\left(t-t_{A}\right)-\boldsymbol{x}^{(r e f)}(t) \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{g}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) d t^{\prime} \\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \Delta \boldsymbol{g}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right), \boldsymbol{p}\right) d t^{\prime}  \tag{D.1}\\
& +\int_{t_{A}}^{t}\left(t-t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \Delta \boldsymbol{x}\left(t^{\prime}\right) d t^{\prime}
\end{align*}
$$

where all variables are as defined during model development (section 3.2.). Figure D. 2 shows the error from the linear approximation shown in eq. (D.1) after three days of observations (for perturbations at $\sigma=0.1 \mathrm{~m}$ ). This error is shown to be less than 0.1 cm , which is an order of magnitude worse than the results after one day (see Figure 3.4). The biggest loss in accuracy is seen immediately after a day of observations, with one order of magnitude lost over the next two days. Based on this, to use longer orbits for model validation would be imprudent.


Figure D.2. Absolute differences between the LHS and RHS of eq. (D.1), for $n_{\max }=60$, and $n_{r e f}=12$ from EGM2008 for 72 hr long orbits at random perturbations, $\sigma=0.1 \mathrm{~m}$.

The next step is to then test eq. (D.1) under perturbations smaller than, $\sigma=0.1 \mathrm{~m}$. Figure D. 3 shows that for perturbations, $\sigma=0.01 \mathrm{~m}$ and $\sigma=1 \mathrm{~mm}$, respectively, model (D.1) accuracy is comparable to when the perturbation is to $\sigma=0.1 \mathrm{~m}$. It is reiterated that, $\sigma=0.01 \mathrm{~m}$ is actually equivalent to the nominal GNSS-error and thus perturbations any lower than that, are impractical in view of using real data. In Tables 3.2 and 4.1 the GNSSperturbation model is tested for various perturbations, and it was shown that $\sigma=0.1 \mathrm{~m}$ had the highest level of accuracy for perturbation levels tested. Table D. 1 shows analogous tests, at lower perturbation, specifically: $\sigma=0.01 \mathrm{~m}$ and $\sigma=1 \mathrm{~mm}$.


Figure D.3. Absolute differences between the LHS and RHS of eq. (D.1), for $n_{\max }=60$, and $n_{\text {ref }}=12$ from EGM2008 for 24 hr long orbits at random perturbations: $\sigma=0.01 \mathrm{~m}$ (top) and $\sigma=1 \mathrm{~mm}$ (bottom)

| Perturb, $\boldsymbol{\sigma}$ <br> $[\mathbf{m}]$ | $\boldsymbol{n}_{\text {max }}$ | $\boldsymbol{n}_{\text {ref }}$ | Absolute maximum error <br> $[\mathbf{m m}]$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{z}$ |
| 0.01 | 24 | 4 | 0.013 | 0.034 | 0.005 |
|  | 36 | 12 | 0.015 | 0.047 | 0.028 |
|  | 60 | 12 | 0.030 | 0.061 | 0.012 |
|  | 120 | 12 | 0.020 | 0.046 | 0.026 |
|  |  |  |  |  |  |
| 0.001 | 24 | 4 | $8.718 \times 10^{-4}$ | 0.007 | 0.016 |
|  | 36 | 12 | $3.788 \times 10^{-4}$ | 0.039 | 0.022 |
|  | 60 | 12 | 0.024 | 0.074 | 0.006 |
|  | 120 | 12 | 0034. | 0.074 | 0.004 |

Table D.1. Absolute maximum differences for LHS and RHS of eq. (D.1) for varying fields and perturbations at the end of a 1-day orbit

## Appendix E Error Analysis

For high-low SST, recall the error at each epoch is,

$$
\begin{equation*}
\boldsymbol{\varepsilon}_{i}=\boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{i}\right)+T \int_{t_{A}}^{t_{B}} K\left(t_{i}, t^{\prime}\right) \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t^{\prime}\right)\right) \boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t^{\prime}\right) d t^{\prime} \tag{E.1}
\end{equation*}
$$

which can be approximated using a simple rectangle rule for numerical integration as

$$
\begin{equation*}
\boldsymbol{\varepsilon}_{i} \approx \boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{i}\right)+\frac{T}{N} \boldsymbol{K}_{i} \mathbf{T} \boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{i}\right) \tag{E.2}
\end{equation*}
$$

with $d t^{\prime}=1 / N$,

$$
\boldsymbol{K}_{i}=\left[\begin{array}{lll}
K\left(t_{i}, t_{0}\right) \boldsymbol{I}_{3} & \ldots & K\left(t_{i}, t_{N-1}\right) \boldsymbol{I}_{3} \tag{E.3}
\end{array}\right]
$$

and

$$
\mathbf{T}=\left[\begin{array}{ccc}
\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t_{0}\right)\right) & \ldots & \mathbf{0}_{3}  \tag{E.4}\\
\vdots & \ddots & \vdots \\
\mathbf{0}_{3} & \ldots & \boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t_{N-1}\right)\right)
\end{array}\right]
$$

For all epochs, eq. (E.2) is

$$
\begin{align*}
& {\left[\begin{array}{c}
\boldsymbol{\varepsilon}_{0} \\
\vdots \\
\boldsymbol{\varepsilon}_{N-1}
\end{array}\right]} \\
& =\left[\begin{array}{c}
\boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{0}\right) \\
\vdots \\
\boldsymbol{\varepsilon}_{\boldsymbol{x}}\left(t_{N-1}\right)
\end{array}\right] \\
& +\frac{T}{N}\left[\begin{array}{ccc}
K\left(t_{0}, t_{0}\right) \boldsymbol{I}_{3} & \ldots & K\left(t_{0}, t_{N-1}\right) \boldsymbol{I}_{3} \\
\vdots & \ddots & \vdots\left(t_{N-1}, t_{0}\right) \boldsymbol{I}_{3} \\
\hline & \ldots & K\left(t_{N-1}, t_{N-1}\right) \boldsymbol{I}_{3}
\end{array}\right]\left[\begin{array}{ccc}
\boldsymbol{\Gamma}^{(r e f)}\left(\boldsymbol{x}^{(r e f)}\left(t_{0}\right)\right) & \ldots & \mathbf{0}_{3} \\
\vdots & \ddots & \vdots \\
\mathbf{0}_{3} & \ldots & \boldsymbol{\Gamma}^{(r e f)}\left(\begin{array}{c}
\boldsymbol{x}^{(r e f)}\left(t_{N-1}\right)
\end{array}\right)
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\varepsilon}_{x}\left(t_{0}\right) \\
\vdots \\
\boldsymbol{\varepsilon}_{x}\left(t_{N-1}\right)
\end{array}\right] \tag{E.5}
\end{align*}
$$

which is compiled as

$$
\begin{equation*}
\boldsymbol{\varepsilon}=\varepsilon_{x}+\frac{T}{N} \boldsymbol{K} \mathbf{T} \varepsilon_{x} \tag{E.6}
\end{equation*}
$$

The error propagation of eq. (E.6) is

$$
\begin{align*}
D\{\boldsymbol{\varepsilon}\} & =D\left\{\boldsymbol{\varepsilon}_{\boldsymbol{x}}+\frac{T}{N} \boldsymbol{K} \mathbf{T} \boldsymbol{\varepsilon}_{\boldsymbol{x}}\right\} \\
& =\sigma_{x}^{2}\left(\boldsymbol{I}_{3(N-1)}+\left(\frac{T}{N}\right)^{2} \boldsymbol{K} \mathbf{T T}^{T} \boldsymbol{K}^{T}\right) \tag{E.7}
\end{align*}
$$

assuming (from eq. (4.3)) $D\{\Delta \boldsymbol{x}\}=\sigma_{x}^{2} \boldsymbol{I}$. Let $\boldsymbol{P}=\sigma_{\boldsymbol{x}}^{-2}\left(\boldsymbol{I}_{3(N-1)}+\left(\frac{T}{N}\right)^{2} \boldsymbol{K} \mathbf{T T}^{T} \boldsymbol{K}^{T}\right)^{-\mathbf{1}}$, then analogous to eq. (5.45), the spherical harmonic solution is

$$
\begin{equation*}
\widehat{\boldsymbol{p}}=\left(\boldsymbol{G}^{T} \boldsymbol{P} \boldsymbol{G}-\boldsymbol{G}^{T} \boldsymbol{P} \boldsymbol{A}\left(\boldsymbol{A}^{T} \boldsymbol{P} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{T} \boldsymbol{P} \boldsymbol{G}\right)^{-1}\left(\boldsymbol{G}^{T} \boldsymbol{P}-\boldsymbol{G}^{T} \boldsymbol{P} \boldsymbol{A}\left(\boldsymbol{A}^{T} \boldsymbol{P} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{T} \boldsymbol{P}\right) \boldsymbol{Y}^{(\boldsymbol{x})} \tag{E.8}
\end{equation*}
$$

Following a similar procedure to chapter 7, eq. (E.8) is used to estimate spherical harmonics up to degree and order $n_{\max }=8$. The results of this estimation are shown in Figure E.1, and show a loss of one order of magnitude in the RMS of the relative error. However, when comparing figures (E.1) and (E.2) no significant difference is found between using the full dispersion matrix and a diagonal dispersion matrix. This illustrates the point made in chapter 5, that for this specific application, it is enough to just use a diagonal dispersion matrix.


Figure E.1. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by using a fully populated dispersion matrix from tracking one satellite along a one day orbit in the true gravitational field defined by $n_{\max }=8$, and $n_{r e f}=4$.


Figure E.2. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by using a diagonal dispersion matrix from tracking one satellite along a one day orbit in the true gravitational field defined by $n_{\max }=8$, and $n_{\text {ref }}=4$.

## Appendix F On the Case of Leakage and Aliasing

As aforementioned, all spherical harmonic estimations were carried out using orbits generated in a field of similar maximum harmonic degree to the required coefficients. This section looks at the case where these maximum fields vary, using GNSS observations that are error free. For this application the observations are kept at $n_{\text {max }}=8$, and are 24 hrs long with $1 s$ interval sample rates.

For the first test, an attempt is made to recover spherical harmonics up to degree and order $n_{\max }=10$, in a reference field of $n_{r e f}=2$. According to Figure F.1, such an estimation is clearly aliased, as it only contain signals up to degree and order 8. However, for $n \leq 8$ the estimations are comparable to those in Figure 6.3, where the estimations involved only the fully available gravitational signal. Theoretically, given such a case with real data, one could always truncate the estimations at the point where the aliasing begins. For the second test, the effort is on recovering spherical harmonics up to degree and order 6. Figure F. 2 shows a deterioration in the accuracy by 4 orders of magnitude compared to the original estimation in Figure 6.3. It is noted that in the event of estimations such as those in Figure F.2, the available gravitational signal has to spread itself among the required coefficients in a procedure analogous to spectral leakage. The largest "leakage" will fall on the nearest harmonics to the maximum field, resulting in those harmonics being the least accurate as shown with the RMS in Figure F.2.


Figure F.1. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking one satellite in a $n_{\max }=8$ field for 24 hrs (with known arc boundaries), then estimating the gravitational field defined by $n_{\max }=10$, and $n_{r e f}=2$.


Figure F.2. SDV (top) and RMS of relative errors (bottom) of harmonic coefficients estimated by tracking one satellite in a $n_{\max }=8$ field for 24 hrs (with known arc boundaries), then estimating the gravitational field defined by $n_{\max }=6$, and $n_{r e f}=2$.

